II. Maxwell Equations

1. Scalar electric potential:

Let's consider the case of the positive point q present in uniform electric field: $\vec{E} = -E\vec{j}$.

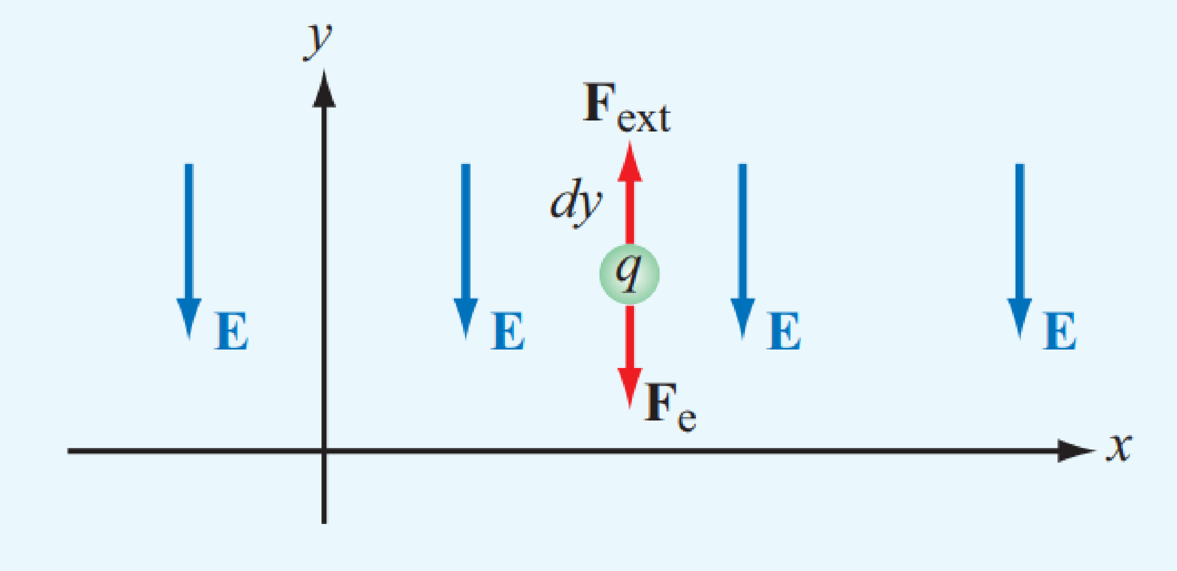
The presence of the electric field will exert a force on the point charge, called Coulomb force given by:

$\vec{F}_e = q\vec{E} = -qE\vec{j}$

To move the charge q in opposite direction with a constant speed ($\sum \vec{f}_i = 0$), one needs to apply an external force $\vec{F}_{ext} = -\vec{F}_e = qE\vec{j}$

This is equivalent to do a work (spend an energy) over the distance element $d\vec{l}$:

$$dW = \vec{F}_{ext}.d\vec{l} = -q\vec{E}.d\vec{l} = qE.dy$$



One can change the point charge by another one q' and the expression of the executed work take the same form:

$$dW' = q'E.dy$$

And so on:

$$dW'' = q''E.dy$$

II. Maxwell Equations

1. Scalar electric potential:

This means that the spent work by unit of charge:

$$\frac{dW}{q} = \frac{dW'}{q'} = \frac{dW''}{q''}[J/C] = \cdots = -\vec{E}.d\vec{l}$$

Will depend only on the scalar expression:

