Chapter 5

Inverse Functions

In this chapter, we introduce the concept of inverse functions and examine their main properties. We then study inverse trigonometric functions, followed by hyperbolic and inverse hyperbolic functions, with particular emphasis on their definitions and fundamental properties.

5.1 Inverse functions and their properties

Definition 5.1.1 *Let $f : A \to B$, $g : B \to A$ be two functions. We say that the function g is an inverse of the function f if an only if*

$$f(a) = b \Leftrightarrow g(b) = a,$$

for every b in the domain of g and every a in the domain of f. We denote g by f^{-1} and we call it the inverse of f.

In fact, the inverse of the function f is the function that sends each $f(a)$ back to a.

$$a \xrightarrow{f} f(a) \xrightarrow{f^{-1}} a$$

We note that if f^{-1} is an inverse of f then f is an inverse of f.

There is a geometrical relationship between inverse functions. If f and f^{-1} are inverse functions of each other, then the graph of $y = f(x)$ is the graph of $y = f^{-1}(x)$, reflected about the line $y = x$.

Proposition 5.1.1 *Suppose that a function f has an inverse. Then the domain of f is equal to the range of f^{-1}, and the range of f is equal to the domain of f^{-1} and we have*

$$f(f^{-1}(y)) = y, \quad \text{for all } y \text{ in the range of } f.$$
$$f^{-1}(f^{-1}(x)) = x, \quad \text{for all } x \text{ in domain of } f.$$

The previous two equations show that when a function has an inverse, the composite of the function and its inverse in either order is the identity function. Recall that an identity function is a function that maps each element in its domain onto itself.

Proposition 5.1.2 *If a function f admits an inverse, then that inverse is unique.*

Definition 5.1.2 *We say that a function f is one-to-one, if its graph satisfies the horizontal line test: no horizontal line intersects the graph more than once.*

No horizontal line intersects a graph more than once is equivalent that no y-value on the graph can come from two different x-values. In fact, f is one-to-one is equivalent to say that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (or $f(x_1) = f(x_2)$ implies $x_1 = x_2$).

Proposition 5.1.3 *If the real function f satisfies the horizontal line test, i.e. f is one-to-one, then f admits an inverse function f^{-1}.*

Example 5.1.1 *The absolute value function $f(x) = |x|$ is not one-to-one, since*

$$f(2) = 2 = f(-2).$$

Thus, this function does not have an inverse.

More generally, an even function never has an inverse function, since $f(-x) = f(x)$.

Remark 5.1.1 *For a function* f, f^{-1} *will denote its inverse, and* -1 *does not mean the exponent :*

$$f^{-1}(x) \neq \frac{1}{f(x)} = f(x)^{-1}.$$

We present now the following algorithm for computing the inverse function (when it exists):

1. Form the equation $y = f(x)$.
2. If possible, solve that equation for x as a function of y, $x = f^{-1}(y)$.
3. If x is required as the name of the independent variable, interchange the names x and y: $y = f^{-1}(x)$.
4. Describe the domain of f^{-1} (which is the range of f).

Example 5.1.2 *Find the inverse of the function* $f : \mathbb{R}-\{-3\} \to \mathbb{R}$ *defined by* $f(x) = \frac{5-x}{3+x}$.

First let us show that f *is one-to one. Let* $x_1, x_2 \in \mathbb{R} - \{-3\}$ *such that* $f(x_1) = f(x_2)$.

Then

$$\begin{aligned}\frac{5-x_1}{3+x_1} = \frac{5-x_2}{3+x_2} &\Rightarrow (5-x_1)(3+x_2) = (5-x_2)(3+x_1)\\ &\Rightarrow 15 - 3x_1 + 5x_2 - x_1x_2 = 15 + 5x_1 - 3x_2 - -x_1x_2\\ &\Rightarrow x_1 = x_2.\end{aligned}$$

So f *is one-to one and* f *has an inverse* f^{-1}. *We have*

$$\begin{aligned}y = \frac{5-x}{3+x} &\Rightarrow y(3+x) = 5 - x\\ &\Rightarrow 3y + xy - 5 + x = 0\\ &\Rightarrow x(1+y) = 5 - 3y\\ &\Rightarrow x = \frac{5-3y}{1+y}, \quad y \neq -1.\end{aligned}$$

Then

$$f^{-1}(x) = \frac{5-3x}{1+x}, \qquad x \in \mathbb{R} - \{-1\}$$

Definition 5.1.3 *A function f is called*

1)strictly increasing if for any x_1, x_2 from its domain,

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

2)strictly decreasing if for any x_1, x_2 from its domain,

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

3) monotonous if it is strictly increasing or strictly decreasing.

Theorem 5.1.1 *A monotonous continuous function is one-to-one and admits an inverse.*

5.1.1 Derivatives of inverse functions

Proposition 5.1.4 ***(Derivative of the inverse function)***

Let f be a bijection from I to $f(I)$ and let f^{-1} be its inverse function. If f is differentiable on I and f' does not vanish on I then f^{-1} is differentiable on $f(I)$ and we have

$$\forall y \in f(I), \qquad \left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{f'(x)},$$

where $y = f(x)$ which is equivalent to $f^{-1}(y) = x$.

Poof. By the definition of the inverse function, we have

$$f^{-1}(f(x)) = x.$$

Differentiate this identity using the chain rule :

$$\begin{aligned}\frac{d}{dx}f^{-1}(f(x)) = \frac{d}{dx}x &\Leftrightarrow \frac{d}{dy}f^{-1}(f(x))\frac{d}{dx}f(x) = 1\\ &\Leftrightarrow \frac{d}{dy}f^{-1}(y) = \frac{1}{\frac{d}{dx}f(x)}\\ &\Leftrightarrow \left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{f'\left(f^{-1}(y)\right)}.\end{aligned}$$

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Example 5.1.3 *Use the formula of the derivative of the inverse function to find the derivative of* $\sqrt[3]{x}$.

