CHAPTER 4: PROBABILITY CALCULUS (PROBABILITY SPACE)

1. Introduction

Probability is a mathematical tool used to describe and measure uncertainty. In many real situations, we cannot predict exactly what will happen, but we can estimate how likely different outcomes are.

Probability is useful in everyday life, science, engineering, medicine, economics, and especially physics. For example, the result of tossing a coin cannot be predicted with certainty, but the possible outcomes and their probabilities can be studied.

$$\mathbf{0 \leq P(A) \leq 1}$$

A probability of 0 means that an event is impossible, while a probability of 1 means that an event is certain.

Example: A coin toss

When a fair coin is tossed, the two possible outcomes are Head and Tail. Each has probability 1/2.

Example: A physics situation

For a radioactive atom, we cannot predict exactly when a particular atom will decay. Probability allows us to describe the likelihood of decay during a given time interval.

2. Sample Space and Event

Definition

A random experiment is an experiment or observation whose result cannot be predicted with certainty before it is performed.

The set of all possible outcomes is called the sample space and is denoted by Ω. An individual outcome is an elementary outcome. An event is a set of one or more outcomes belonging to Ω.

Example: Rolling a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$. The event $A = \{2, 4, 6\}$ means that an even number is obtained.

Example: Tossing two coins (Lancer deux pièces)

$\Omega = \{HH, HT, TH, TT\}$. The event $A = \{HH, HT, TH\}$ means that at least one Head is obtained.

An event containing one outcome is an elementary event. An event containing several outcomes is a compound event. The impossible event is $\emptyset$, and the sure event is Ω.

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2.1 Operations on Events

2.1.1 Union of Events

The union A ∪ B is the event that A occurs or B occurs, including the case where both occur.

A ∪ B = outcomes belonging to A or B

Example

If A = {1, 2, 3} and B = {3, 4, 5}, then A ∪ B = {1, 2, 3, 4, 5}.

2.1.2 Intersection of Events

The intersection A ∩ B is the event that both A and B occur.

A ∩ B = outcomes belonging to both A and B

Example

For A = {1, 2, 3} and B = {3, 4, 5}, A ∩ B = {3}. If A ∩ B = ∅, the events are mutually exclusive.

2.1.3 Complement of an Event

The complement A^c is the event that A does not occur.

A^c = Ω \ A

Example

For a die, if A = {2, 4, 6}, then A^c = {1, 3, 5}.

2.1.4 Probabilizable Space and Probability Space

A probabilizable space consists of a sample space Ω together with a suitable collection of events to which probabilities can be assigned. A probability space adds a probability measure P that assigns a probability to each event.

For this course, remember the three basic ideas: possible outcomes, events, and probabilities.

3. Axioms of Probability

The probability of events is based on three fundamental axioms:

1. For every event A, P(A) ≥ 0.
2. P(Ω) = 1.
3. If A and B are mutually exclusive, P(A ∪ B) = P(A) + P(B).

3.1 Useful Results (Elementary Properties)

- $P(\emptyset) = 0$.
- $P(A^c) = P(\bar{A}) = 1 - P(A)$.

- If A is contained in B, then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If A and B are mutually exclusive, $P(A \cap B) = 0$.

Example

If $P(A) = 0.8$, then $P(A^c) = 1 - 0.8 = 0.2$.

Example

If $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$, then $P(A \cup B) = 0.6 + 0.5 - 0.2 = 0.9$.

3.2 Some Elementary Theorems

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$\mathbf{P(A^c) = 1 - P(A)}$$

If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$.

Example

If A, B and C are mutually exclusive and cover Ω, with $P(A) = 0.2$ and $P(B) = 0.5$, then $P(C) = 1 - 0.2 - 0.5 = 0.3$.

4. Sample Space with Equally Likely Elementary Events

A sample space is equiprobable when all elementary outcomes have the same probability.

$$\mathbf{P(A) = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes}}$$

Example: One die

For a fair die, the probability of obtaining an even number is 3/6 = 1/2 because the favorable outcomes are 2, 4 and 6.

Example: Two coin tosses (Deux lancers de pièce)

The four outcomes HH, HT, TH and TT are equally likely. The probability of exactly two Heads is therefore 1/4.