$$dV = -\vec{E}.d\vec{l} = E.dy$$

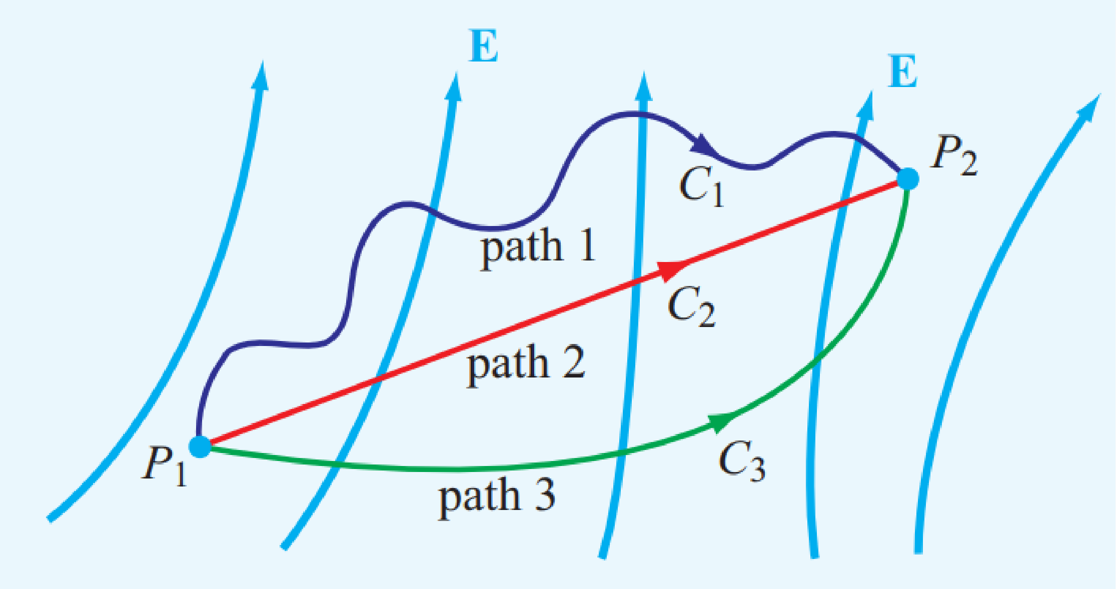
Called *"differential electric potential" (or differential voltage)*

The minus sign means that this physical quantity decrease when electric field increase and vice-versa.

The unit of V is the Volt $[V]$ with $1V = 1J/C$

The potential difference corresponding to moving a point charge from point P_1 *to point* P_2 *is obtained by integrating the last expression along any path between them:*

$$\Delta V = V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \vec{E}.d\vec{l}$$



II. Maxwell Equations

1. Scalar electric potential:

From the following law:

$$\Delta V = V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \vec{E}.d\vec{l}$$

We should note that:

- We have : $V_1 \rightarrow P_1$ *and* $V_2 \rightarrow P_2$
- *Kirchhoff law: For* $P_1 \equiv P_2$

$$\Delta V = 0 \leftrightarrow \int_{P_1}^{P_1} \vec{E}.d\vec{l} = \oint_C \vec{E}.d\vec{l} = 0$$

- *If* $P_1 \rightarrow \infty \leftrightarrow V_1 = 0 \rightarrow V = -\int_{\infty}^{P} \vec{E}.d\vec{l}$
- *The null potential is a referential value (not absolute) , and it is called "ground"*

If we use the Stokes's theorem to convert a surface integral into a line integral to, for any vector field $\vec{A}$:

$$\int_S (\vec{\nabla} \wedge \vec{A}).d\vec{S} = \oint_C \vec{A}.d\vec{l}$$

Where C is a closed contour surrounding S.

Thus, we can obtain the differential form from this integral expression:

$$\vec{\nabla} \wedge \vec{E} = 0$$

Any vector field verifying that its line integral along any closed path is zero, is called conservative or irrotational field.

Hence, the electrostatic field $\vec{E}$ *is conservative.*

II. Maxwell Equations

1. Scalar electric potential:

We should recall that for any scalar f we have:

$$\vec{\nabla}\wedge\left(\vec{\nabla}f\right)=0$$

If we take the former differential equation of electric field:

$$\vec{\nabla}\wedge\vec{E}=0$$

By identification, we can conclude that electric field should be derived from a scalar function, which is the scalar electric potential $(f\equiv V)$

$$\vec{E}=-\vec{\nabla}V$$

This can also derived from the equation:

$$dV=-\vec{E}.d\vec{l}$$

Indeed, if we use the decomposition of any differentiation of a given function $V(x,y,z)$ on the coordinates basis according to partial derivation, and considering an electric field:

$$\vec{E}=E_x\vec{i}+E_y\vec{j}+E_z\vec{k}$$

Over a distance element: $d\vec{l}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$$dV=\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz=-E_x.dx-E_y.dy-E_z.dz$$

After identification:

$$\frac{\partial V}{\partial x}=-E_x;\frac{\partial V}{\partial y}=-E_y;\frac{\partial V}{\partial z}=-E_z$$

