Chapter 2: Differential Equations

1 Differential Equations

1.1 Ordinary differential equations

1.2 Generalities

Definition 1.1 *A differential equation is an equation whose unknown is a function, and in which certain derivatives of the unknown function appear.*

Example 1.1 *Let u be a function, the following equations are differential equations.*

1. $u^{'} = 2u$
2. $u^{''} - 3u^{'} + 1 = 0$
3. $u^{(3)} = u$

Definition 1.2 *Let u be a The unknown function depends on the variable x. We call a differential equation of order n ($n \in \mathbb{N}$) any equation of the form*

$$F\left(x, u, u', u'', \cdots, u^{(n)}\right) = 0, \tag{1}$$

where $u^{'}, u^{''}, \cdots, u^{(n)}$ are the derivatives of u of respective orders $1, 2, \cdots, n$.

Remark 1.1
1. *Equation (1) involves $(n+2)$ variables.*
2. *The unknown function can be denoted by: $y, t, \cdots$*
3. *In a differential equation, when we write $u, u', u'', \ldots, u^{(n)}$, we imply $u(x), u'(x), u''(x), \ldots, u^{(n)}(x)$.*

Definition 1.3 *A solution to equation (1) on the interval I is a function that is n times differentiable on I and satisfies (1).*

Example 1.2 *One can easily verify that the function* $u(x) = ce^{4x}, ; c \in \mathbb{R}$ *is a solution to the differential equation* $u' = 4u$*. Indeed, it is clear that if* $u(x) = ce^{4x}$ *then* $u'(x) = 4ce^{4x} = 4u(x)$*.*

Definition 1.4 *A differential equation of the form* $u^{'} f(u) = g(x)$ *is called an equation with separated variables*

Example 1.3 *The equation* $u^{'} = \dfrac{e^{-u}}{x^2}$ *can be written in the form* $u^{'} e^u = \dfrac{1}{x^2}$*. One can easily find the solutions to this equation. By integrating both sides of the equation, we obtain* $e^u = -\dfrac{1}{x} + k, \quad k \in \mathbb{R}$*. This gives us*

$$u(x) = \ln\Big| -\frac{1}{x} + k\Big|, \quad \textit{such that } \ k, \in \mathbb{R}.$$

Definition 1.5 *The order of a differential equation is the order of the highest derivative appearing in the equation.*

Example 1.4

1. $2xu^{'} + u = 0$ *is a first-order differential equation.*
2. $u^{''} + u^{'} - 3u = ln(x)$ *is a second-order differential equation.*

Definition 1.6

1. *A differential equation of order n is called linear if it is of the form*

$$f_0(x)u + f_1(x)u^{'} + f_2(x)u^{''} + \cdots + f_n(x)u^{(n)} = f(x) \qquad (2)$$

où $f_0, f_1, f_2, \cdots, f_n, f$ *are real functions that are continuous on an interval* $I \subset \mathbb{R}$*.*

2. *If* $f(x) = 0$ *for all* $x \in I$*, then the equation* (2) *is called homogeneous, and it is of the form* $f_0(x)u + f_1(x)u^{'} + f_2(x)u^{''} + \cdots + f_n(x)u^{(n)} = 0$
3. *The equation* (2) *is said to have constant coefficients if the functions* $f_0, f_1, f_2, \cdots, f_n$ *are constants. on* I*. In other words, the equation* (2) *can be written in the form* $a_0u + a_1u^{'} + a_2u^{''} + \cdots + a_nu^{(n)} = f(x)$*, with* $a_0, a_1, a_2, \cdots, a_n$ *real constants.*

Remark 1.2 *In a linear differential equation, there are no exponents for the terms* $u, u^{'}, u^{''}, \cdots, u^{(n)}$*.*

Example 1.5

1. $e^x u + x^{\frac{1}{2}} u^{''} = x^2 + 1$ *it is a linear differential equation, and* $e^x u + x^{\frac{1}{2}} u^{''} = 0$ *it is the associated homogeneous equation.*

2. $2u' - 3u'' + \frac{1}{5}u^{(3)} = x$ *is a linear differential equation with constant coefficients.*

3. *The equation* $\left(u'\right)^2 + u'' + 3u = 0$ *is not a linear differential equation.*

Proposition 1.7 *If* u_1, u_2 *are two solutions of the homogeneous linear equation*

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0, \tag{3}$$

then $\alpha u_1 + \beta u_2$ *is also a solution of* (3).