The function $\sqrt[3]{x}$ *is the inverse of the function* $f(x) = x^3, i.e.\ f^{-1}(x) = \sqrt[3]{x}$.

Begin by evaluating $f'(x)$. *Thus*

$$f'(x) = 3x^2 \text{ and } f'(f^{-1}(x)) = 3\left(\sqrt[3]{x}\right)^2 = 3x^{\frac{2}{3}} \neq 0, \text{ if } x \in \mathbb{R}\backslash\{0\}.$$

Finally

$$\left(f^{-1}\right)'(x) = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3}x^{-\frac{2}{3}}.$$

5.2 Definition of Inverse Trigonometric Functions

Before presenting the definition of inverse trigonometric functions we recall four facts about functions and their inverse functions:

1. A function must be bijective in order to have an inverse function.

2. The graph of an inverse function is the reflection of the original function about the line $y = x$.

3. If (x, y) is a point on the graph of the original function, then (y, x) is a point on the graph of the inverse function.

4. The domain and range of a function and it's inverse are interchanged.

We know that the trigonometric functions are periodic so they are one to one and then they are not bijective. For this each trigonometric function has a restricted domain which on it the trigonometric functions is bijective i.e. for which an inverse function is defined.

Definition 5.2.1 *The inverse function of the restricted sine function on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *exists and is denoted by* arcsin *or* $\sin^{-1}$. *It is called the inverse sine function or the arcsine function. We have*

$$\sin : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \to [-1.1] \quad \textit{then} \quad \arcsin : [-1.1] \to \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \textit{ and}$$
$$\sin x = y \Leftrightarrow \arcsin y = x, \qquad x \in [-1.1]\,,\ y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

Properties.

1. We have $\sin(\arcsin x) = x, \quad \forall x \in [-1.1]\,.$
2. We have $\arcsin(\sin y) = y, \quad \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$
3. We have $\cos(\arcsin x) = \sqrt{1 - x^2}, \quad \forall x \in [-1.1]\,.$
4. The function arcsin is an odd function on $[-1.1]\,.$

Definition 5.2.2 *The inverse function of the restricted cosine function on* $[0, \pi]$ *exists and is denoted by* arccos *or* $\cos^{-1}$. *It is called the inverse cosine function or the arccosine function. We have*

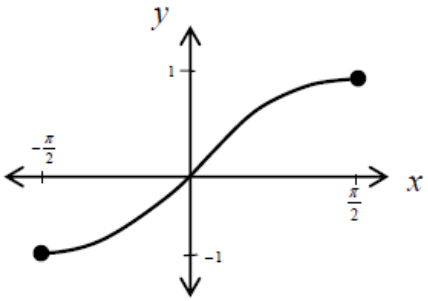
$$\cos : [0, \pi] \to [-1, 1] \quad \textit{then} \quad \arccos : [-1, 1] \to [0, \pi] \textit{ and}$$
$$\cos x = y \Leftrightarrow \arccos y = x, \quad x \in [-1.1]\,, y \in [0, \pi]\,.$$

Properties.

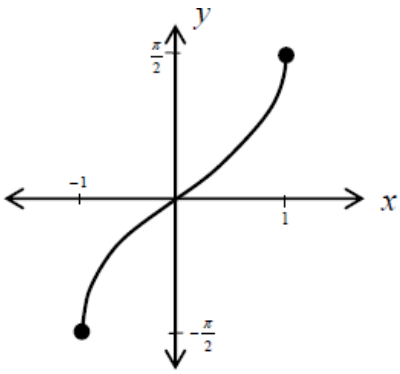
1. We have $\cos(\arccos x) = x, \quad \forall x \in [-1, 1]\,.$
2. We have $\arccos(\cos y) = y, \quad \forall y \in [0, \pi]\,.$
3. We have $\sin(\arccos x) = \sqrt{1 - x^2} \quad \forall x \in [-1, 1]\,.$

Graphs:

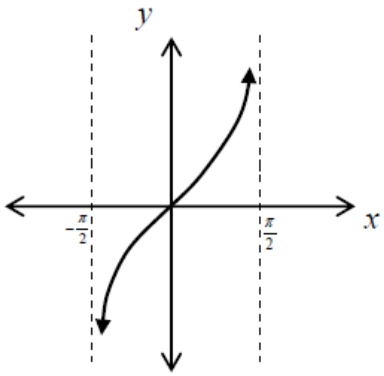
$y = \sin x$:



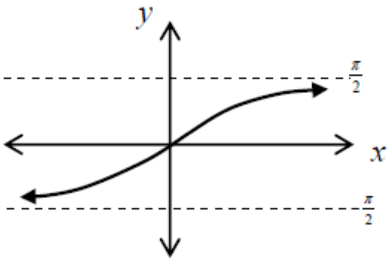
$y = \arcsin x = \sin^{-1} x$:



$y = \tan x$:



$y = \arctan x = \tan^{-1} x$:



Definition 5.2.3 *The inverse function of the restricted tangent function on* $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ *exists and is denoted by* arctan *or* $\tan^{-1}$. *It is called the inverse tangent function or the arctangent function. We have*

$$\tan : [0, \pi] \to \mathbb{R} \qquad \textit{then} \qquad \arctan : \mathbb{R} \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\textit{ and}$$

$$\tan x = y \Leftrightarrow \arctan y = x, \qquad x \in \mathbb{R},\ y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Properties.

1. We have $\tan(\arctan x) = x, \quad \forall x \in]-\infty, +\infty[$.

2. We have $\arctan(\tan y) = y, \quad \forall y \in \mathbb{R}$.

3. We have $\lim\limits_{x \to +\infty} \arctan x = \frac{\pi}{2}$ and $\lim\limits_{x \to -\infty} \arctan = -\frac{\pi}{2}$, and so the lines are horizontal asymptotes of the graph of arctan.