Which equivalent to:

$$\vec{E}=-\vec{\nabla}V$$

II. Maxwell Equations

2. Electric potential of point charges

Knowing that an electric field created in free space by a point charge q could be given by the expression:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}_r$$

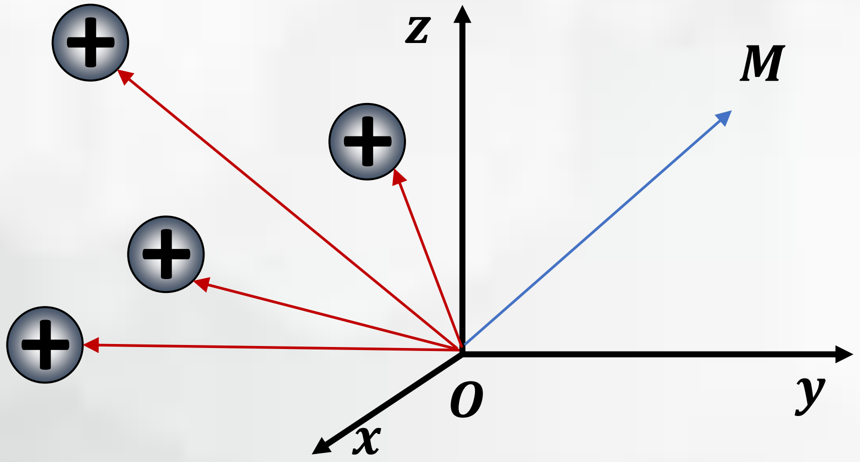
Through a radial line $d\vec{l} \equiv d\vec{r} = dr\vec{u}_r$

We can calculate the electric potential between two points:

$$V = -\int_{\infty}^{r} \vec{E} \, . \, d\vec{r} = -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{r} = k\frac{q}{r}$$

By using the former result of the electric potential resulting from one point charge, we can generalize it for N discrete point charges present at different locations $\vec{r}_1; \vec{r}_2; \vec{r}_3 \ldots \vec{r}_N$, the electric potential in free space resulting from these charge at the measurement point M located by $\vec{r}$ is:

$$V_{tot} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\|\vec{r} - \vec{r}_i\|}$$



II. Maxwell Equations

3. Electric potential of continuous distributions of charge

To obtain expressions for the electric potential V due to continuous charge distributions over a volume v, over a surface S, or along a line l, we replace in the former equation, the point charges q_i by $dq \equiv \rho dv \equiv \sigma dS \equiv \lambda dl$ and convert the summation by the integral:

- **Volume distribution:**

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \frac{\rho dv}{r}$$

- **Surface distribution:**

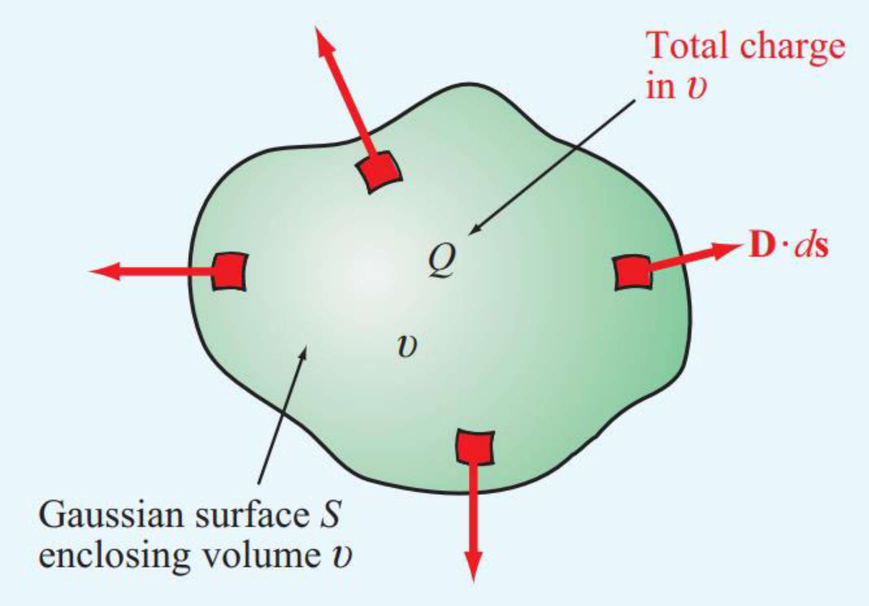
$$V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma dS}{r}$$

