II. Maxwell Equations

5. Maxwell correction of Ampere law:

Let's consider again the electrodynamics set of equations:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & I(Gauss's\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Faraday's\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J} & IV(Ampere's\ Law) \end{cases}$$

When applying the divergent of equations II and IV, we will find:

$$\underbrace{\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{E}\right)}_{=0,\forall \vec{E}} = \vec{\nabla}.\left(-\frac{\partial \vec{B}}{\partial t}\right) = \underbrace{-\frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{B}\right)}_{=0\ (III)} = 0$$

Now, when applying the same action on equation IV:

$$\underbrace{\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{B}\right)}_{=0,\forall \vec{B}} = \vec{\nabla}.\left(\mu_0 \vec{j}\right) = \mu_0 \vec{\nabla}.\vec{j} = \underbrace{\mu_0 \vec{\nabla}.\vec{j}}_{?}$$

In fact, the quantity $\vec{\nabla}.\vec{j}$ does not vanish for all $\vec{j}$, only for special cases corresponding to $\partial\rho/\partial t = 0$, according to charge continuity equation.

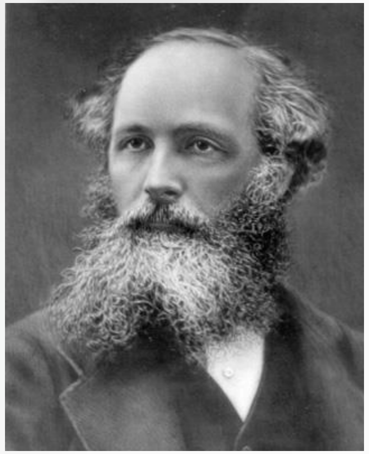
To prevent this, Maxwell proposed to add a term which could cancel the divergent of the current density:

$\vec{J}' = \vec{J} + \vec{G}$ in such a way that:

$$\vec{\nabla}.\vec{J}' = \vec{\nabla}.\vec{J} + \vec{\nabla}.\vec{G} = 0$$

This will give the following result:

$$\vec{\nabla}.\vec{G} = -\vec{\nabla}.\vec{j} = \frac{\partial \rho}{\partial t}$$



James C. Maxwell
1831- 1897, UK

II. Maxwell Equations

5. Maxwell correction of Ampere law:

This new term, will ingeniously ensure the complete relationship (in both senses) between electric and magnetic fields.

According to Gauss's law: $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (Eq. I), the Maxwell condition could be rewritten as:

$$\vec{\nabla}.\vec{G} = -\vec{\nabla}.\vec{J} = \frac{\partial \rho}{\partial t} = \frac{\partial \vec{D}}{\partial t} = \frac{\partial(\varepsilon_0 \vec{\nabla}.\vec{E})}{\partial t} = \vec{\nabla}.\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

This implies that:

$$\vec{G} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Finally, we get:

$$\vec{J'} = \vec{J} + \vec{D} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consequently, the new version of eq. IV:

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J'} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

And, when applying the divergent :

$$\underbrace{\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{B}\right)}_{=0, \forall \vec{B}} = \vec{\nabla}.\left(\mu_0 \vec{J'}\right) = \mu_0 \vec{\nabla}.\vec{J'} = \underbrace{\mu_0 \vec{\nabla}.\vec{J} + \mu_0 \varepsilon_0 \vec{\nabla}.\frac{\partial \vec{E}}{\partial t}}_{=0\ (charge\ continuity)}$$

The term $\vec{G}$, *known also as "Maxwell correction", is called the "displacement current". The reduced form of the equation IV, using both* $\vec{H}$ and $\vec{D}$ fields :

$$\vec{\nabla} \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The new set of electrodynamics equations could be now completed and finalized, as Maxwell's Equations.

II. Maxwell Equations

6. Maxwell's equations:

The electromagnetism now are well described by the set of Maxwell's equations:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & I(Maxwell - Gauss\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} & IV(Maxwell - Ampere\ Law) \end{cases}$$

These equations are also known as Maxwell equations for time-varying fields $\vec{E}(t)$ and $\vec{B}(t)$.