4.1 Method of Calculating Probabilities

1.Identify the random experiment.
2.Determine the sample space or total number of possible outcomes.
3.Identify the favorable outcomes.
4.Count the favorable outcomes.
5.Apply the probability formula.

6. Check that the answer is between 0 and 1.

Example: Selecting students

A class contains 10 students. Two students are selected without assigning roles. There are C(10,2) = 45 possible selections. If 6 students are female, there are C(6,2) = 15 selections containing two females. Therefore the probability that the selected students are $P = \frac{15}{45} = \frac{1}{3}$

Combinatorial methods from Chapter 3 are particularly useful when there are many possible outcomes.

5. Conditional Probability and Independence

Sometimes the probability of an event changes when we know that another event has already occurred. This leads to conditional probability.

5.1 Conditional Probability

The conditional probability of A given B is the probability that A occurs when we know that B has occurred.

$$\mathbf{P(A \mid B) = P(A \cap B) / P(B), \text{ provided } P(B) > 0}$$

Example: Students

In a group of 20 students, 12 are girls and 8 are boys. Among the girls, 6 study physics. If we know that the selected student is a girl, the probability that the student studies physics is 6/12 = 1/2.

5.2 Multiplication Theorem

$$\mathbf{P(A \cap B) = P(B) \times P(A \mid B)}$$

$$\mathbf{P(A \cap B) = P(A) \times P(B \mid A)}$$

Example: Drawing balls

A box contains 5 red balls and 3 blue balls. Two balls are drawn without replacement.

Of course. This example is important because it introduces **conditional probability** and the **multiplication theorem**. The key point is understanding why the second probability changes from 5/8 to 4/7.

Example: Drawing Two Red Balls

A box contains **5 red balls and 3 blue balls**, so there are:

5 + 3 = 8 balls

We draw **two balls without replacement**.

"Without replacement" means that **after drawing the first ball, we do not put it back into the box**.

Step 1: Probability that the first ball is red

Initially, there are:

- 5 red balls
- 3 blue balls
- 8 balls in total

Therefore, the probability of drawing a red ball first is:

P(first red) = 5/8

Step 2: What happens if the first ball is red?

Suppose the first ball drawn is red.

We now have **one fewer red ball** and **one fewer ball in total**.

Initially:

- Red balls = 5
- Total balls = 8

After removing one red ball:

- Red balls = 4
- Total balls = 7

Therefore, the probability that the **second ball is red, knowing that the first ball was red**, is:

P(second red | first red) = 4/7

The symbol **"|"** means **"given that"**.

So:

P(second red | first red)

means:

"The probability of getting a red ball on the second draw, given that we already know that the first ball was red."

Step 3: Probability of getting two red balls

To obtain **two red balls**, two things must happen:

1. The first ball must be red.
2. The second ball must also be red.

Therefore, we multiply the two probabilities:

$$P(\textbf{two red balls}) = P(\textbf{first red}) \times P(\textbf{second red} \mid \textbf{first red})$$

So:

P(two red balls) = 5/8 × 4/7

Therefore:

P(two red balls) = 20/56 = 5/14

So the final answer is:

P(two red balls) = 5/14 ≈ 0.357

or approximately:

35.7%

The important idea

The probability of the second draw is **not 5/8** because the first red ball was **not replaced**.

That is why:

First draw → 5 red out of 8 → 5/8

Second draw, knowing first was red → 4 red out of 7 → 4/7

This is exactly why the example is a good introduction to the **multiplication theorem**:

$$P(A \cap B) = P(A) \times P(B \mid A)$$

Here:

- A = first ball is red

- B = second ball is red
- A ∩ B = both balls are red.

P(first red) = 5/8. After a red ball is removed, P(second red | first red) = 4/7. Therefore P(two red) = (5/8)(4/7) = 5/14.

5.3 Total Probability Theorem

Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive events that cover the whole sample space. Then:

$$\mathbf{P(A) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + \ldots + P(B_n)P(A \mid B_n)}$$

Example*: *Two machines

Machine 1 produces 60% of products and Machine 2 produces 40%. The defective rates are 2% and 5%, respectively.

$$\mathbf{P(D) = 0.60 \times 0.02 + 0.40 \times 0.05 = 0.032}$$

Therefore, the probability that a randomly selected product is defective is 3.2%.

