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Level : 1^{st} year LMD
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Algebra 1

Homework

Exercise 1 : Let $G=\mathbb{R}$. We define on G the internal composition law denoted by $*$ as follows :

$$\forall a, b \in G,\ a * b = a + b + \frac{1}{6}$$

1. Prove that $*$ is commutative.
2. Prove that $(G, *)$ is a group.
3. Let f the map defined by :
$$\begin{aligned} f:(G,*) &\longrightarrow (\mathbb{R},+) \\ x &\longmapsto 3x+\tfrac{1}{2} \end{aligned}$$
   Demonstrate that f is an isomorphism of groups.

Exercise 2 : I) Let $n \geq 2$, we consider :

$$U_n = \{z \in \mathbb{Z}^*; z^n = 1\}$$

Prove that U_n is a sub-group of $(\mathbb{C}^*, \times)$.

Let $\varphi_n : U_n \longrightarrow U_n$ the map defined by $\varphi_n(z) = z^2$.

(a) Prove that φ_n is an endomorphism of groups.

(b) Determine $ker\,(\varphi_3)$. φ_3 is-it injective ?

II) We set :

$$\mathbb{Z}[i] = \{a + ib\ /a, b \in \mathbb{Z}\},$$

where i is the complex number verifying $i^2 = -1$.

1. Prove that $(\mathbb{Z}[i], +, \times)$ is a sub-ring of $(\mathbb{C}, +, \times)$.
2. For $z = a + ib \in \mathbb{Z}[i]$, we set $N(z) = a^2 + b^2$.

   (a) Check that : $N(z) = z\bar{z}$, where $\bar{z} = a - ib$.

   (b) Prove that : $\forall z, z' \in \mathbb{Z}[i], \quad N(zz') = N(z)N(z')$.

   (c) Prove that : $\forall z \in \mathbb{Z}[i]$, z invertible $\Leftrightarrow N(z) = 1$.

   (d) Deduce all the invertible elements of $\mathbb{Z}[i]$.