1. Introduction

This chapter introduces simple methods for counting possible choices and arrangements. These methods are useful in probability, statistics, physics, computer science, and many other fields.

In many problems, we need to know how many different possibilities can occur. For example, how many passwords can be created? How many ways can we arrange students in a row? How many different groups can be selected from a class?

When the number of possibilities is small, we can list them one by one. When the number is large, listing all possibilities becomes difficult. Combinatorial analysis gives us systematic counting rules.

Simple example

Suppose a student has 3 shirts and 2 pairs of trousers. For each shirt, the student can choose either pair of trousers. Therefore, the total number of outfits is $3 \times 2 = 6$.

The main tools studied in this chapter are:

- the Fundamental Counting Principle (FCP)
- permutations
- arrangements (or ordered selections)
- combinations (or unordered selections)
- the Binomial Theorem

2. Fundamental Counting Principle (FCP)

The Fundamental Counting Principle tells us how to count a sequence of choices.

2.1. Restricted version

Suppose an activity is performed in two successive steps. If the first step can be done in $\boldsymbol{m}$ ways and, for each choice of the first step, the second step can be done in $\boldsymbol{n}$ ways, then the whole activity can be done in $\boldsymbol{m} \times \boldsymbol{n}$ ways.

Total number of possibilities = m × n

Example 1: Choosing an outfit

A student has 4 shirts and 3 pairs of trousers. How many different outfits can be formed?

Step 1: Choose a shirt: 4 possibilities.
Step 2: Choose a pair of trousers: 3 possibilities.
Therefore: $4 \times 3 = 12$ different outfits.

Example 2: A two-digit number(Un nombre à deux chiffres)

How many two-digit numbers can be formed using the digits 1, 2, 3, 4 and 5 if repetition is allowed?

The tens digit has 5 choices. The units digit also has 5 choices.
Therefore: 5 × 5 = 25 numbers.

2.2. Generalized Fundamental Counting Principle

The same idea can be extended to three or more successive choices.

$$\textbf{Total number of possibilities} = n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

Example 3: A three-step choice

A laboratory experiment uses 2 possible temperatures, 4 possible materials, and 3 possible measuring instruments. If every temperature can be combined with every material and every instrument, the number of possible experimental settings is:

2 × 4 × 3 = 24.

Thus, there are 24 possible settings.

Important idea: when choices are made successively, multiply the numbers of possibilities.

3. Main Types of Counting

The appropriate counting method depends mainly on two questions:

- Does the order matter?
- Can an element be used more than once?

Method	Does order matter?	Repetition?
Permutation	Yes	Usually no; repetition may be allowed in a special case
Arrangement	Yes	Without or with repetition
Combination	No	Usually without repetition

3.1. Permutations

A permutation is an arrangement of all available elements in a specific order.

3.1.1. Permutations without repetition

If n distinct objects are arranged in a row, the number of possible permutations is:

$$\mathbf{P(n) = n!}$$

Here, $n!$ (read 'n factorial') means:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

By convention, 0! = 1.

Example 4: Arranging books

Four different books are placed on a shelf. How many different arrangements are possible?

For the first position there are 4 choices.
For the second position there are 3 choices.
For the third position there are 2 choices.
For the last position there is 1 choice.

Therefore: $4 \times 3 \times 2 \times 1 = 24$.
So, there are 24 different arrangements.

3.1.2. Permutations with repetition

1. The Basic Idea

When we rearrange objects, we normally use a factorial. However, if some objects are identical, we must correct the counting because exchanging identical objects does not create a new rearrangement.

For example, if two letters are identical, changing the position of one identical letter with the other does not produce a different rearrangement.

2. General Rule

Suppose there are $\boldsymbol{n}$ objects in total and some objects are repeated.

If $\boldsymbol{n_1}$ objects are identical of the first type, $\boldsymbol{n_2}$ of the second type, $\boldsymbol{n_3}$ of the third type, and so on, then:

$$\textbf{Number of different rearrangements} = \frac{n!}{(n_1! \times n_2! \times n_3! \times \ldots \times n_k!)}$$

The denominator corrects the over counting caused by identical objects.

3. Example 1: A Simple Case

How many different rearrangements can be made with the letters of the word "MAMA"?

The word MAMA contains 4 letters:

$$\textbf{M} - \textbf{A} - \textbf{M} - \textbf{A}$$

Step 1: Count the total number of letters. There are **4** letters, so $\boldsymbol{n = 4}$.

Step 2: Identify the repeated letters.

• M appears 2 times.
• A appears 2 times.

If all four letters were different, there would be 4! rearrangements:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

But the two M's are identical. Exchanging the two M's does not create a new rearrangement, so we divide by 2!. The same reasoning applies to the two A's.

$$\text{Number of different rearrangements} = 4! / (2! \times 2!)$$

$$= 24 / (2 \times 2) = 6$$

Therefore, the letters of MAMA can be rearranged in 6 different ways.

4. Example 2: Applying the General Rule

How many different rearrangements can be made with the letters of the word "REARRANGEMENT"?

Step 1: Count the total number of letters.

REARRANGEMENT contains **13** letters.

$$n = 13$$

Step 2: Count the occurrences of each letter:

Letter	Number of occurrences
R	3
E	3
A	2
N	2
G	1
M	1
T	1

G, M and T occur only once. Since 1! = 1, they do not change the calculation.

$$\text{Number of different rearrangements} = 13! / (3! \times 3! \times 2! \times 2!)$$

$$= 6227020800 / (6 \times 6 \times 2 \times 2)$$

$$= 6227020800 / 144$$

$$= 43243200$$

Therefore, the letters of REARRANGEMENT can be rearranged in 43,243,200 different ways.

