Unit 1 - Chapter 1
Mathematical logic

Chapter Structure

1.0 Objectives

Logic rules are used to give definite meaning to any mathematical statements.
Many algorithms and proofs use logical expressions such as:

> "IF p THEN q"
> or
> "If p1 AND p2, THEN q1 OR q2"

Logic rules are used to distinguish between valid and invalid mathematical arguments, i.e. to know the cases in which these expressions are TRUE or FALSE, that is, to know the "truth value" of such expressions.
We discuss the study of discrete mathematics with an introduction to logic in this chapter.
Logic has numerous applications to the computer science field such as, design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways.

1.1 Propositions and logical operations

Lets n begins with an introduction to the basic building blocks of logic—propositions.

1.1.1 Proposition

A proposition (or statement) is a declarative statement which is true or false, but not both. Let’s understand this by taking a few examples.

Consider following statements to find out whether they are propositions are not.

1)	Is it cold outside?	This is not a preposition, as it’s answer can have both the value true and false. So If a statement has a question mark it is not considered as preposition.
2)	Sun is bright.	This is a preposition , as this statement can either be true or false but not both.
3)	2 + 2 = 5	This is a preposition , as this statement can either be true or false but not both.
4)	Read this carefully.	This is not a preposition, as this is not a declarative sentence. It is a command.
5)	5 * X = 25	This is not a preposition, as its validity is dependent on the value of variable X . Hence it’s answer can have both the value true and false.

- We use letters to denote propositional variables (or statement variables).
- Variables that represent propositions are just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s, ...
- The truth value of a proposition is **true**, denoted by **T**, if it is a true proposition, and
- The truth value of a proposition is **false**, denoted by **F**, if it is a false proposition.
- Propositional logic was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago. It deals with propositional calculus.

Exercise:

Are these propositions?Give proper reasoning.

1. The sun is shining.
2. The sum of two prime numbers is even.
3. 3+4=7
4. It rained in Austin, TX, on October 30, 1999.

5. x+y > 10
6. Is it raining?
7. Come to class!
8. n is a prime number.
9. The moon is made of green cheese.

<u>Compound Statements or Compound Propositions:</u>

- Generally mathematical statements are constructed by combining one or more propositions.
- New propositions, called compound propositions, are formed from existing propositions using logical operators.

The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions.

Many propositions are composite, that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound statements or compound propositions.
A proposition is said to be primitive if it cannot be broken down into simpler propositions, that is, if it is not composite.
For example, the below propositions
(i) Ice floats in water.
(ii) 2+2=6
are primitive propositions.

On the other hand, the following two propositions are composite:
(i) “Roses are red and violets are blue.” and
(ii) “John is smart or he studies every night.”

Let P(p, q, . . .) denote an expression constructed from logical variables p, q, . . ., which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and ~ (and others discussed subsequently). Such an expression P(p, q, . . .) will be called a proposition. The letters p,q,r…. denotes propositional variables.
We can write the above compound statements as;
(i) p: Roses are red
q: violets are blue
thus, Compound statement is “p AND q”

(ii) p: John is smart

q: he studies every night
thus, Compound statement is "p OR q"

1.1.2 Basic Logical Operations

In this section we are discussing the three basic logical operations as follow:

1. Conjunction (AND) , symbolically '∧'
2. Disjunction (OR), symbolically '∨'
3. Negation (NOT), symbolically '¬' or '~'

Conjunction (AND) , symbolically '∧'

Definition :

Let p and q be propositions. The conjunction of p and q, denoted by **p ∧ q**, is the proposition read as "p and q." The conjunction p ∧ q is true when both p and q are true and is false otherwise.

Truth Table of p ∧ q:

Since p ∧ q is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Note that, *p ∧ q is true only when both p and q are true.*

Observe the following truth table for conjunction operation:

p	q	p ∧ q (p AND q)
T	T	T
T	F	F
F	T	F
F	F	F

Table: Conjunction / AND logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Mumbai is the capital of India and 1 + 1 = 2."

