Chapter 9
Matrices and Determinants

9.1 Introduction:

In many economic analysis, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation. The concept of determinant and is based on that of matrix. Hence we shall first explain a matrix.

9.2 Matrix:

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called $m \times n$ matrix (read “m by n matrix”) .

An $m \times n$ matrix is expressed as

$$A=\begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------ \\ --- & ---- & ------ \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix}$$

The letters a_{ij} stand for real numbers. Note that a_{ij} is the element in the ith row and jth column of the matrix .Thus the matrix A is sometimes denoted by simplified form as (a_{ij}) or by $\{a_{ij}\}$ i.e., $A = (a_{ij})$

Matrices are usually denoted by capital letters A, B, C etc and its elements by small letters a, b, c etc.

Order of a Matrix:

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3 x 2) read as three by two. In general if m are rows and n are columns of a matrix, then its order is (m x n).

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are matrices of orders (2 x 3), (3 x 1) and (4 x 4) respectively.

9.3 Some types of matrices:

1. Row Matrix and Column Matrix:

A matrix consisting of a single row is called a **row matrix or a row vector**, whereas a matrix having single column is called a **column matrix or a column vector.**

2. Null or Zero Matrix:

A matrix in which each element is ‘0’ is called a Null or Zero matrix. Zero matrices are generally denoted by the symbol O. This distinguishes zero matrix from the real number 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

The matrix O_{mxn} has the property that for every matrix A_{mxn},

$A + O = O + A = A$

3. Square matrix:

A matrix A having same numbers of rows and columns is called a square matrix. A matrix A of order m x n can be written as A_{mxn}. If m = n, then the matrix is said to be a square matrix. A square matrix of order n x n, is simply written as A_n.

Thus $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ are square matrix of order 2 and 3

Main or Principal (leading)Diagonal:

The principal diagonal of a square matrix is the ordered set of elements a_{ij}, where i = j, extending from the upper left-hand corner to the lower right-hand corner of the matrix. Thus, the principal diagonal contains elements a_{11}, a_{22}, a_{33} etc.

For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of elements 1, 2 and 0, in that order.

Particular cases of a square matrix:

(a)Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

For example $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are diagonal matrices.

In general $A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ --- & --- & --- & --- \\ --- & --- & --- & --- \\ a_{n1} & a_{n2} & --- & a_{nn} \end{bmatrix} = (a_{ij})_{nxn}$

is a diagonal matrix if and only if

$a_{ij} = 0$ for $i \neq j$

$a_{ij} \neq 0$ for at least one $i = j$

(b) Scalar Matrix:

A diagonal matrix in which all the diagonal elements are same, is called a scalar matrix i.e.

Thus

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ are scalar matrices

(c) Identity Matrix or Unit matrix:

A scalar matrix in which each diagonal element is 1(unity) is called a unit matrix. An identity matrix of order n is denoted by I_n.

Thus $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the identity matrices of order 2 and 3 .

In general, $A = \begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------- \\ --- & ---- & ------- \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix} = [a_{ij}]_{mxn}$

is an identity matrix if and only if

$a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$

Note: If a matrix A and identity matrix I are comformable for multiplication, then I has the property that

AI = IA = A i.e., I is the identity matrix for multiplication.

4. Equal Matrices:

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B i.e for each i, j, $a_{ij} = b_{ij}$

Thus $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

then A = B because the order of matrices A and B is same and $a_{ij} = b_{ij}$ for every i , j.

Example 1: Find the values of x , y , z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution : By the definition of equality of matrices, we have

$x + 3 = 0$ ………………………………..(1)

$2y + x = -7$ ……………………………(2)

$z - 1 = 3$ ……………………………(3)

$4a - 6 = 2a$ ……………………………(4)

From (1) $x = -3$

Put the value of x in (2) , we get $y = -2$
From (3) $z = 4$
From (4) $a = 3$

5. The Negative of a Matrix:

The negative of the matrix A_{mxn}, denoted by $-A_{mxn}$, is the matrix formed by replacing each element in the matrix A_{mxn} with its additive inverse. For example,

If $$A_{3x2} = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}$$

Then $$-A_{3x2} = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$$

for every matrix A_{mxn}, the matrix $-A_{mxn}$ has the property that

$A + (-A) = (-A) + A = 0$

i.e., $(-A)$ is the additive inverse of A.

