Khemis Miliana University – Djilali BOUNAAMA
Faculty of Science and Technology
Department of Physics

Electromagnetism

L2 Fundamental Physics

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Content

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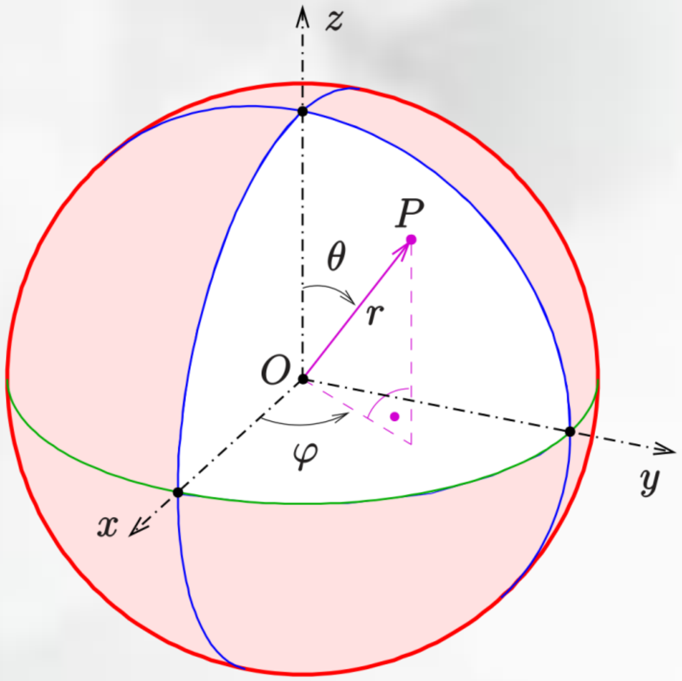
Chapter 01

Useful Maths



I. Useful Maths

For the cartesian system we have:

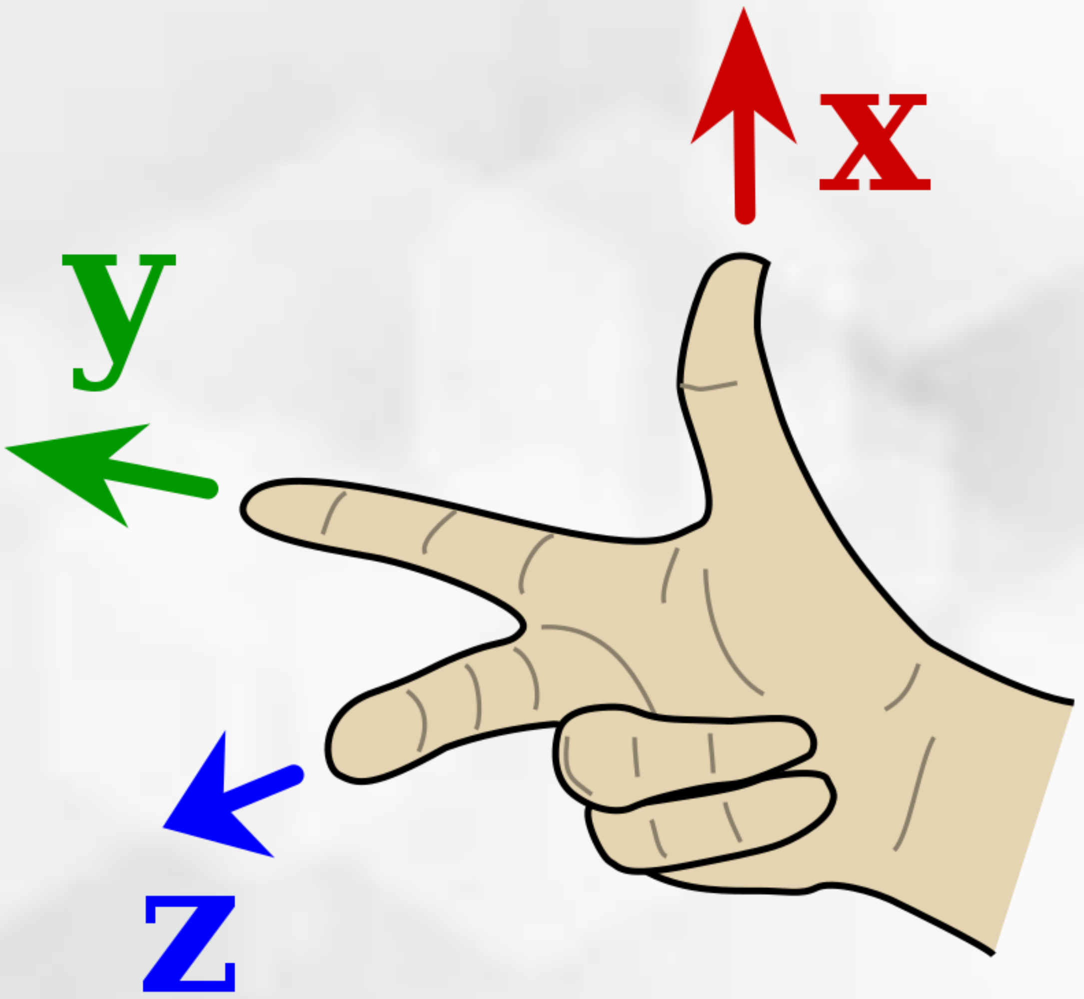
$$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$$

