Series 2 : Sets and Maps

Exercise 1 : Assuming the set $A=\{w,x,y,z\}$, $B=\{x,y\}$, $C=\{x,y,z\}$ and $D=\{x,z\}$ three parts of A. Identify the elements in each set : $B^c, C^c, B\backslash C, B\backslash D, B\cap C, B\cap D, B\cap(C\cup D), (B\cap C)\cup D, B\backslash D, D\backslash B, B\times C, C\times B, B\times D, P(B)$ and $P(C)$.

Exercise 2 : Let A, B and C be three parts of a set E. Prove that :

1) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
2) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Exercise 3 : Let E be a set and A and B two parts of E. Assume that :

$$A\cap B\neq\emptyset,\ A\cup B\neq E,\ A\not\subset B \text{ and } B\not\subset A.$$

Suppose that : $A_1=A\cap B$, $A_2=A\cap B^c$, $A_3=B\cap A^c$, $A_4=(A\cup B)^c$.

1) Prove that A_1, A_2, A_3 and A_4 are not emply.
2) Prove that A_1, A_2, A_3 and A_4 are two by two disjoint.
3) Prove that $A_1\cup A_2\cup A_3\cup A_4=E$.

Exercise 4 : Let A, B and C be three parts of a set E.

1) What do you think about the implication : $(A\cup B\not\subset C)\Longrightarrow(A\not\subset C$ or $B\not\subset C)$?
2) Suppose that we have $A\cup B\subset A\cup C$ and $A\cap B\subset A\cap C$. Prove that $B\subset C$.

Exercise 5 : Let E a set and A and B two parts of E. Demonstrate that :

1) $F\subset G\Longleftrightarrow F\cup G=G$.
2) $F\subset G\Longleftrightarrow F\cap G^c=\emptyset$.

Exercise 6 : Let $f: I\longrightarrow J$ the function defined by $f(x)=x^2$.

1) Give sets I and J such that f will be injective but not surjective.
2) Give sets I and J such that f will be surjective but not injective.
3) Give sets I and J such that f will be neither injective nor surjective.
4) Give sets I and J such that f will be injective and surjective.

Exercise 7 : We consider the map $f:\mathbb{N}\longrightarrow\mathbb{N}$ defined by : for all $n\in\mathbb{N}$, $f(n)=n^2$.

1) Is it exist a map $g:\mathbb{N}\longrightarrow\mathbb{N}$ such that $f\circ g=Id_{\mathbb{N}}$?
2) Is it exist a map $h:\mathbb{N}\longrightarrow\mathbb{N}$ such that $h\circ f=Id_{\mathbb{N}}$?

Exercise 8 : Let E and F two sets and a map $f: E\longrightarrow F$. Let A and B two parts of E. Demonstrate that :

1) $f(A\cup B)=f(A)\cup f(B)$.
2) $f(A\cap B)\subset f(A)\cap f(B)$.

Give an example for the second property. Then prove that f is injective iff for any parts A and B of E, we have $f(A\cap B)=f(A)\cap f(B)$.

Exercise 9 : 1) Let f the map of $\{1,2,3,4\}$ in it self defined by : $f(1)=4$, $f(2)=1$, $f(3)=2$ and $f(4)=2$.
Determine $f^{-1}(A)$ when $A=\{2\}$, $A=\{1,2\}$ and $A=\{3\}$.

2) Let f the map of $\mathbb{R}$ in $\mathbb{R}$ defined by : $f(x)=x^2$. Determine $f^{-1}(A)$ when $A=\{1\}$ and $A=[1,2]$.

Exercise 10 : 1) Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by : $f(x,y)=x$. Determine $f([0,1]\times[0,1])$ and $f^{-1}([-1,1])$.

2) Let $g:\mathbb{R}\longrightarrow[-1,1]$ defined by : $g(x)=\cos(\pi x)$. Determine $g(\mathbb{N})$, $g(2\mathbb{N})$ and $g^{-1}(\{-1,1\})$.

Exercise 11 : Let E and F two sets and a map $f: E\longrightarrow F$. Let C and D two parts not emply of F. Demonstrate that :

1) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Exercise 12 : Let E and F two sets and a map $f : E \longrightarrow F$.
1) Prove that for any part A of E we have : $A \subset f^{-1}(f(A))$.
2) Prove that for any parts B of E we have : $f(f^{-1}(B)) \subset B$.
3) Prove that f is injective iff for any part A of E we have : $A = f^{-1}(f(A))$.
4) Prove that f is surjective iff for any part B of E we have : $f(f^{-1}(B)) = B$.

Exercise 13 : 1) Let $q_1 \in \mathbb{N}_{-\{0,1\}}$ and $q_2 \in \mathbb{N}_{-\{0,1\}}$. Prove that :

$$-\frac{1}{2} < \frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{2}$$

2) Let $f : \mathbb{Z} \times \mathbb{N}_{-\{0,1\}} \longrightarrow \mathbb{Q}$ the map defined by : $f(p, q) = p + \frac{1}{q}$.
a) Prove that f is injective.
b) Is f surjective ?