The sum $B_{m-n} + (-A_{mxn})$ is called the difference of B_{mxn} and A_{mxn} and is denoted by $B_{mxn} - A_{mxn}$.

9.4 Operations on matrices:

(a) Multiplication of a Matrix by a Scalar:

If A is a matrix and k is a scalar (constant), then kA is a matrix whose elements are the elements of A , each multiplied by k

For example, if $A = \begin{bmatrix} 4 & -3 \\ 8 & -2 \\ -1 & 0 \end{bmatrix}$ then for a scalar k,

$$kA = \begin{bmatrix} 4k & -3k \\ 8k & -2k \\ -k & 0 \end{bmatrix}$$

Also, $$3\begin{bmatrix}5 & -8 & 4\\ 0 & 3 & -5\\ 3 & -1 & 4\end{bmatrix} = \begin{bmatrix}15 & -24 & 12\\ 0 & 9 & -15\\ 9 & -3 & 12\end{bmatrix}$$

(b) Addition and subtraction of Matrices:

If A and B are two matrices of same order m×n then their sum A + B is defined as C, m×n matrix such that each element of C is the sum of the corresponding elements of A and B .

for example

If $A = \begin{bmatrix}3 & 1 & 2\\ 2 & 1 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0 & 2\\ -1 & 3 & 0\end{bmatrix}$

Then $C = A + B = \begin{bmatrix}3+1 & 1+0 & 2+2\\ 2-1 & 1+3 & 4+0\end{bmatrix} = \begin{bmatrix}4 & 1 & 4\\ 1 & 4 & 4\end{bmatrix}$

Similarly, the difference A – B of the two matrices A and B is a matrix each element of which is obtained by subtracting the elements of B from the corresponding elements of A

Thus if $A = \begin{bmatrix}6 & 2\\ 7 & -5\end{bmatrix}$, $B = \begin{bmatrix}8 & 1\\ 3 & 4\end{bmatrix}$

then $$A - B = \begin{bmatrix}6 & 2\\ 7 & -5\end{bmatrix} - \begin{bmatrix}8 & 1\\ 3 & 4\end{bmatrix} = \begin{bmatrix}6-8 & 2-1\\ 7-3 & -5-4\end{bmatrix}$$
$$= \begin{bmatrix}-2 & 1\\ 4 & -9\end{bmatrix}$$

If A, B and C are the matrices of the same order mxn

then $A + B = B + A$

and $(A + B) + C = A + (B + C)$ i.e., the addition of matrices is commutative and Associative respectively.

Note: The sum or difference of two matrices of different order is not defined.

(c) Product of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B. Then the product matrix AB has the same number of rows as A and the same number of columns as B.

Thus the product of the matrices A_{mxp} and B_{pxn} is the matrix $(AB)_{mxn}$. The elements of AB are determined as follows:

The element C_{ij} in the ith row and jth column of $(AB)_{mxn}$ is found by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots\ldots\ldots + a_{in}b_{nj}$

for example, consider the matrices

$$A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B_{2x2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Since the number of columns of A is equal to the number of rows of B ,the product AB is defined and is given as

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Thus c_{11} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the first column of B i.e., b_{11} , b_{21} and adding the product.

Similarly , c_{12} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the second column of B i.e., b_{12} , b_{22} and adding the product. Similarly for c_{21} , c_{22} .

Note :

1 . Multiplication of matrices is not commutative i.e., AB $\neq$ BA in general.

2 . For matrices A and B if AB = BA then A and B commute to each other

3 . A matrix A can be multiplied by itself if and only if it is a square matrix.The product A.A in such cases is written as A^2.
Similarly we may define higher powers of a square matrix i.e., $A . A^2 = A^3$, $A^2 . A^2 = A^4$

4. In the product AB, A is said to be pre multiple of B and B is said to be post multiple of A.

Example 1: If A = $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **and B =** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **Find AB and BA.**

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}$$

This example shows very clearly that multiplication of matrices in general, is not commutative i.e., AB $\neq$ BA.

Example 2: If

Example 2: If $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$**, find AB**

Solution:

Since A is a (2 x 3) matrix and B is a (3 x 2) matrix, they are conformable for multiplication. We have

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$$

Remark:

If A, B and C are the matrices of order (m x p), (p x q) and (q x n) respectively, then

i. (AB)C = A(BC) i.e., Associative law holds.

ii. C(A+B) = CA + CB and (A + B)C = AC + BC }i.e distributive laws holds.