Solution:
In this example let's assume,

p: Mumbai is capital of India
q: 1 + 1 = 2

Where we can get proposition p is false and q is true.
As per our truth table, it will give us "p AND q" is false, shown in below table:

p	q	p ∧ q (p AND q)
F	T	F

Disjunction (OR), symbolically 'v '

Definition:

Let p and q be propositions. The disjunction of p and q, denoted by p ∨ q, is the proposition "p or q." The disjunction p ∨ q is false when both p and q are false and is true otherwise.

Truth Table of p ∨ q:

Since p ∨ q is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Note that, *p ∨ q is false only when both p and q are false.*
Observe the following truth table for disjunction operation:

p	q	p ∨ q (p OR q)
T	T	T
T	F	T

F	**T**	**T**
F	**F**	**F**

Table: Disjunction / OR logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Mumbai is the capital of India and 1 + 1 = 2."

<u>Solution</u>:
In this example let's assume,

p: Mumbai is capital of India
q: 1 + 1 = 2

Where we can get proposition p is false and q is true.
As per our truth table, it will give us "p OR q" is false, shown in below table:

p	q	p ∨ q (p OR q)
F	**T**	**T**

<u>Negation (NOT), symbolically '¬' or '~'</u>

Definition :

Let p be a proposition.The negation of p, denoted by ¬p (also denoted by ~p or $\underline{p}$), is the statement "It is not the case that p." The proposition ¬p is read "not p."
The truth value of the negation of p, ¬p, is the opposite of the truth value of p.

Truth Table of ¬p:

The truth value of the negation of p is always the opposite of the truth value of p. It is a unary operator means it requires only one operand(proposition) to perform this operation.
Observe the following truth table for negation operation:

p	¬p (NOT p)
T	**F**
F	**T**

Table: Negation/NOT logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Delhi is the capital of India."

Solution:
In this example let's assume,

p: Delhi is the capital of India

Then the negation of given proposition p is shown in below table:
It can be read as,
¬p : It is not the case that Delhi is the capital of India. OR
¬p : Delhi is **not** the capital of India. OR
¬p : It is not true that Delhi is the capital of India. OR
¬p : It is false that Delhi is the capital of India.

p	¬p (NOT p)
T	F

Tautologies And Contradictions :

Definition:
A proposition P (p, q, . . .) is called a **tautology** if it contains only T in the last column of its truth table or, in other words, if it is True for any truth values of its variables.
Example:

p	~p	p v ~p
T	F	T
F	T	T

Definition:
A proposition P (p, q, . . .) is called a **contradiction** if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables

Example:

p	~p	p ^

		~p
T	**F**	**F**
F	**T**	**F**

Exercise:

1. Let p be “It is hot” and let q be “It is raining”. Give a simple verbal sentence which describes each of the following statements:

   (a) ~p (b) p ∧ q (c) p ∨ q (d) q ∨~p.

2. Find the truth table of ~p ∧ q.
3. Verify that the proposition p ∨~(p ∧ q) is a tautology.
4. Find the truth tables for.

   (a) p ∨~q (b) ~p ∧~q.

5. Verify that the proposition (p ∧ q)∧~(p ∨ q) is a contradiction

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1.2 Conditional Statements

Many mathematical statements are of the form “If p then q.” Such statements are called conditional statements and are denoted by p → q

There are some other common ways to express this conditional statement:

“if p, then q”

“p implies q”

“if p, q”

“p only if q”

“p is sufficient for q”

“a sufficient condition for q is p”

“q if p”

“q whenever p”

“q when p”

“q is necessary for p”

“a necessary condition for p is q”

“q follows from p”

“q unless ¬p

Definition:

Let p and q be propositions. The conditional statement p → q is the proposition “if p, then q.” The conditional statement p → q is false when p is true and q is false, and true otherwise.