Trig function	Restricted domain	Inverse trig function	Principle value range
$y=\sin x$	$-\frac{\pi}{2}\le x\le\frac{\pi}{2}$	$y=\arcsin x$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos x$	$0\le x\le\pi$	$y=\arccos x$	$0\le y\le\pi$
$y=\tan x$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$	$y=\arctan x$	$-\frac{\pi}{2}<y<\frac{\pi}{2}$

Example 5.2.1 *We have :*

a) $\arcsin(1)=\frac{\pi}{2}$ *since* $\sin(\frac{\pi}{2})=1$ *and* $\frac{\pi}{2}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

b) $\arcsin(\sin\frac{3\pi}{2})=\arcsin(-1)=-\frac{\pi}{2}$ *and* $\arcsin(\sin\frac{3\pi}{2})\neq\frac{3\pi}{2}$ *because* $\frac{3\pi}{2}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

c) $\arccos(0)=\frac{\pi}{2}$ *since* $\cos(\frac{\pi}{2})=0$ *and* $\frac{\pi}{2}\in[0,\pi]$.

d) $\arccos(3)$ *is undefined because* $3\notin[-1,1]$.

Example 5.2.2 *Find the domain of definition of the following function:*

$$f(x)=\arcsin(x^2-1).$$

We have

$$\begin{aligned}D_f&=\left\{x\in\mathbb{R},\quad -1\le x^2-1\le 1\right\}\\&=\left\{x\in\mathbb{R},\quad 0\le x^2\le 2\right\}\\&=\left\{x\in\mathbb{R},\quad |x|\le\sqrt{2}\right\}\\&=\left[-\sqrt{2},\sqrt{2}\right].\end{aligned}$$

5.2.1 Derivatives of inverse trigonometric functions

In this section we turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text.

The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. For the first time, we see that the derivative of a function need not be of the same type as the original function

Theorem 5.2.1 *The derivative of the arcsine is equal to :*

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for any } x \in]-1,1[\,.$$

Poof. *The function* $f(x) = \arcsin x$, *where* $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *has an inverse given by*

$$f^{-1}(y) = \arcsin y, \quad \text{where } y \in]-1,1[\,.$$

Using the theorem about derivatives of inverse functions

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)},$$

where $y = f(x), we$ *get in our case*

$$\begin{aligned}(\arcsin y)' &= \frac{1}{(\sin x)'} = \frac{1}{\cos x} \\ &= \frac{1}{\sqrt{1-\sin^2 x}} \qquad (since\ \cos^2 x + \sin^2 x = 1\ and\ x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[). \\ &= \frac{1}{\sqrt{1-y^2}} \qquad (since\ y = \sin x).\end{aligned}$$

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Switch to x we find

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad \text{for any } x \in]-1,1[\,.$$

First we present the following formula which relates arcsin with arccos .

Lemme 5.2.1 *We have the following formula*

$$\arcsin y + \arccos y = \frac{\pi}{2}, \qquad \text{for } y \in [-1, 1]\,.$$

Poof. Let $y = \sin x$, so $x = \arcsin y$. Since $\sin x = \cos(\frac{\pi}{2} - x)$, we get

$$\begin{aligned} y = \cos(\frac{\pi}{2} - x) &\Leftrightarrow \arccos y = \frac{\pi}{2} - x \\ &\Leftrightarrow \arccos y = \frac{\pi}{2} - \arcsin y \\ &\Leftrightarrow \arcsin y + \arccos y = \frac{\pi}{2}. \end{aligned}$$

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Theorem 5.2.2 *The derivative of arccosine is given by*

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \qquad x \in \,]-1, 1[\,.$$

Poof. From the previous lemma we have

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

So

$$\arccos x = \frac{\pi}{2} - \arcsin x.$$

Differentiate both sides of the equation

$$\begin{aligned} (\arccos x)' &= \Big(\frac{\pi}{2} - \arcsin x\Big)' \\ &= -\frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

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Theorem 5.2.3 *The derivative of the arctangent function is equal to :*

$$(\arctan x)' = \frac{1}{1+x^2} \qquad \textit{for any } x \in \mathbb{R}.$$

Poof. *The function* $f(x) = \tan x$, *where* $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *has an inverse given by*

$$f^{-1}(y) = \arctan y, \quad \textit{where } y \in \mathbb{R}.$$

Using the theorem about derivatives of inverse functions

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)},$$

where $y = f(x)$, *we get in our case*

$$\begin{aligned}(\arctan y)' &= \frac{1}{(\tan x)'} = \cos^2 x \\ &= \frac{1}{1+\tan^2 x} \qquad \textit{(using the identity } 1+\tan^2 x = \frac{1}{\cos^2 x}\textit{)}. \\ &= \frac{1}{1+y^2} \qquad \textit{(since } y = \arctan x\textit{)}.\end{aligned}$$

Switch to x *we get*

$$(\arctan x)' = \frac{1}{1+x^2} \qquad \textit{for any } \ x \in \mathbb{R}.$$

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Tables of derivatives of inverse trigonometric

$f(x)$	$f'(x)$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(u(x))$	$\frac{-u'(x)}{\sqrt{1-u^2(x)}}$
$\arcsin(u(x))$	$\frac{-u'(x)}{\sqrt{1-u^2(x)}}$
$\arctan(u(x))$	$\frac{u'(x)}{1+u^2(x)}$

Example 5.2.3 *Find the derivative of f where $f(x) = \arctan(5\sqrt{x}) + \arccos(\ln x)$.*