II. Maxwell Equations

6. Maxwell's equations:

The compact form without electromagnetic constants, by introducing density current $\vec{D}$ and magnetic field $\vec{H}$:

$$\begin{cases} \vec{\nabla}.\vec{D} = \rho & I(Maxwell - Gauss\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & IV(Maxwell - Ampere\ Law) \end{cases}$$

The integral forms of previous equations of Maxwell are given in compact expressions:

$$\begin{cases} \oint_S \vec{D}.d\vec{S} = Q & (I) \\ \oint_C \vec{E}.d\vec{l} = -\int_S \dfrac{\partial \vec{B}}{\partial t}.d\vec{S} & (II) \\ \oint_S \vec{B}.d\vec{S} = 0 & (III) \\ \oint_C \vec{H}.d\vec{l} = \int_S \left(\vec{J} + \dfrac{\partial \vec{D}}{\partial t}\right).d\vec{S} & (IV) \end{cases}$$

II. Maxwell Equations

7. Electromagnetic potential:

Let's now examine the implication of Maxwell's equations on both scalar electric potential V and vector magnetic potential $\vec{A}$.

We know that in static case, Faraday's law reduces to: $\vec{\nabla} \wedge \vec{E} = 0$; which states that electric field $\vec{\mathrm{E}}$ is conservative, and it could be expressed as the derivative of a scalar function (potential V):

$$\vec{E} = -\vec{\nabla} V$$

Whereas, in the dynamic case, Faraday's law becomes:

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Since $\vec{B}$ is derived from a vector potential $\vec{A}$ as:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}$$

Consequently, the former equation of Faraday's law can be expressed as:

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \wedge \vec{A})$$

Which could be rewritten as:

$$\vec{\nabla} \wedge \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \wedge \vec{A}) = \vec{\nabla} \wedge \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

This is equivalent to write:

$$\vec{\nabla} \wedge \vec{E'} = 0; \ \vec{E'} = \vec{E} + \frac{\partial \vec{A}}{\partial t}$$

We should remember that : $\vec{\nabla} \wedge \left(\vec{\nabla} V\right) = 0, \forall V$

Which leads also to write: $\vec{E'} = -\vec{\nabla} V$

II. Maxwell Equations

7. Electromagnetic potential:

Substituting $\overrightarrow{E'} = \vec{E} + \frac{\partial \vec{A}}{\partial t}$ in $\overrightarrow{E'} = -\vec{\nabla} V$, will give the following equation allowing to derive the electric field from a generalized potential (Electromagnetic potential) in the dynamic case:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

This means, that if the scalar potential V and the vector potential $\vec{A}$ are known, it is possible to obtain the electric field from the given equation above, while the magnetic field is obtained from the equation:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}$$

II. Maxwell Equations

8. Lorenz gauge:

If we replace the general expression of electric field derived from electromagnetic potential, in the 1st equation from Maxwell's set, we obtain:

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho}{\varepsilon_0} \qquad (8)$$

An other equation could be obtained from:

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

Knowing that $\vec{B} = \vec{\nabla} \wedge \vec{A}$, we get:

$$\vec{\nabla} \wedge \vec{B} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

Besides that, according to the last Maxwell's equation: $\vec{\nabla} \wedge \vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$, which gives:

$$\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

And again, replacing $\vec{E}$ by $-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$:

$$\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) + \Delta\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \mu_0\vec{J}$$

$$\Delta\vec{A} - \mu_0\varepsilon_0\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\mu_0\varepsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}.\vec{A}\right] = -\mu_0\vec{J} \quad (9)$$

II. Maxwell Equations

8. Lorenz gauge:

Now, both equations are second degree coupled differential equations (V and $\vec{A}$):

$$\begin{cases} \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho}{\varepsilon_0} & (8) \\ \Delta\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\mu_0\varepsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}.\vec{A}\right] = -\mu_0\vec{J} & (9) \end{cases}$$

To decouple these equations, Lorenz sets the following gauge (condition) in dynamics :

$$\mu_0\varepsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}.\vec{A} = 0 \qquad (10)$$

Equivalent to:

$$\vec{\nabla}.\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}$$

This will lead to a decoupled differential equations:

$$\begin{cases} \nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} & (11) \\ \Delta\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} & (12) \end{cases}$$

Which could be rewritten in terms of d'Alembertian operator as:

$$\begin{cases} \Box V = -\frac{\rho}{\varepsilon_0} & (11) \\ \Box\vec{A} = -\mu_0\vec{J} & (12) \end{cases}$$

In absence of charge and current, these equations are reduced to:

$$\Box V = 0; \Box\vec{A} = 0$$