In the conditional statement p → q, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Truth Table:

Note that the conditional p → q is false only when the first part p is true and the second part q is false.Accordingly, when p is false, the conditional p → q is true regardless of the truth value of q.
A conditional statement is also called **an implication**.
Observe the following truth table for conditional operation:

p	**q**	**p → q (if p, then q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Table: conditional operation

Example:
From the implication **p → q** for each of the following:
(i) p:I am girl q: I will dance
can be written as , "If I am a girl, then I will dance".

(ii) p:It is summer q: 2+3= 5
written as , "If it is summer , then 2+ 3=5."

Biconditionals :

Another common statement is of the form "p if and only if q." Such statements are called biconditional statements and are denoted by p ↔ q.
Biconditional statements are also called **bi-implications**.
There are some other common ways to express p ↔ q:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

Definition:

Let p and q be propositions. The biconditional statement p ↔ q is the proposition
"p if and only if q."

The biconditional statement p ↔ q is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

Truth Table:

Note that the biconditional p ↔ q is true whenever p and q have the same truth values and false otherwise.

Observe the following truth table for conditional operation:

p	**q**	**p ↔ q (p if and only if q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Table: conditional operation

Example:
From the bi-implication p ↔ q for each of the following:
p: A polygon is a triangle.
q: A polygon has exactly 3 sides.
p ↔ q : "A polygon is a triangle if and only if it has exactly 3 sides."

1.3 Methods of Proof

Method of mathematical proof is an argument we give logically to validate a mathematical statement. In order to validate a statement, we consider two things: A **statement** and **Logical operators**. A statement is either true or false but not both. Logical operators are AND, OR, NOT, If then, and If and only if. We also use quantifiers like for all(∀) and there exists(∃). We apply operators on the statement to check the correctness of it.

Let's discuss this in more detail:methods you can use to write proofs.

Methods of proof are as follow:

1.3.1 Direct Proof

Any problem solving method has both discovery and proof as integral parts. When you think you have discovered that a certain statement is true, try to figure out why it is true. If you succeed, you will know that your assumption is correct. Even if you fail, the process of trying will give you insight into the nature of the problem and may lead to the discovery that the the statement is false.

Few assumptions which we are going to consider in this section. In this text we assume a familiarity with the laws of basic algebra.

We also use the three properties of equality: For all objects A, B, and C,

- ➔ (1) A = A,
  (2) if A = B then B = A, and
  (3) if A = B and B = C, then A = C.
- ➔ In addition, we assume that there is no integer between 0 and 1 and that the set of all integers is closed under addition, subtraction, and multiplication. This means that sums, differences, and products of integers are integers.
- ➔ Of course, most quotients of integers are not integers. For example, 3 ÷ 2, which equals 3/2, is not an integer, and 3 ÷ 0 is not even a number.

Steps: Method of Direct Proof

1. Express the statement to be proved in the form "∀x ∈ D, if P(x) then Q(x)."
2. Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis P(x) is true. (This step is often abbreviated "Suppose x ∈ D and P(x).")
3. Show that the conclusion Q(x) is true by using definitions, previously established results, and the rules for logical inference.

A direct proof of theorem:

Theorem: The sum of any two even integers is even.
Proof: Suppose m and n are particular but arbitrarily chosen even integers. **We must show that m + n is even.** By definition of even, we can say that:

m = 2*r and
n = 2*s

for some integers r and s.

Then

m + n = 2r + 2s by substitution
= 2(r + s) by factoring out a 2.

Let t = r + s. (Note that t is an integer because it is a sum of integers.) Hence

m + n = 2t where t is an integer.

It follows by **definition of even** that m + n is even. Hence proved.

Theorem:
The sum of any two odd integers is even.

Proof:
Suppose m and n are particular but arbitrarily chosen odd integers.

We must show that m + n is even.

