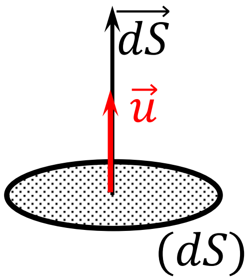
Chapter I: Part II

I. Electrostatic Field Flux- Gauss's Law:

Representation of a surface:

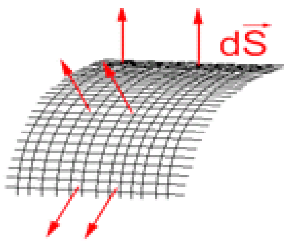
An elementary surface (dS) is represented by a vector $\mathrm{d}\vec{S}$ perpendicular to this surface and whose modulus is equal to the area of this surface.



$$d\vec{S} = dS\vec{u}$$

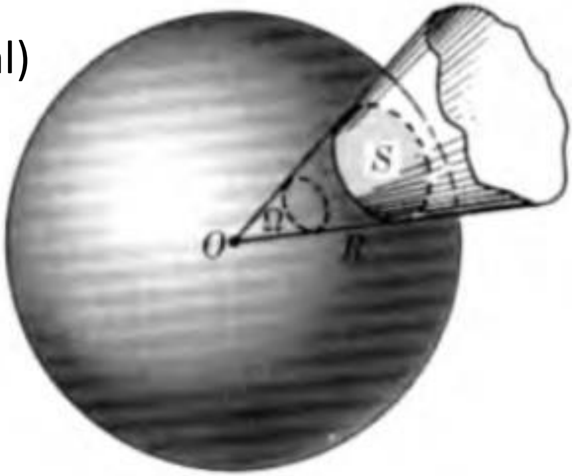
With $|\vec{u}| = 1$

- For a closed surface, The vector $\overrightarrow{\mathrm{d}S}$ is oriented outward..



Solid Angles

- A *solid angle* is the space included inside a conical (or pyramidal) surface.



- Its value, expressed in *steradians* (sr), is obtained by drawing, with arbitrary radius *R* and center at the vertex *O* , a spherical surface and applying the relation: $$\boldsymbol{\Omega = \frac{S}{R^2}}$$

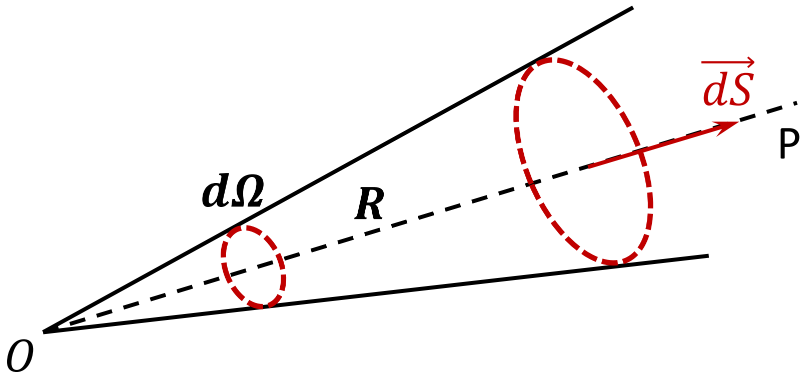
  S *:* area of the spherical cap intercepted by the solid angle.

- Since the surface area of a sphere is $4\pi R^2$, we conclude that the complete solid angle around a point is:

$$\boldsymbol{\Omega = \frac{4\pi R^2}{R^2} = 4\pi}$$

- When the solid angle is small, Then:

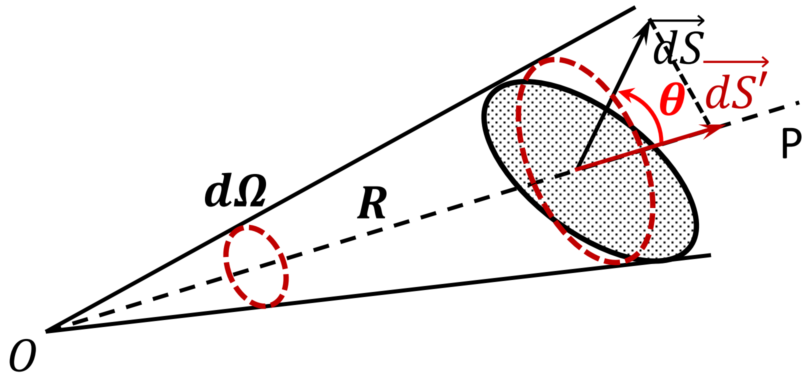
$$d\Omega = \frac{dS}{R^2}$$



- In some cases the surface dS is not perpendicular to $\overrightarrow{OP}$, it makes an angle θ with OP.