Note: that if a matrix A and identity matrix I are conformable for multiplication, then I has the property that

AI = IA = A i.e, I is the identity matrix for multiplication.

Exercise 9.1

Q.No. 1 Write the following matrices in tabular form:

i. A = $[a_{ij}]$, where i = 1, 2, 3 and j = 1, 2, 3, 4
ii. B = $[b_{ij}]$, where i = 1 and j = 1, 2, 3, 4
iii. C = $[c_{jk}]$, where j = 1, 2, 3 and k = 1

Q.No.2 Write each sum as a single matrix:

i. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1 & 3 \end{bmatrix}$

iii. $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$

iv. $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

v. $2\begin{bmatrix} 6 & 1 \\ 0 & -3 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -5 & -1 \end{bmatrix}$

Q.3 Show that $\begin{bmatrix} b_{11} - a_{11} & b_{12} - a_{12} \\ b_{21} - a_{21} & b_{22} - a_{22} \end{bmatrix}$ is a solution of the matrix equation X + A = B, where A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Q.4 Solve each of the following matrix equations:

i. $X + \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

ii. $X + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -2 & 0 \end{bmatrix}$

iii. $3X + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

iv. $X + 2I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

Q.5 Write each product as a single matrix:

i. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

ii. $[3 \quad -2 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}$

iv. $\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

Q.6 If $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find $A^2 + BC$.

Q.7 Show that if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then

(a) $(A + B)(A + B) \neq A^2 + 2AB + B^2$

(b) $(A + B)(A - B) \neq A^2 - B^2$

Q.8 Show that:

(i) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 5 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 2b + 3c \\ 2a + b \\ 3a + 5b - c \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ +\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.9 If $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Show that A and B commute.

Answers 9.1

Q.1(i) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(ii) $\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \end{bmatrix}$

(iii) $\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$

Q.2 (i) $\begin{bmatrix} 8 & 4 & 4 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 6 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 10 \\ 3 \\ -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 4 & 0 \\ -9 & 13 & 7 \end{bmatrix}$

Q.4 (i) $\begin{bmatrix} 2 & 2 \\ -5 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & -\frac{1}{3} \\ -1 & -1 & -1 \\ -\frac{4}{3} & \frac{2}{3} & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Q.5 (i) $\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.6 $\begin{bmatrix} 6 & 17 \\ 8 & 9 \end{bmatrix}$

9.5 Determinants:

The Determinant of a Matrix:

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by det A = |A| = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11} a_{22} - a_{12} a_{21}$$

Example 3: If $A = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ find |A|

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} = 9 - (-2) = 9 + 2 = 11$$

The determinant of the (3 x 3) matrix

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, denoted by $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

is given as, det A = |A|

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note: Each determinant in the sum (In the R.H.S) is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

These determinants are called minors. We take the sign + or $-$, according to $(-1)^{i+j} a_{ij}$

Where i and j represent row and column.

9.6 Minor and Cofactor of Element:

The minor M_{ij} of the element a_{ij} in a given determinant is the determinant of order (n – 1 x n – 1) obtained by deleting the ith row and jth column of A_{nxn}.

For example in the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of the element a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

The minor of the element a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij} of the matrix A.

Note: The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as

$a_{11}M_{11} – a_{12}M_{12} + a_{13}M_{13}$ Or $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors.

The value of the determinant can be found by expanding it from any row or column.

Example 4: If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

find det A by expansion about (a) the first row (b) the first column.

Solution (a)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 3(4 + 6) - 2(0 + 2) + 1\,(0 - 1)$

$= 30 - 4 - 1$

$|A| = 25$

(b) $|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3\,(4 + 6) + 1\,(-4 - 1)$

$= 30 - 5$

$|A| = 25$

9.7 Properties of the Determinant:

The following properties of determinants are frequently useful in their evaluation:

1. Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., $|A| = |A'|$). For example, consider a determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ... (2)

Now again consider

$$|B| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expand it by first column

$|B| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

which is same as equation (2)

so $|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

or $|B| = |A|$

2. If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed but its absolute value is unchanged.

For example if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider the determinant,

$$|B| = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

expand by second row,

$$|B| = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$
$$= -(a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$$