By definition of odd, we can say that:

m = 2*r + 1 and
n = 2*s + 1

for some integers r and s.

Then

m + n = (2r +1) + (2s + 1) by substitution
= 2r + 2s + 2 by associative and commutative laws of addition
= 2(r + s + 1) by factoring out a 2.

Let t = r + s + 1. (Note that t is an integer because it is a sum of integers.) Hence

m + n = 2t where t is an integer.

It follows by **definition of even** that m + n is even. Hence proved.

Exercise:

Prove the following statements using direct method of proof:

1) For all integers m and n, if m is odd and n is even, then m + n is odd.
2) The product of two odd numbers is odd.

1.3.1.1 Counterexample

A counterexample is an example that disproves a universal (“for all ∀ ”) statement. Obtaining counterexamples is a very important part of mathematics. As we do require critical attitude toward claims.If you find a counterexample which shows that the given logic is false, that’s good because progress comes not only through doing the right thing, but also by correcting your mistakes.

Suppose you have a quantified statement: “All x’s satisfy property P”: $\forall x P(x)$.

The second quantified statement says: “There is an x which does not satisfy property P”.
In other words, to prove that “All x’s satisfy property P” is false, you must find an x which does not satisfy property P.

By basic logic, $P \rightarrow Q$ is 1 false when P is true and Q is false. Therefore: To give a counterexample to a conditional statement $P \rightarrow Q$, find a case where P is true but Q is false.

Give a counterexample to the statement :
If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

Proof:

To give a counterexample,

we have to find an integer n such n^2 is divisible by 4, but n is not divisible by 4

the “if” part must be true, but the “then” part must be false.

Consider n = 6.

Then $n^2 = 6^2$
$= 36$

36 is divisible by 4, but n = 6 is not divisible by 4.

Thus, n = 6 is a counterexample to the statement.

Show that the following statement is false:

There is a positive integer n such that n2 + 3n + 2 is prime.

Proof:

Proving that the given statement is false is equivalent to proving its negation is true. The negation is

For all positive integers n, $n^2 + 3n + 2$ is not prime.

Because the negation is universal, it is proved by generalizing from the generic particular.

Suppose n is any particular but arbitrarily chosen positive integer.

We will show that $n^2 + 3n + 2$ is not prime.

We can factor $n^2 + 3n + 2$ to obtain
$n^2 + 3n + 2 = (n + 1)(n + 2)$.

We also note that n + 1 and n + 2 are integers (because they are sums of integers) and that both $n + 1 > 1$ and $n + 2 > 1$ (because $n \geq 1$).
Thus $n^2 + 3n + 2$ is a product of two integers each greater than 1, and so $n^2 + 3n + 2$ is not prime.

Exercise:
Prove following using counterexample method of proof:

1) For all integers m and n, if 2m + n is odd then m and n are both odd.
2) For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

1.3.2 Indirect Proof

In a direct proof you start with the hypothesis of a statement and make one deduction after another until you reach the conclusion. Indirect proofs are more roundabout or winding.

1.3.2.1 Contradiction

One kind of indirect proof, argument by ***contradiction***, is based on the fact that either a statement is true or it is false but not both.

So if you can show that the assumption that a given statement is not true leads logically to a contradiction then that assumption must be false: and, hence, the given statement must be true.

The point of departure for a proof by contradiction is the supposition that the statement to be proved is false. The goal is to reason to a contradiction. Thus proof by contradiction has the following outline:

Steps: Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Theorem
There is no greatest integer.

Proof:

We take the negation of the theorem and suppose it to be true.
That is, suppose there is a greatest integer N.
We must deduce a contradiction.
Then $N \geq n$ for every integer n.

Let $M = N + 1$.

Now M is an integer since it is a sum of integers.

Also $M > N$ since $M = N + 1$.

Thus M is an integer that is greater than N.