5. What Students Should Remember

1. Count the total number of objects. This gives n.

2. Count how many times each identical object is repeated.

3. Write n! for all objects.

4. Divide by the factorial of each repeated group.

5. Calculate the result.

Number of different rearrangements = n! / (n_1! × n_2! × ... × n_k!)

Key idea: identical objects must not be counted as different objects. We divide by the factorial of each repeated group to remove the repeated counting.

3.2. Arrangements

An arrangement is an ordered selection of r elements chosen from n available elements. Unlike a permutation, we do not necessarily use all n elements.

3.2.1. Arrangements without repetition

When repetition is not allowed, the number of ordered selections of **r** elements from **n** distinct elements is:

$$A(n, r) = n \times (n - 1) \times \ldots \times (n - r + 1) = n! / (n - r)!$$

Example 6: Choosing a president and a vice-president

A class has **8** students. We want to choose a president and a vice-president. The same student cannot hold both positions. How many choices are possible?

There are **8** choices for president.
After choosing the president, there are 7 choices for vice-president.

Therefore: **8 × 7 = 56.**

The order matters because choosing **A** as president and **B** as vice-president is different from choosing B as president and A as vice-president.

3.2.2. Arrangements with repetition

If repetition is allowed, each of the r positions can be filled independently by any of the n elements.

Number of arrangements = n^r

Example 7: PIN code

A PIN has **4** digits. Each digit can be any of the digits **0 to 9**, and repetition is allowed.

There are **10** choices for each of the 4 positions.
Therefore: **10 × 10 × 10 × 10 = 10^4 = 10000.**

Thus, **10000** different PIN codes are possible.

3.3. Combinations

A combination is a selection of r elements from n elements where order does not matter.

$$\mathbf{C(n, r)} = \frac{\mathbf{n!}}{\mathbf{r!} \times (\mathbf{n} - \mathbf{r})!}$$

Example 8: Selecting a committee

A class has **8** students. We want to select **3** students to form a committee. The order of selection does not matter.

Selecting Alice, Bob and Sara gives the same committee as selecting Sara, Alice and Bob.

Therefore:
C(8, 3) = 8! / (3! × 5!) = (8 × 7 × 6) / (3 × 2 × 1) = 56.

Thus, 56 different committees can be formed.

The key difference between arrangements and combinations is order:

• Arrangement: order matters.
• Combination: order does not matter.

For example, choosing A then B is different from choosing B then A in an arrangement, but these two choices are the same combination.

$$\mathbf{C(n, r) = C(n, n - r)}$$

3.4. Newton's Binomial Theorem

The Binomial Theorem gives a systematic way to expand a power of a sum such as $(a + b)^n$

$$(\boldsymbol{a + b})^{\boldsymbol{n}} \;=\; \mathbf{C(n, 0)}a^n + \mathbf{C(n, 1)a^{(n-1)}}b + \mathbf{C(n, 1)a^{(n-1)}}b^2 + \cdots + \;\; \mathbf{C(n, n)b^n}$$

The coefficients are combination numbers.

Example 9: Expanding (a + b)^2

Using the coefficients 1, 2, 1:

$$\mathbf{C(2, 0) = 1,} \qquad \mathbf{C(2, 1) = 2, C(n2, 2) = 1}$$

$(a + b)^n = \mathbf{C(2, 0)}a^2 + \mathbf{C(2, 1)a^{(1)}}b + \;\mathbf{C(2, 2)b^2} \;\; = a^2 + \mathbf{ab} + \mathbf{b^2}$

Example 10: Expanding (x + 2)^3
The coefficients for n = 3 are 1, 3, 3, 1.

(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3
= x^3 + 6x^2 + 12x + 8.

Pascal's Triangle is a convenient way to obtain the binomial coefficients:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

4. Worked Comparison Examples

A useful question to ask before solving a counting problem is: 'Does the order matter, and is repetition allowed?' This usually indicates which counting method should be used.

Example 11: Which method should we use?
Question A: In how many ways can 5 different books be arranged on a shelf?
All books are used and order matters → permutation → 5! = 120.

Question B: From 5 students, how many ways can we choose 2 students for a team?
Only a group is selected and order does not matter → combination → C(5,2) = 10.

Question C: From 5 students, how many ways can we choose a president and a vice-president?
Two positions are different, so order matters → arrangement → 5 × 4 = 20.

Exercise 1

A restaurant offers 4 starters, 5 main dishes, and 3 desserts. How many different three-course meals can be prepared?

Exercise 2

How many different arrangements can be made with 5 different books on a shelf?

Exercise 3

How many different arrangements can be made using the letters of the word 'MAMA'?

Exercise 4

A class contains 10 students. In how many ways can a president and a vice-president be chosen, if one student cannot hold both positions?

Exercise 5

A 3-digit code is formed using the digits 0 to 9. Repetition is allowed. How many codes are possible?

Exercise 6

A group of 10 students must choose 4 students to represent the class. How many different groups are possible?

Exercise 7

Expand (x + 1)^4 using the Binomial Theorem.

Exercise 8

A laboratory has 3 possible temperatures, 2 materials, and 4 measuring instruments. How many experimental settings are possible?

Exercise 9

Explain why choosing 2 students from a class is a combination when no roles are assigned, but an arrangement when one student is president and the other is vice-president.

Exercise 10

A box contains 6 different balls. Three balls are selected one after another without replacement, and their order is recorded. Is this a permutation, arrangement, or combination? Calculate the number of possibilities.