The term in the bracket is same as the equation (2)

So $$|B| = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Or $|B| = -|A|$

3. If every element of a row or column of a determinant is zero, the value of the determinant is zero. For example

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= 0(b_2c_3 - b_3c_2) - 0(a_2c_3 - a_3c_2) + 0(a_2b_3 - a_3b_3)$$
$$|A| = 0$$

4. If two rows or columns of a determinant are identical, the value of the determinant is zero. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= a_1(b_1c_3 - b_3c_1) - b_1(a_1c_3 - a_3c_1) + c_1(a_1b_3 - a_3b_1)$$
$$= a_1b_1c_3 - a_1b_3c_1 - a_1b_1c_3 + a_3b_1c_1 + a_1b_3c_1 - a_3b_1c_1$$
$$|A| = 0$$

5. If every element of a row or column of a determinant is multiplied by the same constant K, the value of the determinant is multiplied by that constant. For example if,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a determinant, $|B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|B| = ka_1(b_2c_3 - b_3c_3) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$

$= k(a_1(b_2c_3 - b_3c_3) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$

So $|B| = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Or $|B| = K|A|$

6. The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted from) a constant multiple of the corresponding element of another row or column. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a matrix,

$$|B| = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1+ka_2)(b_2c_3-b_3c_2)-(b_1+kb_2)(a_2c_3-a_3c_2)+(c_1+kc_2)(a_2b_3-a_3b_2)$

$= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= [ka_2(b_2c_3 - b_3c_2) - kb_2(a_2c_3 - a_3c_2) + kc_2(a_2b_3 - a_3b_2)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0) \text{ because row 1st and 2nd are identical}$$

$|B| = |A|$

7. The determinant of a diagonal matrix is equal to the product of its diagonal elements. For example, if

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 2(-15 - 0) - (0 - 0) + 0(0 - 0)$

$= 30$, which is the product of diagonal elements.

i.e., $2(-5)3 = -30$

8. The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that is $|AB| = |A||B|$. for example, if

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$\text{Then } AB = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22})$
$- (a_{11}b_{12} + a_{12}b_{22} - a_{11}b_{22})(a_{21}b_{11} + a_{22}b_{21})$
$= a_{11}b_{11}\, a_{21}b_{12} + a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12}$
$+ a_{12}b_{21}\, a_{22}b_{22} - a_{11}b_{12}\, a_{21}b_{11} - a_{11}b_{12}\, a_{22}b_{21}$
$- a_{12}b_{22}\, a_{21}b_{11} - a_{12}b_{22}\, a_{22}b_{21}$

$|AB| = a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}\, a_{21}b_{12} - a_{11}b_{12}\, a_{22}b_{21}$
$- a_{12}b_{22}\, a_{21}b_{11}$ …………………….. (A)

and $|A| = a_{11}a_{22} - a_{12}a_{21}$

$|B| = b_{11}b_{22} - b_{12}b_{21}$

$|A|\,|B| = a_{11}b_{11}\, a_{22}\, b_{22} + a_{12}b_{21}\, a_{21}\, b_{12} - a_{11}b_{12}\, a_{22}\, b_{21}$
$- a_{12}b_{22}\, a_{21}\, b_{11}$ …………………… (B)

R.H.S of equations (A) and (B) are equal, so

$|AB| = |A|\,|B|$

9. The determinant in which each element in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Expand by first column.

Proof:

L.H.S $= (a_1+\alpha_1)(b_2c_3-b_3c_2)-(a_2+\alpha_2)(b_1c_3-b_3c_1)+(a_3+\alpha_3)(b_1c_2-b_2c_1)$

$= [(a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)]$

$+ [(\alpha_1(b_2c_3-b_3c_2)- \alpha_2(b_1c_2-b_3c_1)+ \alpha_3(b_1c_2-b_2c_1)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

= R.H.S

Similarly

$$\begin{vmatrix} \alpha_1+a_1 & b_1+\beta_1 & c_1 \\ \alpha_2+a_2 & b_2+\beta_2 & c_2 \\ \alpha_3+a_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

And, $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \text{sum of six determinant} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Also $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Example 5: Verify that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Multiply row first, second and third by a, b and c respectively, in the L.H.S., then

$$\text{L.H.S} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Take abc common from 3rd column

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Interchange column first and third

$$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

Again interchange column second and third

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

= R.H.S

Example 6: Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - c)\,(c - a)\,(a - b)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

subtracting row first from second and third row

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

from row second and third taking (b – a) and (c – a) common.