So N is the greatest integer and N is not the greatest integer, which is a contradiction.
This contradiction shows that the supposition is false and, hence, that the theorem is true.

Theorem
There is no integer that is both even and odd.

Proof:

We take the negation of the theorem and suppose it to be true.

That is, suppose there is at least one integer n that is both even and odd.

We must deduce a contradiction.

By definition of even, n = 2a for some integer a, and by definition of odd, n = 2b + 1 for some integer b.

Consequently,

$2a = 2b + 1$ by equating the two expressions for n

$2a - 2b = 1$

$2(a - b) = 1$

$a - b = 1/2$ by algebra.

Now since a and b are integers, the difference a − b must also be an integer.
But a − b = 1/2, and 1/2 is not an integer.

Thus a − b is an integer and a − b is not an integer, which is a contradiction.
This contradiction shows that the supposition is false and, hence, that the theorem is true.

Exercise:

Prove following by contradiction method of proof:

1) The sum of any rational number and any irrational number is irrational.
2) The square root of any irrational number is irrational.
3) For all integers n, if n2 is even then n is even.

1.3.2.2 Contraposition

A second form of indirect argument, argument by contraposition, is based on the logical equivalence between a statement and its contrapositive. To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

Method of Proof by Contraposition

1. Express the statement to be proved in the form ∀ x in D, if P(x) then Q(x).
2. Rewrite this statement in the contrapositive form ∀ x in D, if Q(x) is false then

P(x) is false.

3. Prove the contrapositive by a direct proof.
   a. Suppose x is a particular but arbitrarily chosen element of D such that Q(x) is false.
   b. Show that P(x) is false.

Proposition

For all integers n, if n^2 is even then n is even.

Proof :

Suppose n is any odd integer.
We must show that n^2 is odd.
By definition of odd,
$n = 2k + 1$ for some integer k.

By substitution and algebra,

$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2\cdot$(an integer) + 1, and thus, by definition of odd, n^2 is odd .

For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$

Proof:

We'll prove the contrapositive of this statement.
We must show that for any integers a and b, $a < 8$ and $b < 8$ implies that $a + b < 15$.
By above statement ,
a and b are integers such that $a < 8$ and $b < 8$.
This implies that $a \leq 7$ and $b \leq 7$.

By substitution and algebra we get,

$a + b \leq 14$.

But this implies that $a + b < 15$.

Exercise:

Prove following by contrapositive method of proof:

1) If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.
2) For all integers m and n, if mn is even then m is even or n is even.

1.3.2.3 Two Classical Theorems

This section contains proofs of two of the most famous theorems in mathematics: that $\sqrt{2}$ is irrational and that there are infinitely many prime numbers. Both proofs are examples of indirect arguments.

Theorem : Irrationality of $\sqrt{2}$ $\sqrt{2}$ is irrational.
Proof: We take the negation and suppose it to be true. That is, suppose $\sqrt{2}$ is rational. Then there are integers m and n with no common factors such that $\sqrt{2} = \frac{m}{n}$ by dividing m and n by any common factors if necessary We must derive a contradiction. Squaring both sides of equation gives $2 = m^2/n^2$ $m^2 = 2n^2$ This implies that m^2 is even by definition of even. It follows that m is even . m = 2k for some integer k. $m^2 = (2k)^2$ by substituting $= 4k^2$ $= 2n^2$ Dividing both sides of the right-most equation by 2 gives $n^2 = 2k^2$

Consequently, n^2 is even, and so n is even .

But we also know that m is even.
Hence both m and n have a common factor of 2.
But this contradicts the supposition that m and n have no common factors.
Hence the supposition is false and so the theorem is true.

Proposition:
1 + 3√2 is irrational.

Proof:

Suppose 1 + 3√2 is rational.