Then it is necessary to project dS on a plane perpendicular to OP, which gives us the area: $dS' = dS\cos\theta$

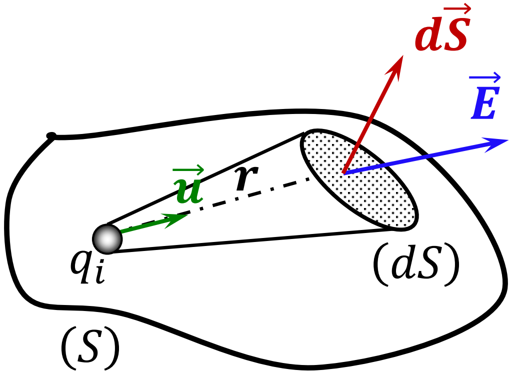
Thus: $$d\Omega = \frac{dS\cos\theta}{R^2}$$



Gauss's law:

Flux of the Electrostatic Field:

Let a closed surface **(S)** inside which there is a charge ***q***.



The flux of the electrostatic field ($\vec{E}$) through the surface *dS* is:

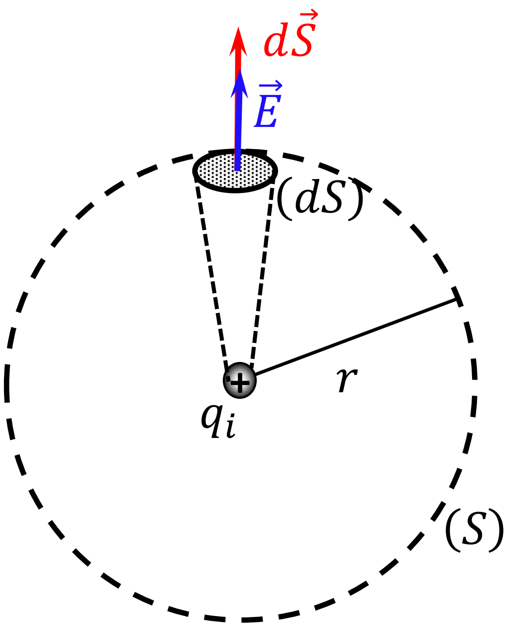
$$d\phi = \vec{E}.d\vec{S}$$

The total flow of $\vec{E}$ across the surface S is therefore:

$$\phi = \oiint d\phi = \oiint_{(S)} \vec{E}.d\vec{S}$$

Gauss's theorem:

- *A simple example allows us to find Gauss's theorem:*



- ✓ Let a positive charge q_i.
- ✓ We choose an "imaginary" spherical surface (S) centered on q_i and having a radius r.

The elementary flux $d\phi$ of the field $\vec{E}$ Created by q_i and passing through the surface dS is given by: $d\phi = \vec{E}.d\vec{S}$ $\left(\vec{E}\ //\ \overrightarrow{dS}\right)$

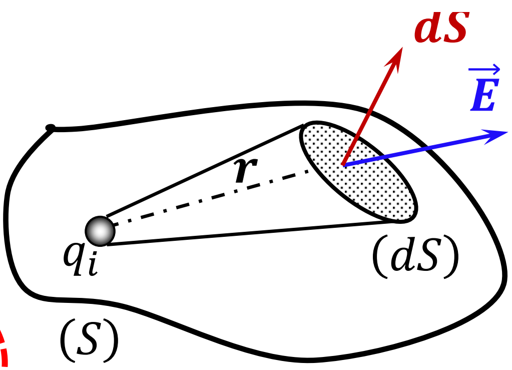
The total flow ϕ of $\overrightarrow{E}$ across the surface (S) is therefore:

$$\phi = \oiint d\phi = \oiint_{(S)} \vec{E}.d\vec{S} = \oiint_{(S)} \frac{kq_i}{r^2}.dS = \frac{q_i}{4\pi\varepsilon_0 r^2} \oiint_0^{4\pi r^2} dS = \frac{q_i}{\varepsilon_0}$$

