D'après les équations d'Euler-Lagrange,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Donc

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \tag{2.40}$$

On a aussi

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi)$$

Donc,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \tag{2.41}$$

En remplaçant les équations (2.40) et (2.41) dans l'équation (2.39), on trouve

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu(\delta x_\nu)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu(\delta x_\nu) \tag{2.42}$$

On a

$$\delta\phi = \delta_o \phi + (\partial_\nu \phi) \delta x_\nu$$

Donc

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\delta_o \phi + (\partial_\nu \phi)(\delta x_\nu)) \right)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) \tag{2.43}$$

Calculons le terme $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right)$:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\partial_\nu \phi)(\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \left((\partial_\nu \phi) \right) (\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu \left(\delta x_\nu \right)$$

En négligeant les termes d'ordre supérieur, on trouve

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\partial_\nu \phi)(\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu \left(\delta x_\nu \right) \tag{2.44}$$

Donc,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)\tag{2.45}$$

En remplaçant l'équation (2.45) dans l'équation (2.42), on trouve

$$\mathcal{L}(\phi',\partial'_\mu\phi')-\mathcal{L}(\phi,\partial_\mu\phi)=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu)$$

$$=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)$$

Donc,

$$\mathcal{L}(\phi',\partial'_\mu\phi')-\mathcal{L}(\phi,\partial_\mu\phi)=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$$

Calculons le terme $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)=\frac{\partial\mathcal{L}}{\partial\phi}(\partial_\nu\phi)(\delta x_\nu)=\frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\phi}\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$$

$$=\frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\partial x_\nu}\delta x_\nu=\frac{\partial\mathcal{L}}{\partial x_\mu}\delta_{\mu\nu}\delta x_\nu=\partial_\mu\mathcal{L}\ \delta x_\mu$$

Finalement, on trouve

$$\mathcal{L}(\phi',\partial'_\mu\phi')-\mathcal{L}(\phi,\partial_\mu\phi)=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu\mathcal{L}\ \delta x_\mu\tag{2.46}$$

La variation de l'action dans l'équation (2.38) devient,

$$\delta S=\int\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu\mathcal{L}\ \delta x_\mu+\partial_\mu(\delta x_\mu)\mathcal{L}\right]d^4x\simeq 0$$

On a

$$\partial_\mu\mathcal{L}\ \delta x_\mu+\partial_\mu(\delta x_\mu)\mathcal{L}=\partial_\mu(\mathcal{L}\,\delta x_\mu)$$

Alors,

$$\delta S=\int\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right)+\partial_\mu(\mathcal{L}\,\delta x_\mu)\right]d^4x\simeq 0$$

$$\delta S = \int \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi \right) + \mathcal{L}\, \delta x_\mu \right] d^4x \simeq 0$$
$$\Rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi + \mathcal{L}\, \delta x_\mu \right] = 0$$

Cette dernière équation peut être écrite sous la forme

$$\partial_\mu J_\mu = 0$$

Avec,

$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi + \mathcal{L}\, \delta x_\mu \longrightarrow \text{Courant de Noether}$$

Exercice 6 :

1. Montrer que la densité lagrangienne du champ scalaire complexe libre est invariante dans la transformation de phase globale suivante,

$$\begin{cases} \phi(x) \longrightarrow \phi'(x) = e^{i\theta}\phi(x) \\ \phi^*(x) \longrightarrow \phi'^*(x) = e^{-i\theta}\phi^*(x) \end{cases}$$

   où θ est un réel indépendants de x_μ.

2. Quels sont les courant et charge qui se conservent?

2.2 Tenseur Énergie-Impulsion du champ scalaire

Étant donner que densité lagrangienne $\mathcal{L}$ ne dépend pas explicitement du quadri-vecteur position x_μ, sa dérivée par rapport à x_μ est donnée par

$$\partial_\mu \mathcal{L} = \partial_\mu \mathcal{L}(\phi, \partial_\mu \phi) \quad \text{où} \quad \partial_\mu = \frac{\partial}{\partial x_\mu} \tag{2.47}$$

Donc,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} \tag{2.48}$$

On a,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial x_\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \frac{\partial(\partial_\nu \phi)}{\partial x_\mu} \tag{2.49}$$

Or, d'après l'équation d'Euler-Lagrange on a

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) = 0 \quad \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \quad \text{pour} \qquad \mu = \nu \tag{2.50}$$

Donc,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \left(\partial_\nu \phi \right) \tag{2.51}$$

On pose,

$$\partial_\mu \left(\partial_\nu \phi \right) = \partial_\nu \left(\partial_\mu \phi \right) \tag{2.52}$$

On trouve,

$$\partial_\mu \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\nu \left(\partial_\mu \phi \right) = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right) \tag{2.53}$$

Le terme $\partial_\mu \mathcal{L}$ peut être écrit aussi sous la forme:

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial x_\nu} \frac{\partial x_\nu}{\partial x_\mu} = \left(\partial_\nu \mathcal{L} \right) \delta_{\mu\nu} = \partial_\nu \left(\mathcal{L} \delta_{\mu\nu} \right) \tag{2.54}$$

Finalement, en comparant les équation (2.53) et (2.54), on trouve

$$\partial_\mu \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right) = \partial_\nu \left(\mathcal{L} \delta_{\mu\nu} \right) \tag{2.55}$$

Donc,

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{2.56}$$

Maintenant, si on remplace ν par μ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{2.57}$$

Cette dernière équation peut être réécrite sous la forme suivante,

$$\partial_{\mu\nu} T_{\mu\nu} = 0 \;\text{ avec } T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \tag{2.58}$$

Où $T_{\mu\nu}$ représente le tenseur énergie-impulsion du champ scalaire.