We have

$$\begin{aligned} f'(x) &= \frac{(5\sqrt{x})'}{1+(5\sqrt{x})^2} - \frac{(\ln x)'}{\sqrt{1-\ln^2 x}} = \frac{\frac{5}{2\sqrt{x}}}{1+25x} - \frac{\frac{1}{x}}{\sqrt{1-\ln^2 x}} \\ &= \frac{5}{2\sqrt{x}\,(1+25x)} - \frac{1}{x\sqrt{1-\ln^2 x}}. \end{aligned}$$

5.2.2 Integrals leading to inverse trigonometric functions

We have the following integrals

$$\begin{aligned} &\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C, \qquad |x| < 1, \\ &\int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\arcsin(\frac{x}{a}) + C, \qquad |x| < a, \quad a > 0, \end{aligned}$$

$$\begin{aligned} &\int \frac{1}{1+x^2}dx = \arctan(x) + C, \\ &\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(\frac{x}{a}) + C, \quad a < 0. \end{aligned}$$

Example 5.2.4 *We have*

$$\begin{aligned}\int (\frac{2x+5}{16+x^2}+\frac{1}{\sqrt{9-x^2}})dx &= \int \frac{2x}{16+x^2}dx+\int \frac{5}{16+x^2}dx+\int \frac{1}{\sqrt{9-x^2}}dx \\ &= \ln(16+x^2)+\frac{5}{4}\arctan(\frac{x}{4})+\frac{1}{3}\arcsin(\frac{x}{3})+C.\end{aligned}$$

5.3 Hyperbolic Functions

In this section we introduce the four main hyperbolic functions. The hyperbolic functions are defined in terms of the exponential function. We also discuss some identities relating these functions, and mention their inverse functions (reciprocal) functions. We will see some connections with trigonometric functions and we show how they can be used to evaluate certain integrals that cannot be solved without the use of hyperbolic functions.

5.3.1 Definitions

a) The hyperbolic functions $\cosh x$ and $\sinh x$ are defined using the exponential function e^x. In fact, **the hyperbolic cosine** function is defined by the relation

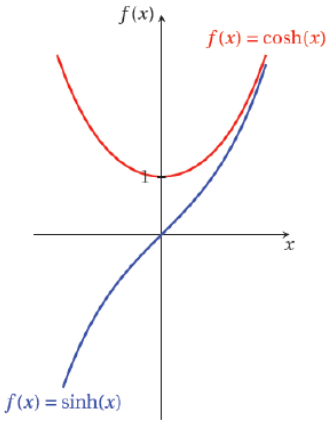
$$\cosh x = \frac{e^x+e^{-x}}{2}, \qquad \text{for all } x \in \mathbb{R}.$$

We remark that the function cosh satisfies the conditions $\cosh 0 = 1$ and $\cosh x = \cosh(-x)$, for all $x \in \mathbb{R}$. The function cosh is even, so its graph is symmetric with respect to the y-axis. In addition, the graph of cosh is always above the graphs of $e^x/2$ and $e^{-x}/2$.

b) The **hyperbolic sine** function is defined by the relation

$$\sinh x = \frac{e^x-e^{-x}}{2}, \qquad \text{for all } x \in \mathbb{R}.$$

The function sinh satisfies the conditions $\sinh 0 = 0$ and $\sinh(-x) = -\sinh x$, for all



$x \in \mathbb{R}$. Thus, the sinh function is odd and its graph is symmetric about the origin.

c) The **hyperbolic tangent** function is defined as the quotient of the hyperbolic sine and hyperbolic cosine :

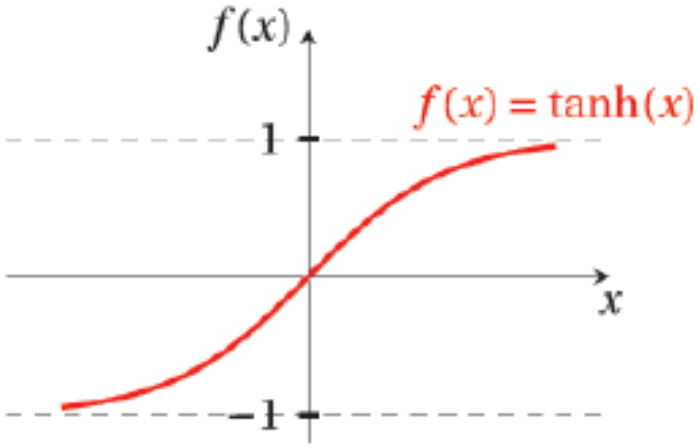
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \text{for all } x \in \mathbb{R}.$$

The hyperbolic tangent is an odd function ($\tanh 0 = 0$ and $\tanh(-x) = -\tanh x$, for all $x \in \mathbb{R}$) that takes real values between -1 and 1, and its graph is symmetric with respect to the origin.

d) The **hyperbolic cotangent** function is defined as the quotient of the hyperbolic cosine and hyperbolic sine :

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \text{for all } x \in \mathbb{R} - \{0\}.$$

The hyperbolic cotangent is an odd function, $\coth(-x) = -\coth x$, for all $x \in \mathbb{R}$, that takes real values in $]-\infty, -1[\cup]1, +\infty[$.



	Domain	Range
cosh	$\mathbb{R}$	$[1, +\infty[$
sinh	$\mathbb{R}$	$\mathbb{R}$
tanh	$\mathbb{R}$	$]-1, 1[$
coth	$\mathbb{R} - \{0\}$	$]-\infty, -1[\cup]1, +\infty[$.

Example 5.3.1 *Solve in* $\mathbb{R}$ *the equation :* $e^{2x} - \cosh x - \sinh x = 6$.