- **Linear distribution:**

$$V = \frac{1}{4\pi\varepsilon_0}\int_{l'} \frac{\lambda dl}{r}$$

II. Maxwell Equations

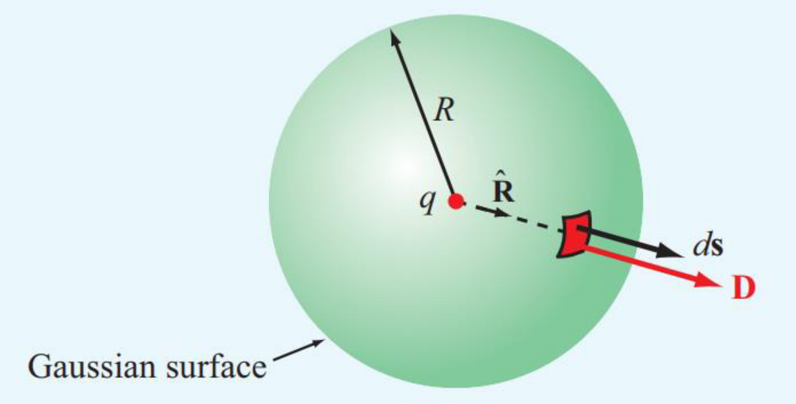
4. Gauss's law and Poisson's equation:



The Gauss law states that for any enclosed charge inside a surface S', one can find the electric field resulting from this charge by calculating its flux:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{Q}{\varepsilon_0} \leftrightarrow \oiint_{S'} \vec{D}\,.\,d\vec{S} = Q$$

Where: $\vec{D} = \varepsilon_0 \vec{E}$ is electric flux density $[C.m^{-2}]$



A good choice of the Gaussian surface will conduct to a simple calculation of the electric field generated by the point charge q:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{q}{\varepsilon_0} \leftrightarrow \oiint_{S'} E_r\, r^2 sin\theta d\theta d\varphi = E_r r^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{q}{\varepsilon_0}$$

$$E_r r^2 4\pi = \frac{q}{\varepsilon_0} \rightarrow E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

II. Maxwell Equations

4. Gauss's law and Poisson's equation:

The Gauss law for a number of discrete charges could also obtained as a generalization of the former law, when the surface S is enclosing N charges Q_i:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{N} Q_i$$

And, for a continuous distributions of charge, we get:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int dq$$

- Linear distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{l'} \lambda dl$
- surface distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{S'} \sigma dS$
- volume distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{v'} \rho dv$

It is interesting to see that for any vector field $\vec{A}$, the divergence theorem allows us to convert a surface integral into a volume integral:

$$\oiint_S \vec{A}.\,d\vec{S} = \int_V (\vec{\nabla}.\vec{A}).\,dv$$

Thus, it is possible to rewrite the left-hand term of the Gauss law with the volume distribution case:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \int_{v'} (\vec{\nabla}.\vec{E}).\,dv = \frac{1}{\varepsilon_0}\int_{v'} \rho dv$$

By identification, we get the differential form of Gauss law (divergent of $\vec{E}$):

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

II. Maxwell Equations

4. Gauss's law and Poisson's equation:

Now, using both equations:

$$\vec{E} = -\vec{\nabla}V \ldots\ldots\ldots(1)$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \ldots\ldots\ldots(2)$$

And, by replacing (1) into (2), we obtain:

$$\vec{\nabla}.\vec{E} = -\vec{\nabla}.\left(\vec{\nabla}V\right) = -\nabla^2 V = -\Delta V = \frac{\rho}{\varepsilon_0}$$

Which could be rewritten as:

$$\Delta V = -\frac{\rho}{\varepsilon_0} \leftrightarrow \Delta V + \frac{\rho}{\varepsilon_0} = 0$$

This a second degree differential equation with source term is known as *"Poisson's equation".*

The special case of absence of electric charges in the free space, the Poisson's equation will be reduced to homogeneous differential equation:

$$\Delta V = \nabla^2 V = 0$$

Known as *"Laplace's equation".*

In rectangular coordinates:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2.\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2.\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