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

expand from first column

$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$

Or L.H.S $= (b-c)(c-a)(a-b)(-1)(-1)$

$$= (b-c)(c-a)(a-b) = \text{R.H.S}$$

Example 7: Without expansion, show that

$$\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -4 & 0 & 2 & 8 \end{vmatrix} = 0$$

Solution:

In the L.H.S Taking 2 common from fourth row, so

$$\text{L.H.S} = 2\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -2 & 0 & 1 & 4 \end{vmatrix}$$

Since rows 2nd and 3rd are identical, so

$$= 2(0) = 0$$

L.H.S = R.H.S

9.8 Solution of Linear Equations by Determinants: (Cramer's Rule)

Consider a system of linear equations in two variables x and y,

$$a_1x + b_1y = c_1 \qquad (1)$$

$$a_2x + b_2y = c_2 \qquad (2)$$

Multiply equation (1) by b_2 and equation (2) by b_1 and subtracting, we get

$$x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad (3)$$

Again multiply eq. (1) by a_2 and eq. (2) by a_1 and subtracting, we get

$$y(a_2b_1 - a_1b_2) = a_2c_1 - a_1c_2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (4)$$

Note that x and y from equations (3) and (4) has the same denominator $a_1b_2 - a_2b_1$. So the system of equations (1) and (2) has solution only when $a_1b_2 - a_2b_1 \neq 0$.

The solutions for x and y of the system of equations (1) and (2) can be written directly in terms of determinants without any algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This result is called Cramer's Rule.

Here $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = |A|$ is the determinant of the coefficient of x and y in equations (1) and (2)

If $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = |A|$

and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = |A|$

Then $x = \dfrac{|A_x|}{|A|}$ and $y = \dfrac{|A_y|}{|A|}$

Solution for a system of Linear Equations in Three Variables:

Consider the linear equations:

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

Hence the determinant of coefficients is

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ if } |A| \neq 0$$

Then by Cramer's Rule the value of variables is:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

and $$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|} = \frac{|A_z|}{|A|}$$

Example 8: Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Solution:

Here the determinant of the coefficients is:

$$|A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

$$= -4(8 + 3) - 2(6 - 1) - 9(-9 - 4)$$

$$= -44 - 10 + 117$$

$$|A| = 63$$

for $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}$$

$$= 2(11) - 2(2) - 9(-47) = 22 - 4 + 423$$

$$\boxed{|A_x| = 441}$$

Similarly,

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}$$

$$= -4(2) - 2(5) - 9(19)$$

$$= -8 - 10 - 171$$

$$\boxed{|A_y| = -189}$$

and

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}$$

$$= -4(47) - 2(19) + 2(-13)$$

$$= -188 - 38 - 26$$

$$\boxed{|A_z| = -252}$$

Hence $x = \dfrac{|A_x|}{|A|} = \dfrac{441}{63} = 7$

$$y = \frac{|A_y|}{|A|} = \frac{-189}{63} = -3$$

$$z = \frac{|A_z|}{|A|} = \frac{-252}{63} = -4$$

So the solution set of the system is {(7, –3, –4)}

Exercise 9.2

Q.1 Expand the determinants

(i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ -2 & 1 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & 1 \\ a & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$

Q.2 Without expansion, verify that

(i) $\begin{vmatrix} -2 & 1 & 0 \\ 3 & 4 & 1 \\ -4 & 2 & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ (iv) $\begin{vmatrix} bc & ca & ab \\ a^3 & b^3 & c^3 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0$

(v) $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix} = 0$

(vi) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

Q.3 Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1x+d_1 & c_2x+d_2 & c_3x+d_3 \end{vmatrix} = x \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Q.4 Show that

(i) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = a^3$

(iii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)$

Q.5 Show that:

(i) $\begin{vmatrix} \ell & a & a \\ a & \ell & a \\ a & a & \ell \end{vmatrix} = (2a+\ell)(\ell-a)^2$

(ii) $\begin{vmatrix} a+\ell & a & a \\ a & a+\ell & a \\ a & a & a+\ell \end{vmatrix} = \ell^2(3a+\ell)$

Q.6 prove that:

(i) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

(ii) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2 (a + b + c + \lambda)$

(iii) $\begin{vmatrix} \text{Sin } \alpha & \text{Cos } \alpha & 0 \\ -\text{Sin } \beta & \text{Cos } \beta & \text{Sin } \gamma \\ \text{Cos } \beta & \text{Sin } \beta & \text{Cos } \gamma \end{vmatrix} = \text{Sin } (\alpha+\beta+\gamma)$

Q.7 Find values of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Q.8 Use Cramer's rule to solve the following system of equations.