We have

$$\begin{aligned} e^{2x} - \cosh x - \sinh x = 6 &\Leftrightarrow e^{2x} - \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right) = 6 \\ &\Leftrightarrow 2e^{2x} - \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 12 \\ &\Leftrightarrow 2e^{2x} - 2e^x - 12 = 0 \\ &\Leftrightarrow t^2 - t - 6 = 0 \\ &\Leftrightarrow t = 3, \quad \textit{or } t = -2 \textit{ (not admissible).} \end{aligned}$$

So $t = e^x = 3 \Rightarrow x = \ln 3$.

5.3.2 Properties of hyperbolic functions

Proposition 5.3.1 *For all $x \in \mathbb{R}$, we have the following identities :*

1. $\cosh x + \sinh x = 1$.
2. $\cosh^2 x - \sinh^2 x = 1$.
3. $1 - \tanh^2 x = \frac{1}{\cosh^2 x}$.

Proof.

1. We have
$$\begin{aligned}\cosh x + \sinh x &= \left(\frac{e^x + e^{-x}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\\ &= e^x.\end{aligned}$$

2. We have
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{(e^{2x} + 2 + e^{-2x})}{4} - \frac{(e^{2x} + 2 + e^{-2x})}{4}\\ &= \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

3. We have
$$\begin{aligned}\cosh x + \sinh x &= \left(\frac{e^x + e^{-x}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\\ &= e^x.\end{aligned}$$

Now we present the main summation (addition) formulas for hyperbolic functions.

Proposition 5.3.2 *For all $x, y \in \mathbb{R}$, we have :*

1. $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$.

2. $\sinh(x-y) = \sinh x \cosh y - \sinh y \cosh x$.

3. $\sinh(2x) = 2\sinh x \cosh x$.

4. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

5. $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$.

6. $\cosh(2x) = \cosh^2 x + \sinh^2 x$.

7. $\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$, $\qquad \tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$.

8. $\tanh(x-y) = \frac{\tanh x - \tanh y}{1-\tanh x \tanh y}$.

The following proposition presents some remarkable limits for hyperbolic functions which are very similar to trigonometric ones.

Proposition 5.3.3 *We have the following limits :*

$$\lim_{x\to 0}\frac{\sinh x}{x} = 1, \qquad \lim_{x\to 0}\frac{\tanh x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cosh x}{x^2} = \frac{-1}{2}.$$

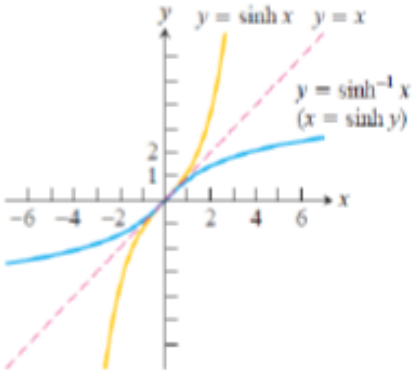
5.3.3 Inverse hyperbolic functions

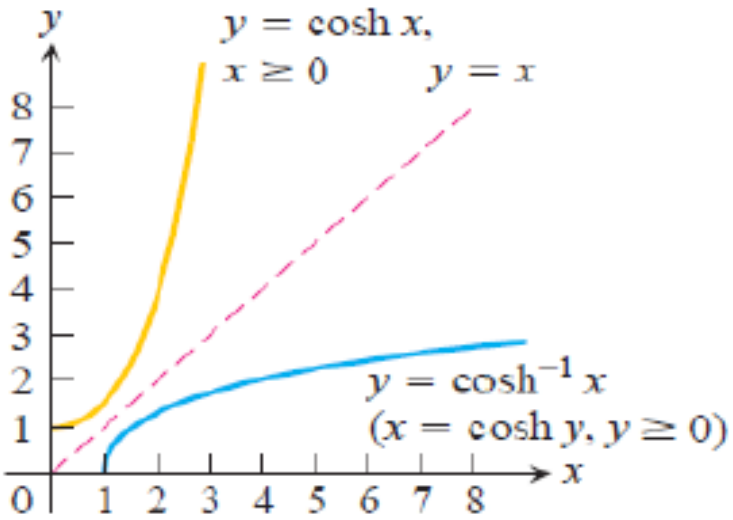
a) The hyperbolic sine is a one-to-one function and therefore it has an inverse and is denoted by $\sinh^{-1}$ or $\arg\sinh$. We have $\sinh : \mathbb{R} \to \mathbb{R}$ and $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$ with :

$$y = \sinh x \Leftrightarrow x = \sinh^{-1} y = x, \quad \forall x, y \in \mathbb{R}.$$

The inverse hyperbolic sine function, $\sinh^{-1}$, is also an odd function.

b) The hyperbolic cosine function is not one-to-one. In order to find its inverse function we need to restrict its domain into $x \geq 0$. Then the function $\cosh x : [0, +\infty[\to [1, +\infty[$





has an inverse, denoted $\cosh^{-1}$ or $\arg\cosh$, where $\cosh^{-1} x : [1, +\infty[\to [0, +\infty[$ and

$$y = \cosh x \Leftrightarrow x = \cosh^{-1} y = x, \quad \forall x \in [0, +\infty[, \quad y \in [1, +\infty[.$$

c) The hyperbolic tangent is a bijective function and thus it admits an inverse and is denoted by $\tanh^{-1}$ or $\arg\tanh$. We have $\tanh : \mathbb{R} \to]-1, 1[$ and $\tanh^{-1} :]-1, 1[\to \mathbb{R}$ with :

$$y = \tanh x \Leftrightarrow x = \tanh^{-1} y = x, \quad \forall x \in \mathbb{R}, \quad y \in]-1, 1[.$$

We mention that the inverse hyperbolic tangent function, $\tanh^{-1}$, is an odd function.