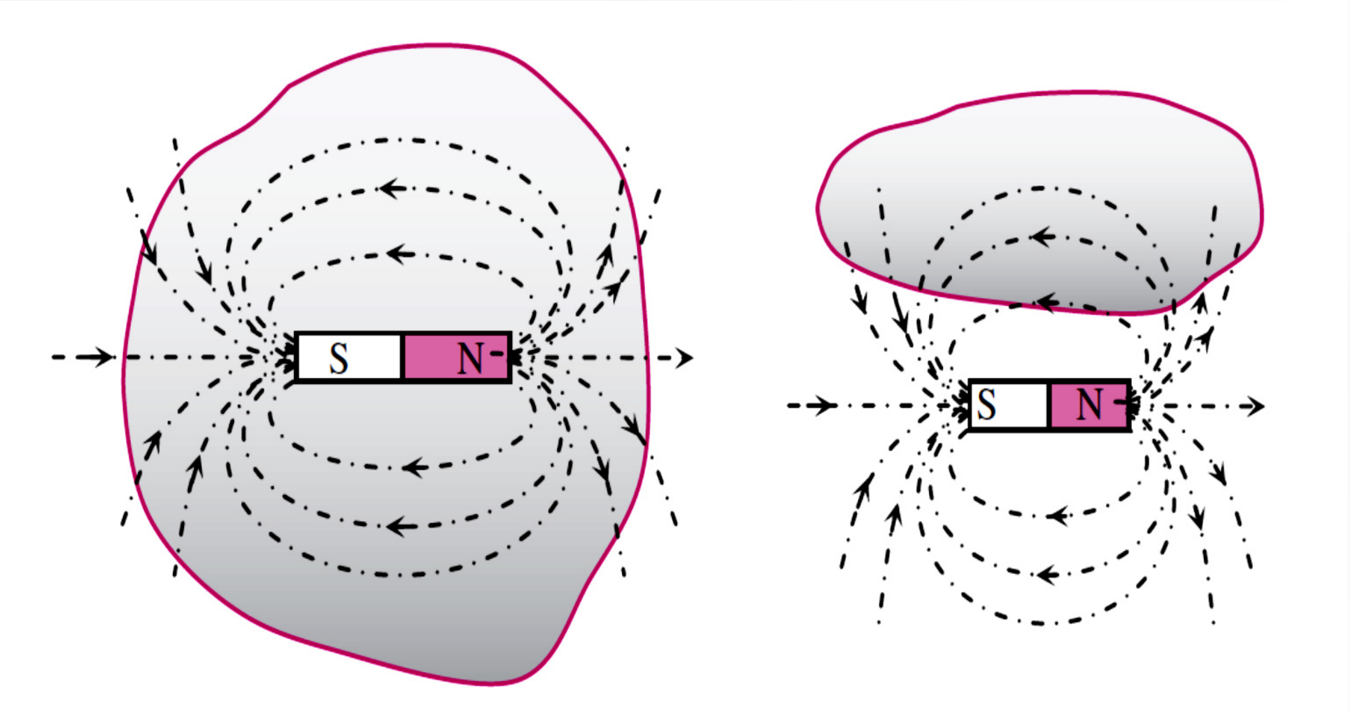
II. Maxwell Equations

5. Gauss law for magnetism:

By considering the flux of magnetic field lines through a given surface enclosing totally or partially these lines, it comes intuitively, due to the nature of the magnetic dipole (permanent or induced) that the same amount of field lines will enter and then exit from that surface. This will imply:

$$\oiint_S \vec{B}\,.\,d\vec{S} = 0$$

This is the equivalent Gauss law for magnetic field in its integral form.



The differential form will be deduced in similar way by using the conversion of surface integral to a volume one:

$$\int_{v'} (\vec{\nabla}.\vec{B}).\,dv = 0 \rightarrow \vec{\nabla}.\vec{B} = 0$$

6. Vector magnetic potential:

Now let's exploit the fact that for any vector field $\vec{A}$ we have always: $\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{A}\right) = 0$

By identification with the former law giving the divergent of magnetic field : $\vec{\nabla}.\vec{B} = 0$

It comes directly that magnetic field could be derived as a curl of a vector field, called *"Vector Magnetic Potential"*:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}$$