We must derive a contradiction.
Then by definition of rational,

$1 + 3\sqrt{2} = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

It follows that

$3\sqrt{2} = \frac{a}{b} - 1$ by subtracting 1 from both sides

$= \frac{a}{b} - \frac{b}{b}$ by substitution

$= \frac{a-b}{b}$

$\sqrt{2} = \frac{a-b}{3b}$ by dividing both sides by 3.

But a − b and 3b are integers since a and b are integers and differences and products of
integers are integers, and $3b \neq 0$ by the zero product property.

Hence √2 is a quotient of the two integers a − b and 3b with 3b = 0, and so √2 is rational by definition of rational.

This contradicts the fact that √2 is irrational. This contradiction shows that the supposition is false.
Hence 1 + 3√2 is irrational.

Unit 1- Chapter 2
Logic

Chapter Structure

2.0 Objective

Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an argument, we mean a sequence of statements that end with a conclusion. By valid, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or premises, of the argument.
That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

2.1 Mathematical Induction

One of the most important techniques to prove many mathematical statements or formulae which cannot be easily derived by direct methods is sometimes derived by using the principle of mathematical induction.

Steps for induction method :
1. *Basic step of induction* : Show it is true for the first one , the smallest integral value of n (n = 1,2,3).
2. *Induction step*: If the statement is true for n=k , where k denotes any value of n , then it must be true for n = k + 1.
3. *Conclusion*: The statement is true for all integral values of n equal to or greater than that for which it was verified in Step 1

Prove that 1 + 2 + 3 + . . . + n = $\frac{n(n+1)}{2}$ for any integer n ≥ 1.

1. Show it is true for n=1

L.H.S. = n = 1

R.H.S. = $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

Hence, L.H.S = R.H.S

2. Assume it is true for n=k that is,

$1 + 2 + 3 + . . . + k = \frac{k(k+1)}{2}$

3. Prove that statement is true for n = k+ 1 that is,

$1 + 2 + 3 + . . . + k + (k + 1) = \frac{k+1(k+1+1)}{2}$.

We have ,

$1 + 2 + 3 + . . . + k = \frac{k(k+1)}{2}$

By putting value of k into (k+1) equation we get,

$\frac{k(k+1)}{2} + (k + 1) = \frac{k+1(k+1+1)}{2}$.

Now we will start solving L.H.S. to check whether it is equal to R.H.S. or not.

Therefore,

L.H.S = 1+2+3+.........k+ (k + 1)

$= \frac{k(k+1)}{2} + (k+1)$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)(k+1+1)}{2}$

= R.H.S

3.Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

Using the principle of mathematical induction, prove that

n(n + 1)(n + 5) is a multiple of 3 for all n ∈ N.

Proof:

Let P(n): n(n + 1)(n + 5) is a multiple of 3.

For n = 1, the given expression becomes (1 × 2 × 6) = 12, which is a multiple of 3.

So, the given statement is true for n = 1, i.e. P(1) is true.

Let P(k) be true. Then,

P(k): k(k + 1)(k + 5) is a multiple of 3

⇒ K(k + 1)(k + 5) = 3m for some natural number m, ... (i)

Now, (k + 1l)(k + 2)(k + 6) = (k + 1)(k + 2)k + 6(k + 1)(k + 2)

= k(k + 1)(k + 2) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5 – 3) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5) – 3k(k + 1) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5) + 3(k + 1)(k +4) [on simplification]

= 3m + 3(k + 1)(k + 4) [using (i)]

= 3[m + (k + 1)(k + 4)], which is a multiple of 3

⇒ P(k + 1): (k + 1)(k + 2)(k + 6) is a multiple of 3

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

Exercise:

Prove following by mathematical induction method of proof:

1) For all n ≥ 1, 1 + 4 + 7 + · · · + (3n − 2) = n(3n − 1) /2
2) For n any positive integer, $6^n - 1$ is divisible by 5.