d) The hyperbolic cotangent is a bijective function and thus it admits an inverse and is denoted by $\coth^{-1}$ or $\arg\coth$. We have $\coth : \mathbb{R} - \{0\} \to]-\infty, -1[\cup]1, +\infty[$ and $\coth^{-1} :]-\infty, -1[\cup]1, +\infty[\to \mathbb{R} - \{0\}$ with :

$$y = \coth x \Leftrightarrow \coth^{-1} y = x, \quad \forall x \in \mathbb{R}, \quad y \in]-\infty, -1[\cup]1, +\infty[.$$

We mention that the inverse hyperbolic tangent function $\coth^{-1}$ is an odd function.
Using the logharithmic form, the previous inverse hyperbolic functions can be written as :

$$\begin{aligned}
\sinh^{-1} x &= \ln(x + \sqrt{x^2+1}), \qquad x \in \mathbb{R}, \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2-1}), \qquad x > 1, \\
\tanh^{-1} x &= \frac{1}{2}\ln(\frac{1+x}{1-x}), \qquad |x| < 1, \\
\coth^{-1} x &= \frac{1}{2}\ln(\frac{1-x}{1+x}), \qquad |x| > 1.
\end{aligned}$$

5.3.4 Derivatives and integrals for hyperbolic functions

The four hyperbolic functions presented above, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined. The derivatives of hyperbolic functions are as follows:

$$\begin{aligned}
\cosh'(x) &= \sinh(x), & \sinh'(x) &= \cosh(x), \\
\tanh'(x) &= \frac{1}{\cosh^2(x)}, & \coth'(x) &= -\frac{1}{\sinh^2(x)}.
\end{aligned}$$

Again, there are similarities with trigonometric functions. Applying the chain rule, we get the following formulas, where u is a function of x:

$$\begin{aligned}
\cosh'(u) &= u' \sinh(u), & \sinh'(u) &= u' \cosh(u), \\
\tanh'(u) &= \frac{u'}{\cosh^2(u)}, & \coth(u) &= -\frac{u'}{\sinh^2(u)}.
\end{aligned}$$

Example 5.3.2 *a) If* $f(x) = \sinh(x^4 + 7x)$ *then*

$$f'(x) = (x^4 + 7x)' \cosh(x^4 + 7x) = (4x^3 + 7)\cosh(x^4 + 7x).$$

b) If $g(x) = \tanh(\sqrt{x^3 + 1}) + \cosh(e^{3x} - 5x)$ then

$$\begin{aligned} g'(x) &= \frac{(\sqrt{x^3 + 1})'}{\cosh^2(\sqrt{x^3 + 1})} + (e^{3x} - 5x)' \sinh(e^{3x} - 5x) \\ &= \frac{3x^2}{2\sqrt{x^3 + 1}\cosh^2(\sqrt{x^3 + 1})} + (3e^{3x} - 5)\sinh(e^{3x} - 5x). \end{aligned}$$

The derivatives of inverse hyperbolic functions are given as follows:

$$\begin{aligned} &\frac{d}{dx}(\sinh^{-1} x) = \frac{d}{dx}(\ln(x + \sqrt{x^2 + 1})) = \frac{1}{\sqrt{x^2 + 1}}, \qquad x \in \mathbb{R}, \\ &\frac{d}{dx}(\cosh^{-1} x = \frac{d}{dx}(\ln(x + \sqrt{x^2 - 1})) = \frac{1}{\sqrt{x^2 - 1}}, \qquad x > 1, \\ &\frac{d}{dx}(\tanh^{-1} x = \frac{d}{dx}(\frac{1}{2}\ln(\frac{1 + x}{1 - x})) = \frac{1}{1 - x^2}, \qquad |x| < 1, \\ &\frac{d}{dx}(\coth^{-1} x = \frac{d}{dx}(\frac{1}{2}\ln(\frac{1 - x}{1 + x})) = \frac{1}{1 - x^2}, \qquad |x| > 1. \end{aligned}$$

Using the derivative formulas, we derive the following integral expressions:

$$\begin{aligned} &\int \sinh(x)dx = \cosh(x) + C, && \int \cosh(x)dx = \sinh(x)dx + C, \\ &\int \frac{1}{\cosh^2(x)}dx = \tanh(x) + C, && -\int \frac{1}{\sinh^2(x)}dx = \coth(x) + C, \\ &\int \frac{1}{\sqrt{x^2 + 1}}dx = \sinh^{-1} x + C, && \int \frac{1}{\sqrt{x^2 - 1}}dx = \cosh^{-1} x + C, \end{aligned}$$

where C is a constant in $\mathbb{R}$.

Example 5.3.3 1) *For then integral* $H = \int x\cosh 5x dx$ *we use integration by parts with*

$$u = x, \qquad du = dx, \qquad dv = \cosh 5x dx, \qquad v = \frac{1}{5}\sinh 5x.$$

Substituting in the formula of integration by parts

$$\begin{aligned} H = \int udv = uv - \int vdu \\ \int x\cosh 5x dx = \frac{1}{5}x\sinh 5x - \frac{1}{5}\int \sinh 5x dx \\ = \frac{1}{5}x\cosh 5x - \frac{1}{25}\sinh 5x + C. \end{aligned}$$

2) We have

$$K = \int \frac{\cosh^2 x}{\tanh x}dx = \ln|\tanh x| + C.$$

5.3.5 Integrals leading to inverse hyperbolic functions

Inverse hyperbolic functions appear when the integrand involves expressions like $\sqrt{x^2+a^2}, \sqrt{x^2-a^2}, a^2-x^2$. Using a the substitution $u = \frac{x}{a}$ we can deduce the following integrals :