- ***Consider now a charge q inside an arbitrary closed surface S***

❑ ***Consider now a charge q inside an arbitrary closed surface S***

We have : $d\phi = \vec{E}.d\vec{S}$



$$\phi = \oiint d\phi = \oiint_{(S)} \vec{E}.d\vec{S} = \oiint_{(S)} E.dS\cos\theta$$

$$= \oiint_{(S)} \frac{kq_i}{r^2}.dS.\cos\theta = \frac{q_i}{4\pi\varepsilon_0}\oiint_{(S)} \frac{dS\cos\theta}{r^2}$$

$\dfrac{dS\cos\theta}{r^2} = d\Omega$: solid angle subtended by the surface element *dS* as viewed from the charge *q*

Since the total solid angle around any point is **4π** steradians. Then:

$$\phi = \frac{q_i}{4\pi\varepsilon_0}\int_0^{4\pi} d\Omega = \frac{q_i}{\varepsilon_0}$$

Theorem: *The Flow of the Field $\overrightarrow{E}$ through a closed surface surrounding charges q_i is :*

$$\phi = \oiint_{(S)} \vec{E}.d\vec{S} = \frac{\sum q_i}{\varepsilon_0}$$

Method of applying Gauss's theorem:

Gauss's theorem provides a very useful method for calculating the field $\vec{E}$ when it has particular symmetry properties.

1. Find a closed surface passing through the point "M" where we want to calculate the field.

   The chosen surface must be composed of parts where:

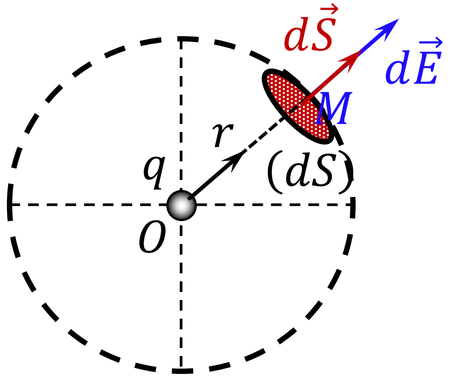
   - Let the field be null
   - Let the field be constant in modulus and equal to E_M and parallel to dS
   - Let the field be perpendicular to dS

2. Write the definition of the flow $\phi = \oiint_{(S)} \vec{E}.d\vec{S}$

3. Apply Gaussian's theorem after counting the algebraic charge inside the surface.

Applications

Example 1: Determine the expression of the field created by a punctual charge ***q*** at a point ***M*** in space at a distance ***r*** from the charge ***q***.



- The problem has spherical symmetry,
- The field is radial $\vec{E}(M) = E(M)\vec{u}_r$,
- The surface of Gauss (S) to chosed is a sphere with center O and radius r.
- At each point of (S) the field is:
  - ✓ Perpendicular to (S)
  - ✓ Constant in modulus

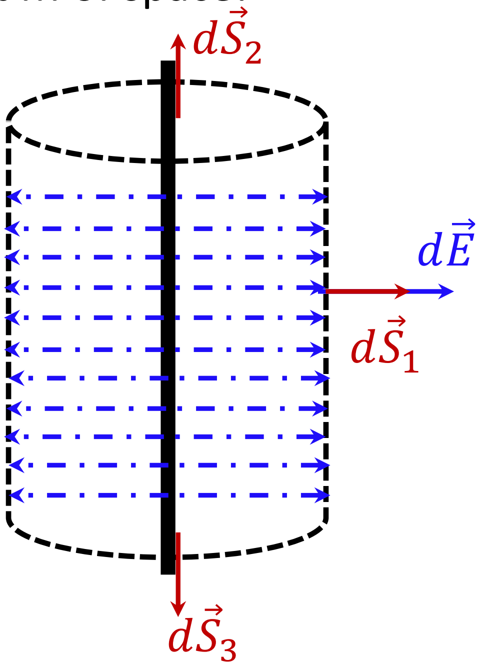
$$\phi = \oiint_{(S)} \vec{E}(M).d\vec{S} \quad = E(M)\oiint_{(S)} ds \;= E(M).S \;= E(M).4\pi r^2 = \frac{\sum q_i}{\varepsilon_0} \;= \frac{q}{\varepsilon_0}$$