We consider the linear differential equation (2) and the associated homogeneous equation (3). The following proposition allows us to find the general solution of the equation (2).

Proposition 1.8 *If* u_0 *is a particular solution of* (2) *and* u_1 *is a solution of the homogeneous equation* (3) *then* $u = u_1 + u_0$ *is a general solution of the equation* (2).

Remark 1.3 *Let us recal that a particular solution of the equation* (2) *is the given function which is* n *times differentiable and satisfies* (2).

2 First-order differentials equations

2.1 Linear differential equation of the first order without second member

We consider the equation

$$a_1(x)u' + a_0(x)u = 0 \text{ tel-que } a_1(x) \neq 0, \tag{4}$$

which is equivalent to the equation with separate variables,

$$u' + a(x)u = 0 \quad \text{tel-que} \quad a(x) = \frac{a_0(x)}{a_1(x)}. \tag{5}$$

To solve the equation (5), we follow the following method:

$$
\begin{aligned}
u^{'} + a(x)u = 0 &\iff u^{'} = -a(x)u, \\
&\iff \frac{du}{dx} = -a(x)u, \\
&\iff \frac{du}{u} = -a(x)\,dx \\
&\iff \int \frac{du}{u}\,dx = -\int a(x)\,dx \\
&\iff \ln|u| = -A(x) + k \\
&\iff |u| = e^{(-A(x)+k)} \\
&\iff u = Ke^{-A(x)},
\end{aligned}
$$

where $A(x)$ is a primitive of $a(x)$ and $K = \pm e^k,\ k \in \mathbb{R}$.

Example 2.1 *The solution of the equation*

$$u^{'} - \sqrt{x}u = 0, \quad x > 0$$

is given by

$$u(x) = Ke^{-A(x)}$$

with $K = \pm e^k$ *and* $-A(x) = \int \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3}$.

2.2 Linear differential equation of the first order with second member

We consider the equation

$$a_1(x)u^{'} + a_0(x)u = f_1(x) \text{ such that } \quad a_1(x) \neq 0, \tag{6}$$

which is equivalent to the equation

$$u^{'} + a(x)u = f(x) \quad \text{such that } a(x) = \frac{a_0(x)}{a_1(x)} \text{ and } f(x) = \frac{f_1(x)}{a_1(x)}. \tag{7}$$

According to Proposition 1.8, the solution of the equation (7) is of the form $u(x) = u_0(x)+u_1(x)$, where $u_1(x) = Ke^{-A(x)}$ is the solution of the associated homogeneous equation(7), and u_0 is the particular solution of the equation (7).

Example 2.2 *Let us consider the equation*

$$u^{'} - \sqrt{x}u = 1 - x\sqrt{x}, \quad x > 0. \tag{8}$$

According to the previous example $u_1(x) = Ke^{\frac{2}{3}\sqrt{x^3}}$, *is a solution of the associated homogeneous equation to the equation* (8). *On the other hand, we can easily check that* $u_0(x) = x$ *is a particular solution of the equation* (8). *So the general solution of the equation* (8) *est*

$$u(x) = x + Ke^{\frac{2}{3}\sqrt{x^3}} = x + Ke^{\frac{2}{3}x\sqrt{x}}, \quad K \in \mathbb{R}.$$

The question we ask ourselves is: how to find a particular solution?