2.2 Mathematical Statements

- To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments.
- Rules of inference are our basic tools for establishing the truth of statements.
- In this section we will look at arguments that involve only compound propositions.
- We will define what it means for an argument involving compound propositions to be valid.
- Then we will introduce a collection of rules of inference in propositional logic.
- After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements.
- We will describe how these rules of inference can be used to produce valid arguments.
- These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.
- Finally, we will show how rules of inference for propositions and for quantified statements can be combined.
- These combinations of rules of inference are often used together in complicated arguments.

Atomic and Molecular Statements

- A statement is any declarative sentence which is either true or false.
- A statement is atomic if it cannot be divided into smaller statements, otherwise it is called molecular.
- Propositional logic is the simplest form of logic. Here the only statements that are considered are propositions, which contain no variables.
- Because propositions contain no variables, they are either always true or always false. Examples of propositions:
  • $2 + 2 = 4$. (Always true).
  • $2 + 2 = 5$. (Always false).
- Examples of non-propositions:
  • $x + 2 = 4$. (May be true, may not be true; it depends on the value of x.)
  • $x \cdot 0 = 0$. (Always true, but it's still not a proposition because of the variable.)

Predicate logic

- Using only propositional logic, we can express a simple version of a famous argument:
  • Socrates is a man.
  • If Socrates is a man, then Socrates is mortal.
  • Therefore, Socrates is mortal.
- This is an application of the inference rule called modus ponens, which says that from p and $p \rightarrow q$ you can deduce q. The first two statements are axioms (meaning we are given them as true without proof), and the last is the conclusion of the argument.

- What if we encounter Socrates's infinitely more logical cousin Spocrates?
  We'd like to argue
  • Spocrates is a man.
  • If Spocrates is a man, then Spocrates is mortal.
  • Therefore, Spocrates is mortal.

Quantifiers

What we really want is to be able to say when H or P or Q is true for many different values of their arguments. This means we have to be able to talk about the truth or falsehood of statements that include variables. To do this,we bind the variables using quantifiers, which state whether the claim we are making applies to all values of the variable (universal quantification), or whether it may only apply to some (existential quantification).

Universal quantifier

- The universal quantifier ∀ called as "for all".
- It means that a statement must be true for all values of a variable within some universe of allowed values (which is often implicit).
- For example, "all humans are mortal" could be written as:
  $\forall x : Human(x) \rightarrow Mortal(x)$ and
- "if x is positive then x + 1 is positive" could be written as:
  $\forall x : x > 0 \rightarrow x + 1 > 0$.

- If you want to make the universe explicit, use set membership notation.
- An example would be $\forall x \in Z : x > 0 \rightarrow x + 1 > 0$.
- This is logically equivalent to writing,
  $\forall x : x \in Z \rightarrow (x > 0 \rightarrow x + 1 > 0)$ or

  to writing $\forall x : (x \in Z \wedge x > 0) \rightarrow x + 1 > 0$,

  but the short form makes it more clear that the intent of $x \in Z$ is to restrict the range of x.

- The statement $\forall x : P(x)$ is equivalent to a very large AND;
- for example, $\forall x \in N : P(x)$ could be rewritten (if you had an infinite amount of paper) as $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$.
- Normal first-order logic doesn't allow infinite expressions like this, but it may help in visualizing what $\forall x : P(x)$ actually means.

- Another way of thinking about it is to imagine that x is supplied by some adversary and you are responsible for showing that P(x) is true; in this sense, the universal quantifier chooses the worst case value of x

Existential quantifier

- The existential quantifier ∃ called as "there exists". It means that a statement must be true for at least one value of the variable.
- So "some human is mortal" becomes $\exists x : Human(x) \wedge Mortal(x)$.
- Note that we use AND rather than implication here;

  the statement $\exists x : Human(x) \rightarrow Mortal(x)$

  makes the much weaker claim that "there is something x, such that if x is human, then x is mortal," which is true in any universe that contains an immortal purple penguin since it isn't human, Human(penguin) → Mortal(penguin) is true.

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