(i) $x - y = 2$, $x + 4y = 5$ (ii) $3x - 4y = -2$, $x + y = 6$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$ (iv) $2x + 2y + z = 1$, $x - y + 6z = 21$, $3x + 2y - z = -4$

(v) $x + y + z = 0$, $2x - y - 4z = 15$, $x - 2y - z = 7$ (vi) $x - 2y - 2z = 3$, $2x - 4y + 4z = 1$, $3x - 3y - 3z = 4$

Answers 9.2

Q.1 (i) –41 (ii) 0 (iii) x^3

Q.7 (i) x = -2 , 3 (ii) x = 3 , 4

Q.8 (i) $\left\{\left(\frac{13}{5}, \frac{3}{5}\right)\right\}$ (ii) $\left\{\left(\frac{22}{7}, \frac{20}{7}\right)\right\}$

(iii) {(1, 1, 0)} (iv) {(1, –2, 3)}

(v) {(3, –1, –2) (vi) $\left\{\left(-\frac{1}{3}, -\frac{25}{24}, -\frac{5}{8}\right)\right\}$

9.9 Special Matrices:

1. Transpose of a Matrix

If $A = [a_{ij}]$ is mxn matrix, then the matrix of order n x m obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted A^t or A'.

Example if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. Symmetric Matrix:

A square matrix A is called symmetric if $A = A^t$

for example if

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \quad \text{then} \quad A^t = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A$$

Thus A is symmetric

3. Skew Symmetric:

A square matrix A is called skew symmetric if $A = -A^t$

for example if $B = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$,then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$B^t = -B$$

Thus matrix B is skew symmetric.

4. Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$, for example if

$$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, \text{ then } |A| = 0, \text{ Hence A is singular}$$

and if $A = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}$, then $|A| \neq 0$,

Hence A is non-singular.

Example: Find k If $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular

Solution: Since A is singular so $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$(k-2)(k+2) - 5 = 0$

$k^2 - 4 - 5 = 0$

$k^2 - 9 = 0 \Rightarrow \quad K = \pm 3$

5. Adjoint of a Matrix:

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is

$$C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

$$C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

Hence adj A = C^t $= \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6. Inverse of a Matrix:

If A is a non-singular square matrix , then $A^{-1} = \dfrac{\text{adj } A}{|A|}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then adj $A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Alternately:

For a non singular matrix A of order (n x n) if there exist another matrix B of order (n x n) Such that their product is the identity matrix I of order (n x n) i.e., AB = BA = I

Then B is said to be the inverse (or reciprocal) of A and is written as $B = A^{-1}$

Example 9: If $A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ then show that AB = BA = I and therefore, $B = A^{-1}$

Solution:

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\quad BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence AB = BA = I

and $\quad$ therefore $B = A^{-1} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example 10: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$|A| = 0 + 2(-2 + 3) - 3(-2 + 3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0$,	$A_{12} = 1$,	$A_{13} = 1$
$A_{21} = 2$,	$A_{22} = -3$,	$A_{23} = 2$
$A_{31} = 3$,	$A_{32} = -3$,	$A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

9.11 Solution of Linear Equations by Matrices:

Consider the linear system:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \text{------} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{------} + a_{2n}x_n = b_2 \\ \quad | \qquad\quad | \qquad\qquad\qquad | \\ a_{n1}x_1 + a_{n2}x_2 + \text{------} + a_{nn}x_n = b_n \end{array}\right\} \quad \text{……………..} \quad (1)$$

It can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$

Then latter equation can be written as,

$$AX = B$$

If $B \neq 0$, then (1) is called non-homogenous system of linear equations and if B = 0, it is called a system of homogenous linear equations.