$$\Longrightarrow \boldsymbol{E(M) = \frac{q}{4\pi\varepsilon_0 r^2}}$$

Example 2: Determine the expression of the field created by a charge distributed uniformly on an infinite straight wire with a constant linear density $\lambda > 0$

- The field $\vec{E}$ is radial and is perpendicular to the wire at any point M of space:

$$\vec{E}(M) = E(M)\vec{u}_r$$



- The problem has cylindrical symmetry
- The chosen Gaussian surface *(S)* is a closed cylinder of radius ***r*** and length L.
- The surface *(S)* consists of the 3 surfaces: S_1, S_2, S_3
  - ✓ S_1: represent by $d\vec{S}_1$ with $d\vec{E}//d\vec{S}_1$
  - ✓ S_2: represent by $d\vec{S}_2$ with $d\vec{E} \perp d\vec{S}_2$
  - ✓ S_3: represent by $d\vec{S}_3$ with $d\vec{E} \perp d\vec{S}_3$

$$\phi = \oiint_{(S)} \vec{E}(M).d\vec{S} = \oiint_{(S)} \vec{E}(M).d\vec{S}_1 + \underbrace{\cancel{\oiint_{(S)} \vec{E}(M).d\vec{S}_2}}_{0} + \underbrace{\cancel{\oiint_{(S)} \vec{E}(M).d\vec{S}_3}}_{0}$$

$$\Longrightarrow \phi = \oiint_{(S)} E(M).\, dS_1 = E(M) \oiint_{(S)} dS_1 = E(M) 2\pi r L \quad = \frac{\sum q_i}{\varepsilon_0} \;= \frac{\lambda L}{\varepsilon_0}$$

$$\Longrightarrow E(M) = \frac{\lambda}{2\pi r \varepsilon_0}$$

From this, we can deduce the expression of the potential from the relation:

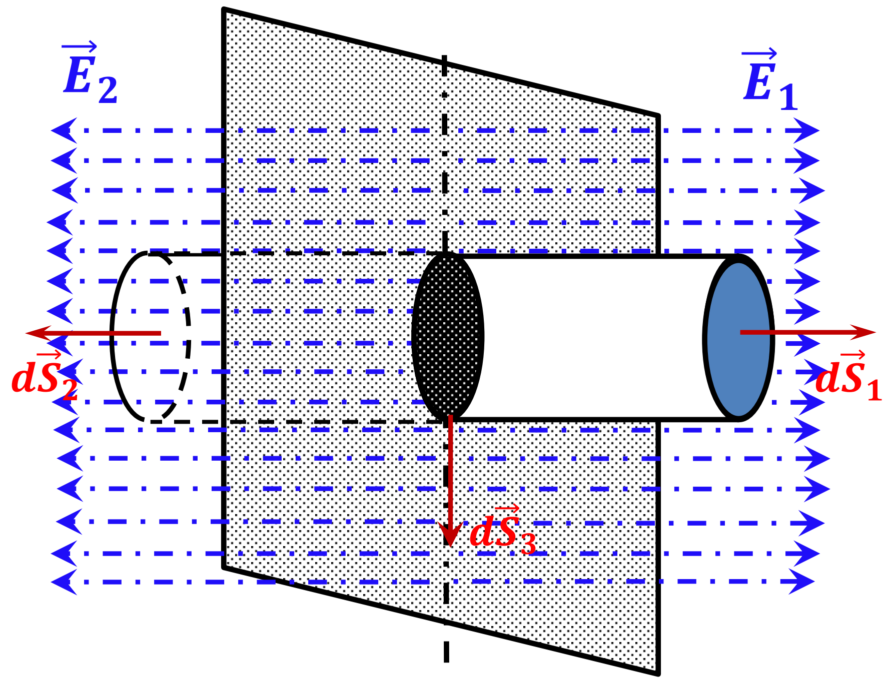
$$\vec{E} = -\overrightarrow{grad} V \Longrightarrow E_r \vec{u}_r + \cancelto{0}{E_\theta} \vec{u}_\theta + \cancelto{0}{E_z} \vec{u}_z = -\frac{\partial V}{\partial r} \vec{u}_r - \cancelto{0}{\frac{1}{r}\frac{\partial V}{\partial \theta}} \vec{u}_\theta - \cancelto{0}{\frac{\partial V}{\partial z}} \vec{u}_z$$