2.3 Search for a particular solution: Method of variation of parameters

We know that the solution of the associated homogeneous equation to (7) is of the form $u_1(x) = Ke^{-A(x)}, \ K \in \mathbb{R}$. The method of variation of parameters consists of seeking a particular solution of the equation (7) in the form $u_0(x) = K(x)e^{-A(x)}$, where $K(x)$ is a function of the variable x instead a constante. Stating that $u_0(x) = K(x)e^{-A(x)}$ is a solution of (7) means that

$$u_0^{'}(x) + a(x)u_0(x) = f(x), \quad \text{with } \ A^{'}(x) = a(x). \tag{9}$$

$$\begin{aligned}
u_0^{'}(x) + a(x)u_0(x) = f(x) \quad &\Longleftrightarrow \quad \left(K(x)e^{-A(x)}\right)^{'} + a(x)K(x)e^{-A(x)} = f(x) \\
&\Longleftrightarrow \quad K^{'}(x)e^{-A(x)} - K(x)A^{'}(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\
&\Longleftrightarrow \quad K^{'}(x)e^{-A(x)} - K(x)a(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\
&\Longleftrightarrow \quad K^{'}(x)e^{-A(x)} = f(x) \\
&\Longleftrightarrow \quad K^{'}(x) = f(x)e^{A(x)} \\
&\Longleftrightarrow \quad K(x) = \int f(x)e^{A(x)}\, dx.
\end{aligned}$$

So the particular solution of (7) is written in the form

$$u_0(x) = \left(\int f(x)e^{A(x)}\, dx\right) e^{-A(x)}$$

Exercise 2.1 *Find the solutions of the equation*

$$u^{'} - 2u = e^{3x+1} \tag{10}$$

Proof. The search for the solution of the homogeneous equation:
The solution of the homogeneous equation $u^{'} = 2u$ associated with equation (10) is given by

$$\begin{aligned} u^{'} = 2u &\iff \frac{du}{dx} = 2u \\ &\iff \frac{du}{u} = 2dx \\ &\iff \ln|u| = 2x + k, \quad k \in \mathbb{R} \\ &\iff u = Ke^{2x},\ K = \pm e^{k}. \end{aligned}$$

Then

$$u_1(x) = Ke^{2x}$$

The search for the particular solution:
We look for $K(x)$ such that the particular solution of the equation (10) is of the form $u_0(x) = K(x)e^{2x}$.

$$\begin{aligned} u_0^{'}(x) - 2u_0(x) = e^{3x+1} &\iff \left(K(x)e^{2x}\right)^{'} - 2K(x)e^{2x} = e^{3x+1} \\ &\iff K^{'}(x)e^{2x} + 2K(x)e^{2x} - 2K(x)e^{2x} = e^{3x+1} \\ &\iff K^{'}(x)e^{2x} = e^{3x+1} \\ &\iff K^{'}(x) = e^{3x+1}e^{-2x} \\ &\iff K(x) = \int e^{x+1}\,dx \\ &\implies K(x) = e\int e^{x}\,dx \\ &\implies K(x) = e^{x+1}. \end{aligned}$$

The solution of equation (10) is in the form

$$u(x) = e^{x+1}e^{2x} + Ke^{2x} = \left(e^{x+1} + K\right)e^{2x}, \quad K \in \mathbb{R}.$$

■

2.4 Linear first-order differential equation with constant coefficients

We consider equations of the form

$$a_1u^{'} + a_0u = f_1(x). \tag{11}$$

This is a particular case of equations (6). We solve equation (11) in the same manner as (6).

2.5 Bernoulli's differential equation

Definition 2.2 *Any equation of the form*

$$u^{'} + a(x)u + b(x)u^n = 0 \tag{12}$$

is called an equation of **Bernoulli**.

Solving Bernoulli's equation

1. if $n = 0$, the equation (12) becomes of the forme (7).
2. if $n = 1$, equation (12) becomes of the forme (5)
3. if $n \neq 0$ and $n \neq 1$, we seek to transform the equation (12) into a first order linear differential equation. To do this, we follow the following method:
   we divide by u^n and the equation (12) becomes

   $$u^{-n}u^{'} + a(x)u^{1-n} + b(x) = 0 \tag{13}$$

   we set, $y = u^{1-n}$ then $\frac{1}{1-n}y^{'} = u^{-n}u^{'}$. Therefore the equation (13) becomes

   $$\frac{1}{1-n}y^{'} + a(x)y + b(x) = 0. \tag{14}$$

   The equation (14) is of the form (6).