$$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$$

$$\vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = 0$$

$$\vec{i} \wedge \vec{j} = \vec{k};\ \vec{j} \wedge \vec{k} = \vec{i};\ \vec{k} \wedge \vec{i} = \vec{j}$$

This is an orthonormal system



I. Useful Maths

Let's define two vectors $\vec{A}$ and $\vec{B}$ in cartesian referential (O, x, y, z):

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$$

Both, obey the following rules:

- ***Constant multiplication:***

$$\alpha.\vec{A} = \alpha A_x\vec{i} + \alpha A_y\vec{j} + \alpha A_z\vec{k}$$

- ***Sum and difference:***

$$\vec{A} \pm \vec{B} = (A_x \pm B_x)\vec{i} + (A_y \pm B_y)\vec{j} + (A_z \pm B_z)\vec{k}$$

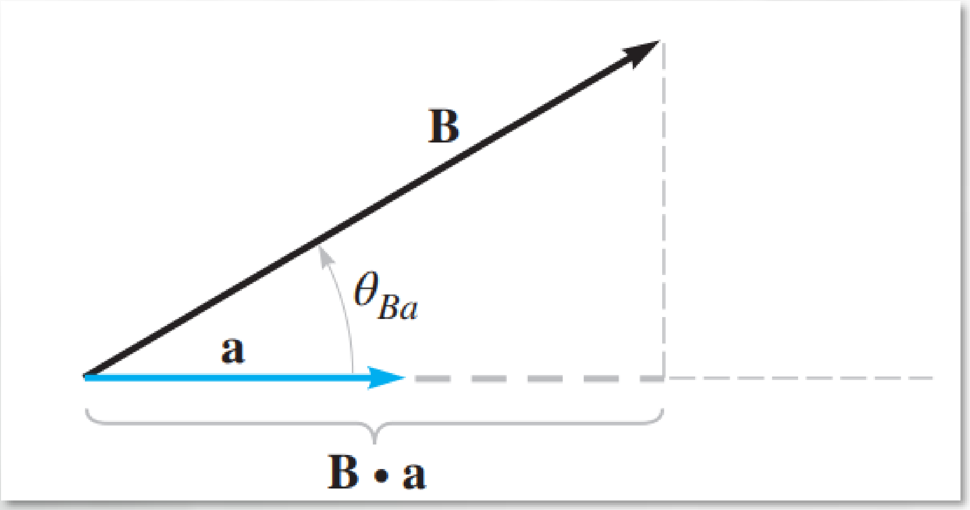
- ***Dot product (Scalar product):***

$$\vec{A}.\vec{B} = S = A_x.B_x + A_y.B_y + A_z.B_z$$

$$\vec{A}.\vec{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

$$A = \|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2} \in \mathbb{R}^+$$