$$\Longrightarrow E_r = -\frac{\partial V}{\partial r} \quad \Longrightarrow dV = -E dr \;= -\frac{\lambda}{2\pi r \varepsilon_0} dr$$

$$\Longrightarrow V(M) = -\frac{\lambda}{2\pi \varepsilon_0} \int \frac{dr}{r} \;\; \boldsymbol{= -\frac{\lambda}{2\pi \varepsilon_0} \ln r + Cte}$$

Example 3: Determine the electric field created by a uniform surface charge distribution on an infinite plane with a surface density $\sigma > 0$

- ❑ $\vec{E}$ is perpendicular to the plane
- ❑ $\vec{E}$ is uniform on both sides of the plane but the direction changes: $\vec{E}_1 = -\vec{E}_2$
- ❑ We take as the Gaussian surface a cylinder of height h and base area S, symmetrical with respect to the plane.



- ❑ The surface *(S)* consists of the 3 surfaces represented by $\overrightarrow{dS}_1, \overrightarrow{dS}_2, \overrightarrow{dS}_3$

$$\phi = \oiint_{(S)} \vec{E}(M).d\vec{S} = \oiint_{(S)} \vec{E}(M).d\vec{S}_1 + \oiint_{(S)} \vec{E}(M).d\vec{S}_2 + \underbrace{\oiint_{(S)} \vec{E}(M).d\vec{S}_3}_{0} = \frac{\sum q_i}{\varepsilon_0}$$

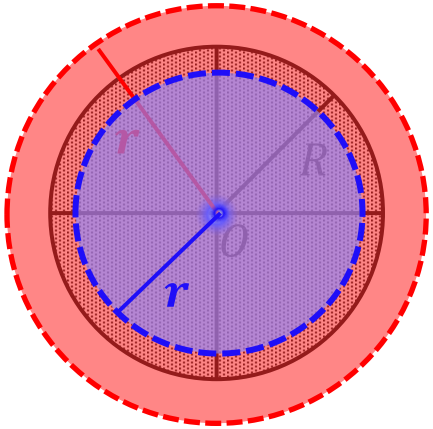
$$\Rightarrow \phi = 2E(M) \oiint_{(S)} dS = 2E(M).S = \frac{\sigma S}{\varepsilon_0} \qquad \Rightarrow \boldsymbol{E(M) = \frac{\sigma}{2\varepsilon_0}}$$

Example 4:

Let be a solid ball with center O and radius R uniformly charged by volume (with charge density ρ) . Calculate the electrostatic field created by this ball inside, on the surface, and outside the ball.

- We take, as a Gaussian surface, a sphere with center O and radius r.

$$\phi = \oiint_{(S)} \vec{E}(M).d\vec{S} = \frac{\sum q_i}{\varepsilon_0} \Longrightarrow E(M).4\pi r^2 = \frac{\rho V_i}{\varepsilon_0}$$



- **Inside the ball:**

✓ $r < R$: $\sum q_i = \rho \frac{4}{3}\pi r^3 \Longrightarrow E(M) = \frac{\rho r}{3\varepsilon_0}$

- **On the surface and outside of the ball:**

✓ $r \geq R$: $\sum q_i = \rho \frac{4}{3}\pi R^3 \Longrightarrow E(M) = \frac{\rho R^3}{3\varepsilon_0 r^2}$