5.4 Bayes' Theorem

Bayes' theorem allows us to reverse a conditional probability. It is useful when an observed result is known and we want to determine which cause is most likely.

$$\mathbf{P(B \mid A) = P(B)P(A \mid B) / P(A)}$$

Example: Defective product

Using the previous example, if a product is defective, the probability that it came from Machine 2 is:

$$\mathbf{P(M_2 \mid D) = (0.40 \times 0.05) / 0.032 = 0.625}$$

Thus, given that the product is defective, there is a 62.5% probability that it came from Machine 2.

6. Independence

Two events are independent when the occurrence of one event does not change the probability of the other event.

Independence is different from mutual exclusivity. Mutually exclusive events cannot occur together, whereas independent events can occur together.

6.1 Independence of Two Events

Definition

Two events A and B are independent if:

$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

Equivalently, when $P(B) > 0, P(A \mid B) = P(A)$.

Example: Two coin tosses

The first toss does not affect the second toss. Therefore 'Head on the first toss' and 'Head on the second toss' are independent.

$$\boldsymbol{P(H_1 \cap H_2) = 1/2 \times 1/2 = 1/4}$$

Example: Drawing without replacement

If two balls are drawn without replacement, the first draw changes the composition of the box. Therefore the events 'first ball is red' and 'second ball is red' are generally dependent.

6.2 Mutual Independence of Three Events

Three events A, B and C are mutually independent when the occurrence of any event does not affect the others. We require:

$$\boldsymbol{P(A \cap B) = P(A)P(B)}$$

$$\boldsymbol{P(A \cap C) = P(A)P(C)}$$

$$\boldsymbol{P(B \cap C) = P(B)P(C)}$$

$$\boldsymbol{P(A \cap B \cap C) = P(A)P(B)P(C)}$$

Exercises

Exercise 1

A fair die is rolled once.

1. Give the sample space.
2. Define A = 'an even number is obtained'.
3. Define B = 'a number greater than 4 is obtained'.
4. Determine A ∩ B and A ∪ B.

Exercise 2

A fair coin is tossed three times(Une pièce équilibrée est lancée trois fois).

. 1. Write the sample space.

2. Determine the probability of at least two Heads(Pile).

3. Determine the probability of at most two Head(Pile) .

Exercise 3

A die is rolled once. Let A = {1, 2, 3} and B = {3, 4, 5}. Determine $A \cup B$, $A \cap B$, A^c and B^c.

Exercise 4

In a class of 30 students, 18 are female and 12 are male. Five female students and three male students study physics. Determine P(student studies physics) and P(student studies physics | student is female).

Exercise 5

A box contains 4 red balls and 6 blue balls. One ball is drawn at random. Determine the probability of drawing a red ball and the probability of not drawing a red ball.

Exercise 6

A class contains 12 students, including 7 girls. Two students are selected at random without assigning roles. Determine the probability that both selected students are girls.

Exercise 7

A box contains 5 red balls and 3 blue balls. Two balls are drawn without replacement. Determine the probability of drawing two red balls using the multiplication theorem.

Exercise 8

A factory has two production lines. Line A produces 70% of the products and Line B produces 30%. The defective rates are 2% for A and 4% for B. Determine the probability that a randomly selected product is defective.

Exercise 9

Using Exercise 8, a product is found to be defective. Determine the probability that it came from Line B.

Exercise 10

A fair coin is tossed twice. Let A be 'Head on the first toss' and B be 'Head on the second toss'. Are A and B independent? Verify using the definition.

Exercise 11

A box contains 6 red balls and 4 blue balls. Two balls are drawn without replacement. Are the events 'first ball is red' and 'second ball is red' independent? Explain.

Exercise 12

Three independent events A, B and C satisfy P(A)=0.5, P(B)=0.4 and P(C)=0.2. Calculate P(A ∩ B), P(A ∩ C), P(B ∩ C) and P(A ∩ B ∩ C).