Example 2.3 *Solve the following equation*

$$u^{'} + e^x u + e^x u^3 = 0. \tag{15}$$

Proof. We divide by u^3 , we will have

$$u^{-3}u^{'} + e^x u^{-2} = -e^x. \tag{16}$$

By making the change of variable $y = u^{-2}$ and by deriving both sides, we obtain $y^{'} = -2u^{'}u^{-3}$. In other words, $u^{'}u^{-3} = -\frac{1}{2}y$. This change of variable allows us to write the equation (16) in the form

$$-\frac{1}{2}y + e^x y = -e^x. \tag{17}$$

It is a first-order linear equation with a second term, the resolution of which proceeds through the previous steps. ∎

2.6 Équation différentielle homogène

Soit H une fonction numérique définie et continue sur un domaine $D \subset \mathbb{R}$.

Definition 2.3 *We call a homogeneous differential equation any differential equation of the form* $F\left(x, u, u^{'}\right) = 0$ *and which remains unchanged when replaced* x *by* αx *and* u *by* αu *and left* $u^{'}$ *as it is. These equations are of the form:*

$$u^{'} = H\left(\frac{u}{x}\right). \tag{18}$$

The resolution of equation (18) generally reduces to solving a simple equation by using the change of variable $t = \frac{u}{x}$, where $tx = u$ and $u^{'} = t^{'}x + t$. The solutions are in the form (x, u).

Example 2.4 *Solve the equation* $2xuu^{'} = u^2 - x^2$.

Proof. When remplacing x by αx and u by αu and left $u^{'}$ unchanged, we obtain:

$$2\alpha^2 xuu^{'} = \alpha^2\left(u^2 - x^2\right)$$

which is exactly

$$2xuu^{'} = u^2 - x^2.$$

So the equation is homogeneous. We use the change of variable $t = \frac{u}{x}$, with $tx = u$ and $u^{'} = t^{'}x + t$. The given equation becomes:

$$\begin{aligned}
2xuu^{'} = u^2 - x^2 &\iff 2uu^{'} = \frac{u^2}{x} - x \\
&\iff 2tx\left(t^{'}x + t\right) = \left(\frac{u}{x}\right)u - x \\
&\iff 2x^2tt^{'} + 2t^2x = t^2x - x \\
&\iff 2x^2tt^{'} = -\left(t^2 + 1\right) \\
&\iff \frac{2t}{t^2 + 1}\,dt = -\frac{1}{x}\,dx \\
&\iff \ln\left(t^2 + 1\right) = -\ln|x| + k, \quad k \in \mathbb{R} \\
&\iff \ln\left(t^2 + 1\right)|x| = k \\
&\iff x = \frac{K}{t^2 + 1},\ u = \frac{Kt}{t^2 + 1},\ K = \pm e^k.
\end{aligned}$$

■

3 Exercices corrigés

Exercise 3.1 *Solve the following differential equation:*

$$u^2u^{'} + x^2 = 0,\ u \neq 0,\ u(0) = 4 \tag{19}$$

Solution:
It is an homogeneous first order differential equation with separate variables.

$$\begin{aligned} u^2u^{'} + x^2 = 0 &\iff u^2u^{'} = -x^2 \\ &\iff u^2\frac{du}{dx} = -x^2 \\ &\iff u^2\,du = -x^2\,dx \\ &\Longrightarrow \int u^2\,du = -\int x^2\,dx \\ &\Longrightarrow \frac{1}{3}u^3 = -\frac{1}{3}u^3 + k, \quad k \in \mathbb{R}, \\ &\Longrightarrow u = \sqrt[3]{-x^3 + k}, \quad k \in \mathbb{R}. \end{aligned}$$

$$u(0) = 4 \iff \sqrt[3]{-(0)^3 + k} = 4$$
$$\sqrt[3]{+k} = 4 \Longrightarrow k = 64.$$

Donc $u(x) = \sqrt[3]{-x^3 + 64}$.