If now $B \neq 0$ and A is non-singular then A^{-1} exists.

Multiply both sides of AX = B on the left by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$1X = A^{-1}B$$
$$\text{Or} \quad X = A^{-1}B$$

Where $A^{-1}B$ is an n x 1 column matrix. Since X and $A^{-1}B$ are equal, each element in X is equal to the corresponding element in $A^{-1}B$. These elements of X constitute the solution of the given linear equations.

If A is a singular matrix, then of course it has no inverse, and either the system has no solution or the solution is not unique.

Example 11: Use matrices to find the solution set of

$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6z = 9$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Since $|A| = 3 + 21 - 24 = 0$

Hence the solution of the given linear equations does not exists.

Example 12: Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$

Since $|A| = -32 + 48 + 17 = 61$

So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since,

$X = A^{-1}B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

$$= \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: {(x, y, z)} = {(–2, 0, 2)}

Exercise 9.3

Q.1 Which of the following matrices are singular or non-singular.

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Q.2 Which of the following matrices are symmetric and skew-symmetric

(i) $\begin{bmatrix} 2 & 6 & 7 \\ 6 & -2 & 3 \\ 7 & 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Q.3 Find K such that the following matrices are singular

(i) $\begin{vmatrix} K & 6 \\ 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & K \\ -4 & 2 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ k & 3 & -6 \end{vmatrix}$

Q.4 Find the inverse if it exists, of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$, $x + 4y = 5$

(ii) $x + y = 2$, $2x - z = 1$, $2y - 3z = -1$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$

(iv) $-4x + 2y - 9z = 2$, $3x + 4y + z = 5$, $x - 3y + 2z = 8$

(v) $x + y - 2z = 3$, $3x - y + z = 0$, $3x + 3y - 6z = 8$

Answers 9.3

Q.1 (i) Non-singular (ii) Singular
(iii) Singular

Q.2 (i) Symmetric (ii) Skew-symmetric
(iii) Symmetric

Q.3 (i) 8 (ii) 5 (iii) 3

Q.4 (i) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{-7} \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{4}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

(iv) A^{-1} does not exist.

Q.5 (i) {(1, 1)} (ii) {(1, 1, 1)} (iii) {(1, 1, 0)}
(iv) {(7, -3, -4)} (v) no solution

Summary

1. If $A = [a_{ij}]$, $A = [b_{ij}]$ of order m x n. Then $A + B = [a_{ij} + b_{ij}]$ is also m x n order.
2. The product AB of two matrices A and B is conformable for multiplication if No of columns in A = No. of rows in B.
3. If $A = [a_{ij}]$ is m x n matrix, then the n x m matrix obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A^t.
4. Symmetric Matrix:
   A square matrix A is symmetric if $A^t = A$.
5. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then,

   (i) adj $A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$, a_{ij} are the co-factor elements.

   And inverse of A is:

(ii) $A^{-1} = \dfrac{\text{adj } A}{|A|}$

6. A square matrix A is singular if $|A| = 0$.

Short Questions

Write the short answers of the following:

Q.1: Define row and column vectors.
Q.2: Define identity matrix.
Q.3: Define symmetric matrix.

Q.4: Define diagonal matrix.

Q.5: Define scalar matrix.

Q.6: Define rectangular matrix.

Q.7: Show that $A = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$ is singular matrix

Q.8: Show that $A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -3 \\ 3 & -3 & 6 \end{vmatrix}$ is symmetric

Q.9: Show that $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = b^3 + a^3$

Q.10: Evaluate $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

Q.11: Without expansion show that $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Q.12: Find x and y if $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix}$

Q.13: Find x and y if $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

Q.14: If $A = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{vmatrix}$, find A – B

Q.15: Find inverse of $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

Q.16: If A is non-singular, then show that $(A^{-1})^{-1} = A$

Q.17: If A is any square matrix then show that AA^t is symmetric.

Q.18: Find K if $A = \begin{bmatrix} 4 & k & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$ is singular matrix

Q.19: Define the minor of an element of a matrix.

Q.20: Define a co-factor of an element of a matrix.

Q.21: Without expansion verify that $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

Q.22: What are the minor and cofactor of 3 in matrix.

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Q.23: What are the minor and cofactor of 4 in matrix.

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Q.24: If $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$, Then find k.