$$\vec{A}.\vec{B} = A.B\cos(\widehat{AB})$$

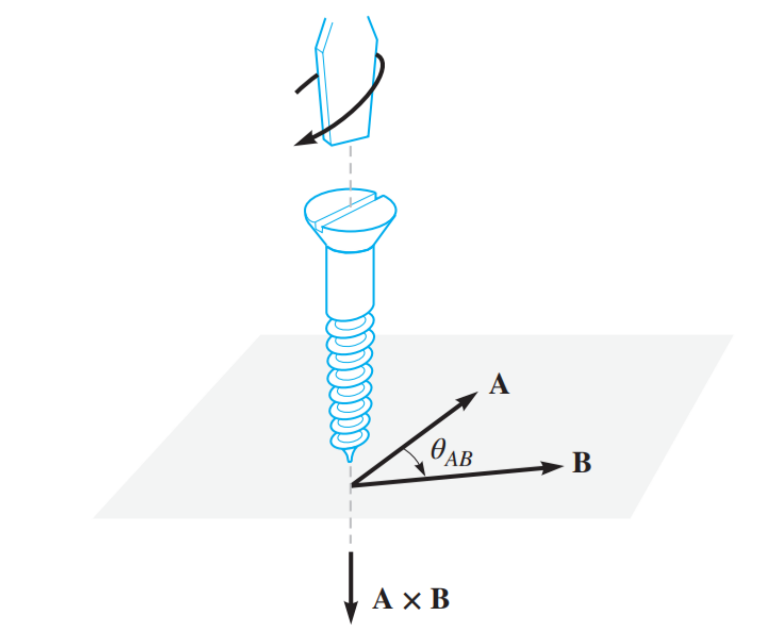


I. Useful Maths

- Cross product

$$\vec{A} \wedge \vec{B} = \vec{A} \otimes \vec{B} = \vec{C} = \text{A. B} \sin(\widehat{AB})\vec{u}_C$$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



It is important to note that:

- ***Dot product is commutative***

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

- ***Dot product is associative***

$$\vec{A}.\left(\vec{B} \pm \vec{C}\right) = \vec{A}.\vec{B} \pm \vec{A}.\vec{C}$$

- ***Cross product is anti-commutative***

$$\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$$

- ***Cross product is associative***

$$\vec{A} \wedge \left(\vec{B} \pm \vec{C}\right) = \vec{A} \wedge \vec{B} \pm \vec{A} \wedge \vec{C}$$

Following rules are applied too:

- $\vec{A}.\left(\vec{B} \wedge \vec{C}\right) = \vec{B}.\left(\vec{C} \wedge \vec{A}\right) = \vec{C}.\left(\vec{A} \wedge \vec{B}\right)$

- $\vec{A} \wedge \left(\vec{B} \wedge \vec{C}\right) = \left(\vec{A}.\vec{C}\right)\vec{B} - \left(\vec{A}.\vec{B}\right)\vec{C}$