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+a^2}}dx &= \sinh^{-1}(\frac{x}{a}) + C, \quad a > 0, \\ \int \frac{1}{\sqrt{x^2-a^2}}dx &= \cosh^{-1}(\frac{x}{a}) + C, \quad x > a > 0, \\ \int \frac{1}{a^2-x^2}dx &= \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C, \quad \text{if } x^2 < a^2, \\ \int \frac{1}{a^2-x^2}dx &= \frac{1}{a}\coth^{-1}(\frac{x}{a}) + C, \quad \text{if } x^2 > a^2. \end{aligned}$$

Example 5.3.4 *Evaluate the following integral*

$$\int_0^1 \frac{4dx}{\sqrt{3+4x^2}}.$$

Solution. *Let $u = 2x$ and $a = \sqrt{3}$, then $du = 2dx$ and when $x = 0$ then $u = 0$ and when*

$x = 1$ then $u = 2$. We have

$$\begin{aligned}\int_0^1 \frac{4dx}{\sqrt{3+4x^2}} &= \int_0^2 \frac{2du}{\sqrt{a^2+u^2}} \\ &= 2\sinh^{-1}(\frac{u}{a})\Big]_0^2 \\ &= 2\sinh^{-1}(\frac{u}{\sqrt{3}})\Big]_0^2 \\ &= 2\sinh^{-1}(\frac{2}{\sqrt{3}}) - 2\sinh^{-1}(\frac{0}{\sqrt{3}}) \\ &= 1.9733.\end{aligned}$$

5.4 Exercises

Exercise 5.4.1 *Find the domain of definition of the following functions:*

1. $f(x) = \arcsin(x^2 - 1)$,
2. $g(x) = \frac{\arccos(5x+3)}{x^2+x}$,
3. $h(x) = \frac{\arctan(\ln x)}{\sqrt{4-x}}$.

Solution.

Exercise 5.4.2 *Find the derivative of the following functions*:

1. $f(x) = \arctan(5\sqrt{x})$,
2. $f(t) = \left[e^t + \arcsin(t^4)\right]^5$,
3. $f(t) = \arccos(e^{2t})$.

Solution.

1. We have
$$f'(x) = \frac{(5\sqrt{x})'}{1+(5\sqrt{x})^2} = \frac{5}{2\sqrt{x}(1+25x)}.$$

2. We have
$$\begin{aligned} f'(t) &= \left(e^t + \arcsin(t^4)\right)' \left[e^t + \arcsin(t^4)\right]^4 \\ &= \left(e^t + \frac{4t^3}{\sqrt{1-t^8}}\right) \left[e^t + \arcsin(t^4)\right]^4. \end{aligned}$$

3. We have
$$f'(t) = -\frac{2e^{2t}}{\sqrt{1-e^{4t}}}.$$

Exercise 5.4.3 *Evaluate the following integrals*

$I = \int \frac{dx}{49+x^2}dx, \qquad J = \int \frac{e^{3x}}{\sqrt{1-e^{6x}}}dx, \qquad K = \int \frac{(\arctan(3x))^6}{1+9x^2}dx.$

Solution. The first integral is equal to
$$I = \int \frac{dx}{49+x^2}dx = \frac{1}{7}\arctan\left(\frac{x}{7}\right) + C.$$

For the second one, we have
$$\begin{aligned} J &= \int \frac{e^{3x}}{\sqrt{1-e^{6x}}}dx = \frac{1}{3}\int \frac{(e^{3x})'}{\sqrt{1-(e^{3x})^2}}dx \\ &= \frac{1}{3}\arcsin(e^{3x}) + C. \end{aligned}$$

Concerning the last integral, we have
$$\begin{aligned} K &= \int \frac{(\arctan(3x))^6}{1+9x^2}dx = \frac{1}{3}\int (\arctan x)'(\arctan(3x))^6 dx \\ &= \frac{1}{21}(\arctan(3x))^7 + C. \end{aligned}$$

Exercise 5.4.4 *Solve the following equations :*

$$a)\ \cosh x = 2, \quad b)\ \sinh x = \tfrac{3}{4},$$
$$c)\ \tfrac{\sinh 2x}{1+\cosh 2x} = \tfrac{1}{2}, \quad d)\ (\cosh x + 7)\,(3\cosh x - 1) = 0.$$

Solution. $a)$ We know that $\cosh x \in [1, +\infty[$. Since $2 \in [1, +\infty[$, then the equation $\cosh x = 2$ has solutions. We have

$$\cosh x = 2 \Leftrightarrow \frac{e^x + e^{-x}}{2} = 2 \Leftrightarrow e^x + e^{-x} = 4$$
$$\Leftrightarrow e^{2x} + 1 = 4e^x. \qquad (1)$$

Let $t = e^x > 0$, then (1) becomes $t^2 - 4t + 1 = 0$. So $t = 2 + \sqrt{3}$ or $t = 2 - \sqrt{3} > 0$. Therefore $t = e^x \Rightarrow x = \ln t$ so $x = \ln(2 + \sqrt{3})$ or $x = \ln(2 - \sqrt{3})$.

$b)$ To solve the equation $\sinh x = \frac{3}{4}$.

We use the identity $\cosh^2 x = \sinh^2 x + 1$ and $\cosh x > 0$, then

$$\cosh x = \sqrt{\sinh^2 x + 1} = \sqrt{1 + \left(\frac{3}{4}\right)^2} = \frac{5}{4}.$$

Using the formula $\cosh x + \sinh x = e^x$, we get $\frac{3}{4} + \frac{5}{4} = x = e^x$. So $e^x = 2$, then $x = \ln 2$. (We can solve it in the same way as question a)).