Since the magnetic field unit in S.I is Tesla :

$$1[T] = 1\left[Weber.m^{-2}\right] = 1\left[Wb.m^{-2}\right]$$

Consequently the S.I unit for the vector magnetic potential will be : $Wb.m^{-1} \equiv \frac{Wb}{m}$

If now we rewrite the Ampere law of magnetic field induced by a set of currents:

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \sum_k I_k$$

Passing from summation to integral and using surface density of current:

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \iint_{S'} \vec{j}.d\vec{S}$$

If we use the Stokes's theorem to convert a line integral into a surface integral:

$$\int_S \left(\vec{\nabla} \wedge \vec{B}\right).d\vec{S} = \oint_C \vec{B}.d\vec{l}$$

We get by identification:

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j}$$

II. Maxwell Equations

7. Vector Poisson's equation:

In the same way as we achieve it for electric field, let's exploit both equations:

$$\vec{B} = \vec{\nabla} \wedge \vec{A} \dots\dots\dots (3)$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} \dots\dots\dots (4)$$

And replace (3) into (4):

$$\vec{\nabla} \wedge \vec{B} = \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \mu_0 \vec{j}$$

We know (chapter 01) that for any vector $\vec{A}$, the Laplacian of $\vec{A}$ obeys the vector identity given by:

$$\nabla^2 \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A})$$

This implies:

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{j}$$

An appropriate and simplest choice about the term $\vec{\nabla}.\vec{A}$ is to take (Coulomb gauge):

$$\vec{\nabla}.\vec{A} = 0$$

To avoid any conflicting with the requirement of equation (3).

Using this choice leads to the *"Vector Poisson's equation"*:

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}$$

Which is very similar to the Poisson's equation for the scalar electric potential:

$$\Delta V = -\frac{\rho}{\varepsilon_0}$$

7. Vector Poisson's equation:

Using the definition for $\nabla^2\vec{A}$, the vector Poisson's equation can be decomposed into three scalar Pois son's equations

$$\begin{cases} \dfrac{\partial^2 A_x}{\partial x^2} = -\mu_0 J_{Sx} \\ \dfrac{\partial^2 A_x}{\partial y^2} = -\mu_0 J_{Sy} \\ \dfrac{\partial^2 A_x}{\partial z^2} = -\mu_0 J_{Sz} \end{cases}$$

As for Poisson's equation for scalar potential, it is possible to get back into vector potential components:

$$A_x = \frac{\mu_0}{4\pi}\int_{S'} \frac{J_{Sx}}{r} dS$$

Similar solutions could be found for the remain components y and z:

Volume density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_V}{r} dv$$

Surface density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{J}_S}{r} dS$$

Linear density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{l'} \frac{I}{r} d\vec{l}$$

The vector magnetic potential provides a 3rd approach for computing the magnetic field due to current-carrying conductors in addition to the methods suggested by Biot-Savart and Ampère law.

II. Maxwell Equations

1. Electro-magnetostatics laws:

According to previous sections we could gather all the differential and integral equations of both electric and magnetic fields:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla} \wedge \vec{E} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \wedge \vec{B} = \mu_0 I \end{cases} \leftrightarrow \begin{cases} \vec{\nabla}.\vec{D} = \rho \\ \vec{\nabla} \wedge \vec{D} = 0 \\ \vec{\nabla}.\vec{H} = 0 \\ \vec{\nabla} \wedge \vec{H} = I \end{cases} \leftrightarrow \begin{cases} \oint \vec{D}\,.\,d\vec{S} = Q \\ \oint_C \vec{D}\,.\,d\vec{l} = 0 \\ \oint \vec{H}\,.\,d\vec{S} = Q \\ \oint_C \vec{H}\,.\,d\vec{l} = I \end{cases}$$

Where: $\vec{D} = \varepsilon\vec{E};\ \vec{B} = \mu\vec{H}$

in the case of free space: $\varepsilon = \varepsilon_0, \mu = \mu_0$

Both fields are derived from potentials:

- **Scalar electric potential:**

$$V = -\int \vec{E}.d\vec{l} \leftrightarrow \vec{E} = -\vec{\nabla}V$$

- **Vector magnetic potential:**

$$\vec{B} = \vec{\nabla} \wedge \vec{A}$$

Volume density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_V}{r} dv$$

Surface density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{J}_S}{r} dS$$

Linear density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{l'} \frac{I}{r} d\vec{l}$$