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Level : 1^{st} year LMD
Module: Algebra 2

Series 1: Vector spaces

Exercise 1.
1. Is the subset $E=\{(x,y)\in\mathbb{R}^2, y=2x\}$ of $\mathbb{R}^2$, equipped with the usual laws of the vector space $\mathbb{R}^2$, a vector space on $\mathbb{R}$?
2. Is the subset $F=\{(x,y,z)\in\mathbb{R}^3, y^2=2x, z=0\}$ of $\mathbb{R}^3$, equipped with the usual laws of the vector space $\mathbb{R}^3$, a subspace of $\mathbb{R}^3$?

Exercise 2.
Let $u_1=(1,2,3,4)$ and $u_2=(1,-2,3,-4)$ be two vectors of $\mathbb{R}^4$.
Can one determine x and y so that $(x,1,y,1)\in Vect\,(u_1,u_2)$? And so that $(x,1,1,y)\in Vect\,(u_1,u_2)$?

Exercise 3.
In $\mathbb{R}^4$, let's consider the sub-set E of vectors $(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$ satisfying $x_1+x_2+x_3+x_4=0$.
Is the subset E a subspace of $\mathbb{R}^4$? If it is, give a basis.

Exercise 4.
Let $a=(2,3,-1), b=(1,-1,-2), c=(3,7,0)$ and $d=(5,0,-7)$ be four vectors of $\mathbb{R}^3$.
Let $E=Vect(a,b)$ and $F=Vect(c,d)$ be subspaces of $\mathbb{R}^3$. Prove that $E=F$.

Exercise 5.
Let $E=Vect(a,b,c,d)$ be a subspace of $\mathbb{R}^3$ with:

$$a=(2,-1,-1);\quad b=(-1,2,3);\quad c=(1,4,7) \text{ and } d=(1,1,2)$$

1. Is $\{a,b,c,d\}$ a basis of $\mathbb{R}^3$?
2. Prove that $\{a,b\}$ is a base of E.
3. Determine one or more equations characterizing E.
4. Complete a basis of E to a basis of $\mathbb{R}^3$.

Exercise 6.
Let E et F be two subsets of $\mathbb{R}^3$ defined by:

$$E=\{(x,y,z)\in\mathbb{R}^3 \mid x+y-2z=0 \text{ and } 2x-y-z=0\},$$

and

$$F=\left\{(x,y,z)\in\mathbb{R}^3 \mid x+y-z=0\right\}$$

We admit that F is a subspace of $\mathbb{R}^3$. Let $a=(1,1,1), b=(1,0,1)$ and $c=(0,1,1)$ be three vectors of $\mathbb{R}^3$.

1. Prove that E is a subspace of $\mathbb{R}^3$.
2. Determine a generating family (spanning set) of E and show that it is a basis.
3. Prove that $\{b, c\}$ is a basis of F.
4. Prove that $\{a, b, c\}$ is linearly independent in $\mathbb{R}^3$.
5. Do we have $E \oplus F = \mathbb{R}^3$?
6. Let $u = (x, y, z)$, express u in the basis $\{a, b, c\}$.

Exercise 7.

Let $E = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 = 0 \text{ and } x_3 - x_4 = 0\}$

We admit that F is a subspace of $\mathbb{R}^4$.

1. Determine a basis of E.
2. Complete this basis of E to a basis of $\mathbb{R}^4$.

Exercise 8.

Let $E = \{P \in \mathbb{R}_2[X], P(1) = 0\}$.

1. Prove that E is a subspace of $\mathbb{R}_2[X]$.
2. Give a basis of E and deduce its dimension.

Exercise 9.

Let $E = \{P \in \mathbb{R}_3[X], P(-1) = 0 \text{ and } P(1) = 0\}$

1. Prove that E is a subspace of $\mathbb{R}_3[X]$.
2. Determine a basis and the dimension of E.

Exercise 10.

In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, are the following three functions $x \mapsto \sin(x), x \mapsto \sin(2x)$ and $x \mapsto \sin(3x)$ linearly independent?

Exercise 11.

Let $f(x) = \cos(x), g(x) = \cos(x)\cos(2x)$ and $h(x) = \sin(x)\sin(2x)$. Determine $Vect(f, g, h)$.

Exercise 12.

Let's consider the vectors $v_1 = (1, 0, 0, 1), v_2 = (0, 0, 1, 0), v_3 = (0, 1, 0, 0), v_4 = (0, 0, 0, 1)$ and $v_5 = (0, 1, 0, 1)$ of $\mathbb{R}^4$.

1. Is $Vect\,(v_1, v_2)$ supplementary to $Vect\,(v_3)$ in $\mathbb{R}^4$?
2. Same question for $Vect\,(v_1, v_3, v_4)$ and $Vect\,(v_2, v_5)$.
3. Same question for $Vect\,(v_1, v_2)$ and $Vect\,(v_3, v_4, v_5)$.