I. Useful Maths

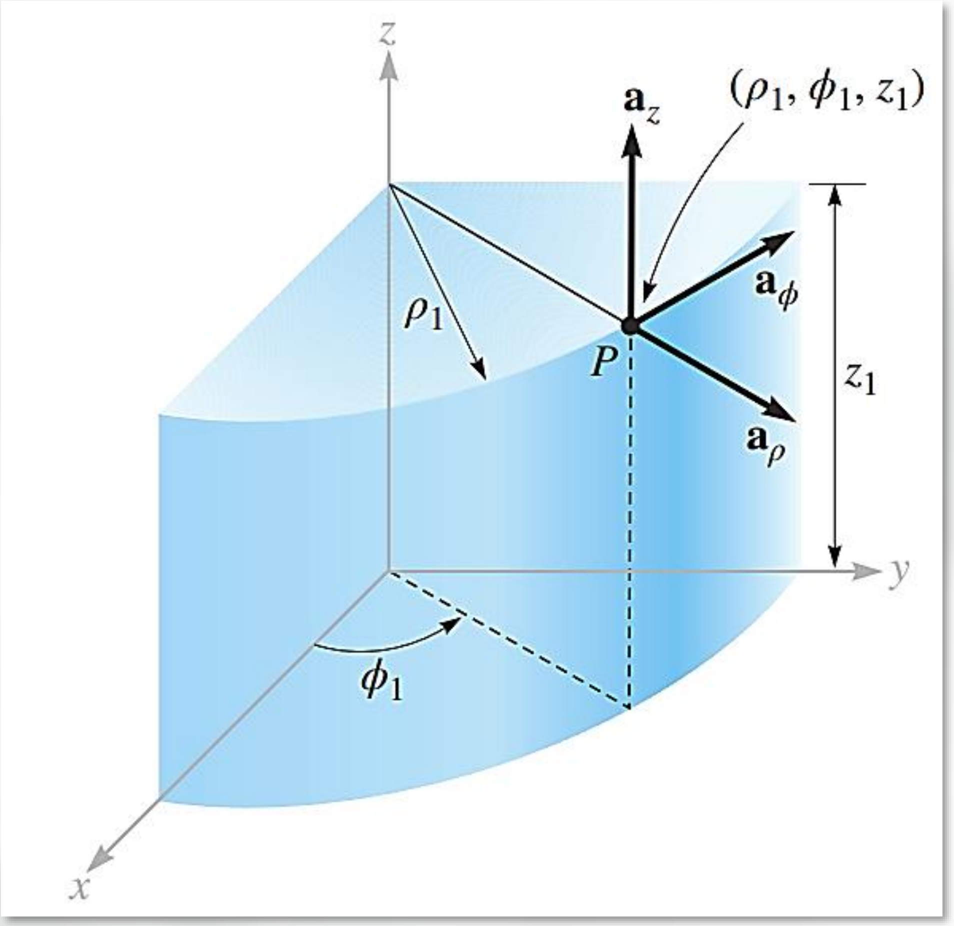
Other systems of coordinates are also useful to express vectors:

- ***Cylindrical coordinates (3D):***

$$(x, y, z) \rightarrow (\rho, \varphi, z): \begin{cases} x = \rho.cos\varphi \\ y = \rho.sin\varphi \\ z \end{cases} \leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ tan\varphi = y/x \end{cases}$$

$$\vec{A} = A_\rho \vec{u}_\rho + A_\varphi \vec{u}_\varphi + A_z \vec{k}$$

With the interval of the azimutal angle: $\varphi \in [0, 2\pi]$



Cylindrical coordinates

I. Useful Maths

Other systems of coordinates are also useful to express vectors:

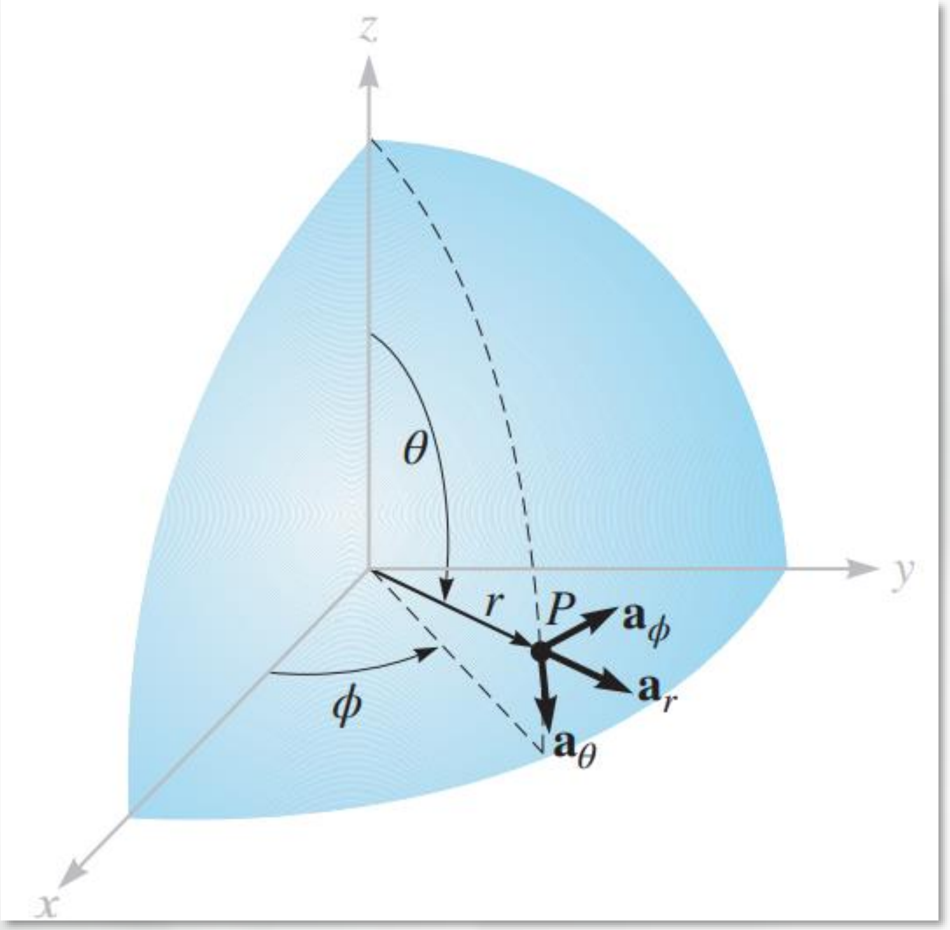
- ***Spherical coordinates (3D):***

$$(x, y, z) \rightarrow (r, \varphi, \theta): \begin{cases} x = r.\sin\theta.\cos\varphi \\ y = r.\sin\theta.\sin\varphi \\ z = r.\cos\theta \end{cases} \leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan\varphi = y/x \\ \cos\theta = z/r \end{cases}$$

$$\vec{A} = A_r\vec{u}_r + A_\varphi\vec{u}_\varphi + A_\theta\vec{u}_\theta$$

With the interval of the azimutal angle: $\varphi \in [0, 2\pi]$

And the interval of polar angle : $\theta \in [0, \pi]$



Spherical coordinates

I. Useful Maths

Conversion rules betwwen different systems of coordinates

		From		
		Cartesian	**Cylindrical**	**Spherical**
To	**Cartesian**	$x = x$ $y = y$ $z = z$	$x = \rho \cos\varphi$ $y = \rho \sin\varphi$ $z = z$	$x = r \sin\theta \cos\varphi$ $y = r \sin\theta \sin\varphi$ $z = r \cos\theta$
	Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$	$\rho = \rho$ $\varphi = \varphi$ $z = z$	$\rho = r \sin\theta$ $\varphi = \varphi$ $z = r \cos\theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	$r = r$ $\theta = \theta$ $\varphi = \varphi$

I. Useful Maths

❑ Line element:

- *Cartesian coordinates:*

$$d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- *Cylindrical coordinates:*

$$d\vec{l} = d\rho\vec{u}_\rho + \rho d\varphi\vec{u}_\varphi + dz\vec{k}$$

- *Spherical coordinates:*

$$d\vec{l} = dr\vec{u}_r + r.\sin\theta d\varphi\vec{u}_\varphi + rd\theta\vec{u}_\varphi$$

❑ Surface element:

- *Cartesian coordinates:*

$$OZ:\ dS = dx.dy$$

$$OY:\ dS = dx.dz$$

$$OX:\ dS = dy.dz$$

- *Cylindrical coordinates:*

$$radial:\ dS = \rho d\varphi dz$$

$$axial:\ dS = \rho d\varphi d\rho$$

- *Spherical coordinates:*

$$radial: dS = r^2 \sin\theta d\varphi d\theta$$

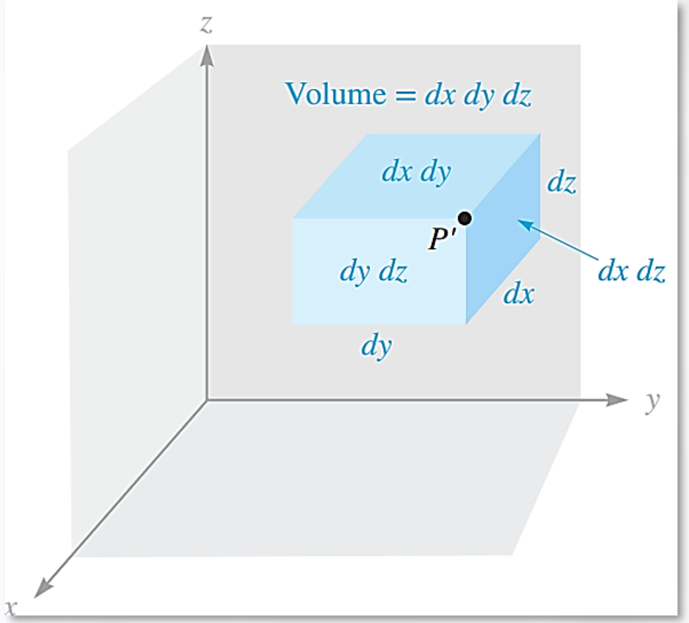
I. Useful Maths

Elementary measures

❑ Volume element:

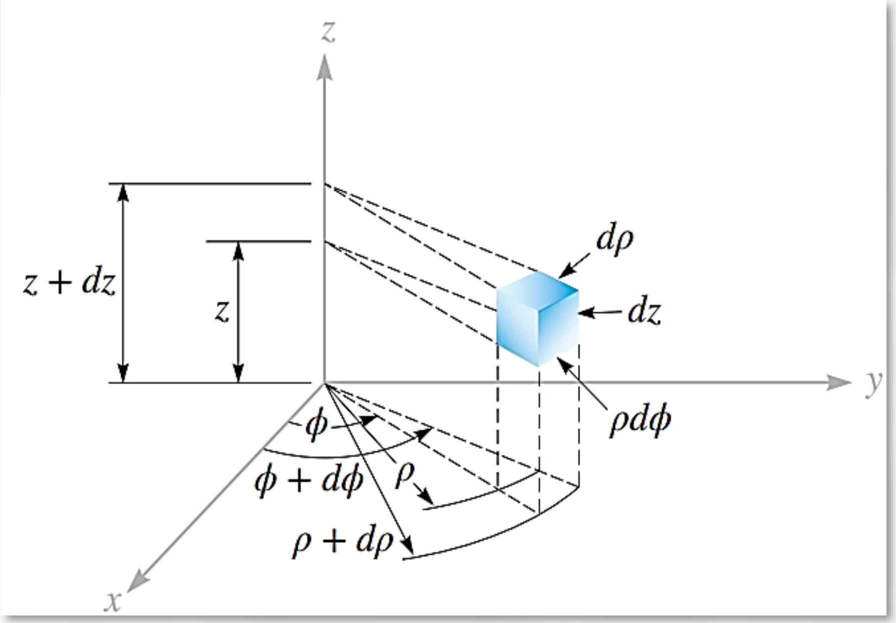
- ***Cartesian coordinates:***

$$dV = dx.dy.dz$$



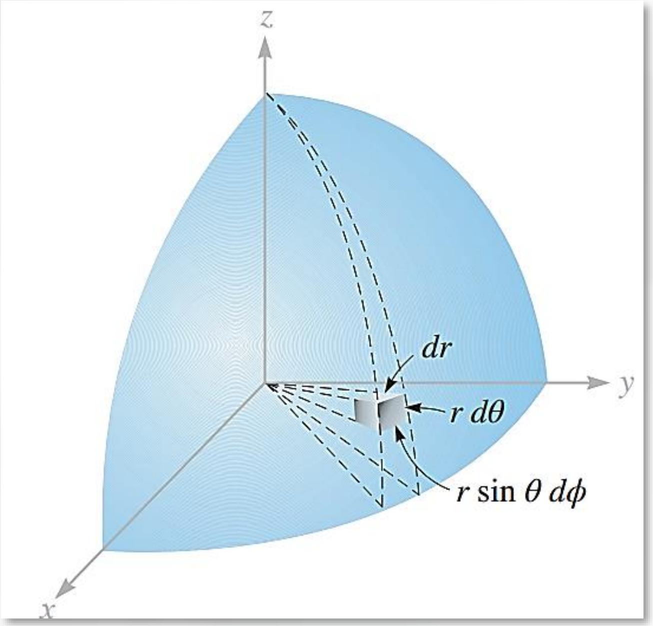
- ***Cylindrical coordinates:***

$$dV = \rho d\rho.d\varphi.dz$$



- ***Spherical coordinates:***

$$dV = r^2 sin\theta dr.d\varphi.d\theta$$



I. Useful Maths

For a given variable x, we recall that partial derivation noted:

$$\frac{\partial}{\partial x} = \partial_x = \left.\frac{d}{dx}\right|_{y=z=t=Cte}$$

We define the vector operator Nabla:

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

In such case, when applied on a given vector $\vec{A}$, we obtain the divergent of $\vec{A}$ (Scalar):

$$\vec{\nabla}.\vec{A} = div.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Similarly, the curl of $\vec{A}$ (Vector) is given by:

$$\vec{\nabla} \wedge \vec{A} = rot\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

When Nabla operator is applied on scalar function $f(x, y, z)$ it gives the gradient of f:

$$\vec{\nabla}f = \overrightarrow{grad}f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

The Nabla operator could be applied twice on the same operand (scalar or vector function):

- *The scalar function*

$$(\vec{\nabla}.\vec{\nabla})f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This squared operator is called Laplacian.

I. Useful Maths

- *The vector function:*

$$\nabla^2\vec{A} = \nabla^2 A_x\vec{i} + \nabla^2 A_y\vec{j} + \nabla^2 A_z\vec{k}$$

Also, given as follows:

$$\nabla^2\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A})$$

Besides that, it is possible to demonstrate that alternate application of Nabla with dot and cross product on scalar of vector function will give always null result:

- $\vec{\nabla}.(\vec{\nabla}\wedge\vec{A}) = 0, \forall\vec{A}$
- $\vec{\nabla}\wedge(\vec{\nabla}f) = 0, \forall f$

When applied with dot or cross product, Nabla operator obeys some arithmetic rules:

- $\vec{\nabla}.(\vec{A}\pm\vec{B}) = \vec{\nabla}.\vec{A}\pm\vec{\nabla}.\vec{B}$
- $\vec{\nabla}\wedge(\vec{A}\pm\vec{B}) = \vec{\nabla}\wedge\vec{A}\pm\vec{\nabla}\wedge\vec{B}$
- $\vec{\nabla}(f\pm g) = \vec{\nabla}f\pm\vec{\nabla}g$
- $\vec{\nabla}(f\vec{A}) = \vec{A}.(\vec{\nabla}f) + f(\vec{\nabla}.\vec{A})$
- $\vec{\nabla}\wedge(f\vec{A}) = (\vec{\nabla}f)\wedge\vec{A} + f(\vec{\nabla}\wedge\vec{A})$
- $\vec{\nabla}(fg) = g(\vec{\nabla}f) + f(\vec{\nabla}g)$
- $\vec{\nabla}.(\vec{A}\wedge\vec{B}) = \vec{B}.\vec{\nabla}\wedge\vec{A} - \vec{A}.\vec{\nabla}\wedge\vec{B}$
- $\vec{\nabla}.(\vec{A}.\vec{B}) = (\vec{A}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A} + \vec{A}\wedge(\vec{\nabla}\wedge\vec{B}) + \vec{B}\wedge(\vec{\nabla}\wedge\vec{A})$
- $\vec{\nabla}\wedge(\vec{A}\wedge\vec{B}) = \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A}) + (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B}$

I. Useful Maths

The Nabla operator could be written in other systems of coordinates:

- ***Cylindrical coordinates:***

$$\vec{\nabla} = \vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \vec{k}\frac{\partial}{\partial z}$$

When applied on a scalar function $f(\rho, \varphi, z)$:

- *Gradient of a scalar:*

$$\vec{\nabla} f = \vec{u}_\rho \frac{\partial f}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \vec{k}\frac{\partial f}{\partial z}$$

