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Level : 1^{st} year LMD
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Algebra 1

Series 3 : Binary relations

Exercise 1 : Let R be a relation defined on $\mathbb{R}$ by :

$$xRy \iff x^2 - y^2 = x - y$$

1) Prove that R is an equivalence relation.
2) Determine the equivalence classe $[x]$ for all $x \in \mathbb{R}$.
3) Determine the quotient set $\mathbb{R}/R$.

Exercise 2 : Let R be a relation defined on $\mathbb{Z} \times \mathbb{N}^*$ by :

$$(a,b)R(a',b') \iff ab' = a'b$$

1) Prove that R is an equivalence relation.
2) Let $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$ with $p \wedge q = 1$. Write its equivalence classe $[(p,q)]$.

Exercise 3 : Let $<$ be a relation defined on $\mathbb{N}^2$ by :

$$(a,b) < (a',b') \iff \begin{cases} a < a' \\ or \\ a = a' \text{ and } b \leq b' \end{cases}$$

Prove that $<$ is an order relation. Is it total or partial ?

Exercise 4 : On $\mathbb{N}^*$, we define a relation $\ll$ by assuming that for all $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$:

$$k \ll l \iff \text{There exists } n \in \mathbb{N}^* \text{ such that } l = k^n$$

1) Prove that $\ll$ is a partial order relation.
2) We consider in the rest of the exercise that $\mathbb{N}^*$ is ordered by the relation $\ll$. Let $A = \{2, 4, 16\}$, determine the greatest element and the smallest element of A.

Exercise 5 : Let E and F two sets and $f : E \longrightarrow F$ a map. we define a relation R on E by assuming that for all $(x,x') \in E^2$:

$$xRx' \iff f(x) = f(x')$$

1) Prove that R is an equivalence relation.
2) Determine the equivalence classe $[x]$ for all $x \in E$.
3) Why the map :

$$\begin{array}{ccc} E/R & \longrightarrow & F \\ [x] & \longmapsto & f(x) \end{array}$$

is well defined ? Show that it is injective.