Q.25: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find A + B

Q.26: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find A – B

Q.27: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find AB

Q.28: If $\begin{vmatrix} 2 & 3 \\ 4 & k \end{vmatrix}$ is singular , Then find k.

Q.29: Find A^{-1} if $A = \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix}$

Answers

Q10. 9 Q12. x = –1 , y = 2 Q13. x = – 5, y = – 2

Q14. $\begin{vmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{vmatrix}$ Q15. D^{-1} does not exist Q18. k = 3

Q22. $M_{11} = 16$, $C_{11} = 16$ Q23. $M_{32} = 26$, $C_{32} = -26$

Q24. $K = \pm 3$ Q25. $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ Q26 $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

Q27. $\begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$ Q28. k = 6 Q29. $\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$

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Objective type Exercise

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. The order of the matrix $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is

(a) 2 x 1 (b) 2 x 2 (c) 3 x 1 (d) 1 x 3

___2. The order of the matrix [1 2 3] is

(a) 1 x 3 (b) 3 x 1 (c) 3 x 3 (d) 2 x 3

___3. The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called

(a) Identity (b) scalar (c) diagonal (d) Null

___4. Two matrices A and B are conformable for multiplication if

(a) No of columns in A = No of rows in B
(b) No of columns in A = No of columns in B
(c) No of rows in A = No of rows in B
(d) None of these

___5. If the order of the matrix A is pxq and order of B is qxr, then order of AB will be:

(a) pxq (b) qxp (c) pxr (d) rxp

___6. In an identity matrix all the diagonal elements are:

(a) zero (b) 2 (c) 1 (d) none of these

___7. The value of determinant $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ is:

(a) 6 (b) – 6 (c) 1 (d) 0

___8. If two rows of a determinant are identical then its value is

(a) 1 (b) zero (c) – 1 (d) None of these

___9. If A = $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ is a matrix, then Cofactor of 4 is

(a) – 2 (b) 2 (c) 3 (d) 4

___10. If all the elements of a row or a column are zero, then value of the determinant is:

(a) 1 (b) 2 (c) zero (d) None of these

___11. Value of m for which matrix $\begin{bmatrix} 2 & 3 \\ 6 & m \end{bmatrix}$ is singular.

(a) 6 (b) 3 (c) 8 (d) 9

___12. If $[a_{ij}]$ and $[b_{ij}]$ are of the same order and $a_{ij} = b_{ij}$ then the matrix will be

(a) Singular (b) Null (c) unequal (d) equal

13. Matrix $[a_{ij}]_{mxn}$ is a row matrix if:

(a) $i = 1$ (b) $j = 1$ (c) $m = 1$ (d) $n = 1$

14. Matrix $[c_{ij}]_{mxn}$ is a rectangular if:

(a) $i \neq j$ (b) $i = j$ (c) $m = n$ (d) $m - n \neq 0$

15. If $A = [a_{ij}]_{mxn}$ is a scalar matrix if :

(a) $a_{ij} = 0 \quad \forall\ i \neq j$ (b) $a_{ij} = k\ \forall\ i = j$

(c) $a_{ij} = k \quad \forall\ i \neq j$ (d) (a) and (b)

16. Matrix $A = [a_{ij}]_{mxn}$ is an edentity matrix if :

(a) $\forall\ i = j$, $a_{ij} = 0$ (b) $\forall\ i = j$, $a_{ij} = 1$

(c) $\forall\ i \neq j$, $a_{ij} = 0$ (d) both (b) and (c)

17. Which matrix can be tectangular mayrix ?

(a) Diagonal (b) Identity (c) Scalar (d) None

18. If $A = [a_{ij}]_{mxn}$ then order kA is:

(a) m x n (b) km x kn (c) km x n (d) m x kn

19. $(A - B)^2 = A^2 - 2AB + B^2$, if and only if :

(a) $A + B = 0$ (b) $AB - BA = 0$ (c) $A^2 + B^2 = 0$ (d) (a) and c

20. If A and B ARE symmetric , then AB =

(a) BA (b) $A^t B^t$ (c) $B^t A^t$ (d) (a) and (c)

Answers

Q.1 (1) c (2) a (3) d (4) a (5) c (6) c
(7) a (8) b (9) a (10) c (11) d (12) d
(13) c (14) d (15) d (16) d (17) d (18) a
(19) b (20) d