When applied on a vector function (field):

$$\vec{A}(\rho, \varphi, z) = A_\rho \vec{u}_\rho + A_\varphi \vec{u}_\varphi + A_z \vec{k}$$

- *Dot product (Divergent of $\vec{A}$):*

$$\vec{\nabla}.\vec{A} = \left(\vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \vec{k}\frac{\partial}{\partial z}\right) A_r \vec{u}_\rho + A_\varphi \vec{u}_\varphi + A_z \vec{k}$$

And here we should take in consideration that:

$$\frac{\partial}{\partial \varphi}\vec{u}_\rho = \vec{u}_\varphi; \frac{\partial}{\partial \varphi}\vec{u}_\varphi = -\vec{u}_\rho; \frac{\partial}{\partial \varphi}\vec{k} \quad = 0$$

$$\partial_\rho \vec{u}_{\rho,\varphi,z} = 0; \partial_z \vec{u}_{\rho,\varphi,z} = 0$$

We obtain the divergent of $\vec{A}$:

$$\vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

I. Useful Maths

- ***Cross product (Curl of $\vec{A}$):***

$$\vec{\nabla} \curlywedge \vec{A} = \left(\vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \vec{k}\frac{\partial}{\partial z}\right) \curlywedge \vec{A}$$

In similar way, one can demonstrate that:

$$\vec{\nabla} \curlywedge \vec{A} = \vec{u}_\rho \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)$$

$$+\vec{u}_\varphi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \vec{k}\left(\frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right)$$

- ***Laplacian:***

$$\nabla^2 = \left(\vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \vec{k}\frac{\partial}{\partial z}\right).\left(\vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \vec{k}\frac{\partial}{\partial z}\right)$$

It gives in the cylindrical coordinates:

$$\nabla^2 = \Delta = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

- ***Spherical coordinates:***

$$\vec{\nabla} = \vec{u}_r \frac{\partial}{\partial r} + \vec{u}_\varphi \frac{1}{r.\sin\theta}\frac{\partial}{\partial \varphi} + \vec{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$$

When applied on a scalar function $f(\rho, \varphi, \theta)$:

- ***Gradient of a scalar:***

$$\vec{\nabla} f = \vec{u}_r \frac{\partial f}{\partial r} + \vec{u}_\varphi \frac{1}{r.\sin\theta}\frac{\partial f}{\partial \varphi} + \vec{u}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta}$$

I. Useful Maths

When applied on a vector function (field):

$$\vec{A}(\rho,\varphi,z)=A_r\vec{u}_r+A_\varphi\vec{u}_\varphi+A_\theta\vec{u}_\theta$$

- ***Dot product (divergent of $\vec{A}$):***

$$\vec{\nabla}.\vec{A}=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r.sin\theta}\frac{\partial A_\varphi}{\partial\varphi}+\frac{1}{r.sin\theta}\frac{\partial(A_\theta sin\theta)}{\partial\theta}$$

- ***Cross product (Curl of $\vec{A}$):***

$$\vec{\nabla}\wedge\vec{A}=\vec{u}_r\frac{1}{r.sin\theta}\left(\frac{\partial(A_\varphi sin\theta)}{\partial\theta}-\frac{\partial A_\theta}{\partial\varphi}\right)$$

$$+\vec{u}_\varphi\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)+\vec{u}_\theta\frac{1}{r}\left(\frac{1}{sin\theta}\frac{\partial A_r}{\partial\varphi}-\frac{\partial(rA_\varphi)}{\partial r}\right)$$

- ***Laplacian:***

$$\nabla^2=\left(\vec{u}_r\frac{\partial}{\partial r}+\vec{u}_\varphi\frac{1}{r.sin\theta}\frac{\partial}{\partial\varphi}+\vec{u}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right).\left(\vec{u}_r\frac{\partial}{\partial r}+\vec{u}_\varphi\frac{1}{r.sin\theta}\frac{\partial}{\partial\varphi}+\vec{u}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

It gives in the cylindrical coordinates:

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2.sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2.sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$


The End.