$c)$ We have

$$\frac{\sinh 2x}{1 + \cosh 2x} = \frac{1}{2} \Rightarrow 2\sinh 2x = 1 + \cosh 2x$$
$$\Rightarrow e^{2x} - e^{-2x} = 1 + \frac{e^{2x} + e^{-2x}}{2}$$
$$\Rightarrow 2(e^{2x} - e^{-2x}) = 2 + e^{2x} + e^{-2x}.$$

Multiplying by e^{2x} we get $e^{4x} - 2e^{2x} - 3 = 0$ (2). Let $t = e^{2x} > 0$, (2) becomes

$t^2-2t-3=0$ then $t=3$ and $t=-1$ (is not admissible). Therefore the solution is $x=\frac{\ln 3}{2}$, since $t=e^{2x}=3$.

d) $(\cosh x+7)(3\cosh x-1)=0 \Rightarrow \cosh x=-7$ (not admissible) or $\cosh x=\frac{1}{3}$(not admissible also because $\cosh x\geq 1$). So our problem hasn't a solution.

Exercise 5.4.5 *Find the derivative of the following functions* :

1. $f(x)=\sinh^4(x^7+3x)$,
2. $g(x)=\sqrt{4x+\cosh^2 5x}$,
3. $h(x)=\tanh\sqrt{x^3+1}$,
4. $l(x)=\arg\sinh(x^6)$.

Solution.

1. For the derivative of the first function we have

$$\begin{aligned} f'(x) &= 4\left(\sinh(x^7+3x)\right)'\sinh^3(x^7+3x) \\ &= 4(x^7+3x)'\cosh(x^7+3x)\sinh^3(x^7+3x) \\ &= 4(7x^6+3)\cosh(x^7+3x)\sinh^3(x^7+3x). \end{aligned}$$

2. The derivative of the function g is given by :

$$\begin{aligned} g'(x) &= \left(\sqrt{4x+\cosh^2 5x}\right)' \\ &= \frac{\left(4x+\cosh^2 5x\right)'}{2\sqrt{4x+\cosh^2 5x}} \\ &= \frac{4+10\sinh 5x\cosh 5x}{2\sqrt{4x+\cosh^2 5x}} \\ &= \frac{2+5\sinh 5x\cosh 5x}{\sqrt{4x+\cosh^2 5x}}. \end{aligned}$$

3. For the function h we have :

$$h'(x) = \frac{\left(\sqrt{x^3+1}\right)'}{\left(\cosh\sqrt{x^3+1}\right)^2} = \frac{3x^2}{2\sqrt{x^3+1}\left(\cosh\sqrt{x^3+1}\right)^2}.$$

4. The derivative of the function l is :

$$l'(x) = \left(\arg\sinh(x^6)\right)' = \frac{6x^5}{\sqrt{1+(x^6)^2}}.$$

Exercise 5.4.6 *Evaluate the following integrals :*

$$H = \int x\sinh 3x dx, \quad I = \int \tanh(5x)dx, \qquad J = \int \cosh(3x+1)dx,$$
$$K = \int (\frac{1}{\cosh^2 x})e^{\tanh x}dx.$$

Solution.

1) For $H = \int x\sinh 3x dx$ we use integration by parts with

$$u = x, \quad du = dx, \qquad dv = \sinh x dx, \quad v = \frac{1}{3}\cosh 3x.$$

Substituting in the formula of integration by parts

$$\begin{aligned} H = \int udv &= uv - \int vdu \\ \int x\sinh 3x dx &= \frac{1}{3}x\cosh 3x - \frac{1}{3}\int \cosh 3x dx \\ &= \frac{1}{3}x\cosh x - \frac{1}{9}\sinh 3x + C. \end{aligned}$$

2) For the second integral we have

$$\begin{aligned} I &= \int \tanh(5x)dx = \int \frac{\sinh(5x)}{\cosh(5x)}dx \\ &= \frac{1}{5}\int \frac{5\sinh(5x)}{\cosh(5x)}dx = \frac{1}{5}\int \frac{(\cosh(5x))'}{\cosh(5x)}dx \\ &= \ln|\cosh(5x)| + C. \end{aligned}$$

3) To evaluate J, we use substitution method. Let $z = 3x + 1$ then $dz = dx$. We have

$$\begin{aligned} J &= \int \cosh(3x+1)dx = \frac{1}{3}\int \cosh z dz \\ &= \frac{1}{3}\sinh z + C = \frac{1}{3}\sinh(3x+1) + C. \end{aligned}$$

4) We have

$$K = \int (\frac{1}{\cosh^2 x})e^{\tanh x}dx = \int (\tanh x)' e^{\tanh x}dx = e^{\tanh x} + C.$$

Exercise 5.4.7 *Find* $\int \frac{1}{\sqrt{x^2+2x+5}}dx$.

Solution. Completing the square : $x^2 + 2x + 5 = (x+1)^2 + 4$.

We use the following substitution : $u = x + 1$, $du = dx$. Then :

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+2x+5}}dx &= \int \frac{1}{\sqrt{u^2+4}}dx \\ &= \sinh^{-1}(\frac{u}{2}) + C \\ &= \sinh^{-1}(\frac{x+1}{2}) + C. \end{aligned}$$

Exercise 5.4.8 *Evaluate the following integral :* $\int \frac{1}{x^2+4x+1}dx$, *with* $(x+2)^2 < 3$.

Completing the square on $x^2 + 4x + 4$. We have

$$x^2 + 4x + 1 = x^2 + 4x + 4 - 3 = (x + 2)^2 - 3.$$

Then

$$\begin{aligned}
\int \frac{1}{x^2 + 4x + 1} dx &= \int \frac{dx}{(x + 2)^2 - 3} \\
&= \int \frac{dx}{(x + 2)^2 - (\sqrt{3})^2} \\
&= -\int \frac{dx}{(\sqrt{3})^2 - u^2} \\
&= -\sqrt{3} \tanh^{-1}(\frac{u}{\sqrt{3}}) + C \\
&= -\sqrt{3} \tanh^{-1}(\frac{x + 2}{\sqrt{3}}) + C.
\end{aligned}$$