Exercise 3.2 *Solve the following differential equation:*

$$xu^{''} + 2u^{'} = 0,\ u \neq 0,\ u(1) = 0, u(2) = 1. \tag{20}$$

Solution:
This is a second-order homogeneous equation. We transform it into a first-order equation by making the change of variable $y = u^{'}$, and the equation becomes

$$xy^{'} + 2y = 0.$$

$$
\begin{aligned}
xy^{'}+2y=0 &\iff xy^{'}=-2y\\
&\iff \frac{y^{'}}{y}=-\frac{2}{x}\\
&\iff \frac{1}{y}\frac{dy}{dx}=-\frac{2}{x}\\
&\iff \frac{dy}{y}=-2\frac{dx}{x}\\
&\Longrightarrow \ln|y|=-2\ln|x|+k, k\in\mathbb{R}\\
&\Longrightarrow y=\pm e^{\frac{1}{x^2}+k}\\
&\Longrightarrow y=\frac{K}{x^2}, K=\pm e^{k}.
\end{aligned}
$$

$$
\begin{aligned}
y=u^{'} &\iff u^{'}=\frac{K}{x^2}\\
&\Longrightarrow u=-\frac{K}{x}+\tau,\ K,\tau\in\mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
Takingintoaccounttheinitialconditions, wehave: u(1)=0 &\iff -K+\tau=0\Longrightarrow K=\tau,\\
u(2)=1 &\iff \frac{-K}{2}+\tau=1\Longrightarrow K=2.
\end{aligned}
$$

So the sought(cherchee) solution is

$$u(x)=2\left(1-\frac{1}{x}\right).$$

Exercise 3.3 *Solve the following differential equation:*

$$u^{''}+3u^{'}+2u=xe^{-x}. \tag{21}$$

Solution:

1. We seek the general solution of the homogeneous equation

   $$u^{''}+3u^{'}+2u=0 \tag{22}$$

   The characteristic equation associated to(22) is

   $$r^2+3r+2=0 \tag{23}$$

   $\Delta=9-8=1\Longrightarrow r_1=-2, r_2=-1.$
   The general solution of (22) est

   $$u_1=\lambda_1 e^{-2x}+\lambda_2 e^{-x}$$

2. We seek a particular solution of the equation (21).
   As the second term of equation (21) is in the form $h(x)e^{-x}$ where $h(x)$ is a polynomial of degree $n = 1$, and since $(-1)^2 + 3(-1) + 2 = 0$ then we seek the particular solution in the form $u_0 = k(x)e^{-x}$ where $k(x)$ is a polynomial of degree $m = n+1 = 2$. we set $u_0 = \left(ax^2 + bx + c\right) e^{-x}$. A simple calculation gives us

   $$u_0^{'} = e^{-x} \left(-ax^2 + (2a - b) + (b - c)\right)$$

   and

   $$u_0^{''} = e^{-x} \left(ax^2 + (b - 4a) + (2a - 2b + c)\right).$$

   By replacing these results into (21) and by identification, we find $a = \frac{1}{2}, b = -1, c = 0$. Then ,

   $$u_0 = \left(\frac{1}{2}x^2 - x\right) e^{-x}$$

3. The general solution of the equation (21) is:

   $$u = \left(\frac{1}{2}x^2 - x\right) e^{-x} + \lambda_1 e^{-2x} + \lambda_2 e^{-x} = \left(\frac{1}{2}x^2 - x + \lambda_1\right) e^{-x} + \lambda_2 e^{-x}.$$

Exercise 3.4 *Solve the following differential equation:*

$$u^{''} + u^{'} + u = x^2 + 1, \; u(0) = 1, u^{'}(0) = 0. \tag{24}$$