Chapter 4

Matrices

4.1 Matrices

Matrices are arrays of numbers. The solution of a certain number of problems in linear algebra boils down to manipulations on matrices. As we will see in this chapter, this is true for the solution of linear systems."

4.1.1 Notions and definitions

Throughout this chapter, $\mathbb{k}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$, and n and p denote nonzero natural numbers.

Definition 4.1.1 *A matrix A of dimension (size) $n \times p$ with coefficients in $\mathbb{k}$ is a rectangular array of numbers in $\mathbb{k}$ with n rows and p columns. These numbers are called the coefficients of matrix A.*

Notations

1. Matrices are represented by capital letters ($A, B, C, \cdots$ etc.).

2. A matrix A is represented between two parentheses or two brackets.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{bmatrix}$$

or even $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$ or $A = (a_{ij})$.

3. We denote bya_{ij} the element (or coefficient) of the matrix located in the $i^{ème}$ row and $j^{ème}$ column (these indices indicate the address or position of each element).

4. The dimension (size, order) of the matrix is denoted by $\dim A$: $\dim A = n \times p,\ \ n$, where n denotes the number of rows and p the number of columns.

5. The set of matrices of dimension $n \times p$ with coefficients in $\mathbb{k}$ is denoted by $M_{n,p}(\mathbb{k})$.

Remark 4.1.1 *We must not confuse a_{ij} which is an element with (a_{ij}) which is a matrix whose elements are the a_{ij}.*

Example 4.1.1 *Let* $A = \begin{pmatrix} 1 & 0 & \sqrt{3} \\ \pi & -2 & 1+\sqrt{5} \end{pmatrix}$ *and* $B = \begin{pmatrix} i & 3+2i & 0 \\ 1 & 0 & e \\ -7 & 0 & 11 \end{pmatrix}$.

We have $\dim A = 2 \times 3$ *and* $\dim B = 3 \times 3$. *We have for example* $a_{12} = 0$, $a_{23} = 1+\sqrt{5}$, $b_{11} = i$, $b_{12} = 1$, $b_{33} = 11$.

4.1.2 Particular matrices

1. The zero matrix of dimension $n \times p$ is the matrix whose entries are all zero; it is denoted $0_{n,p}$. For example :

$$0_{3,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2. The matrix A is a row matrix (row vector) if $\dim A = 1 \times p$ $(n = 1)$: $A = (a_{11}, a_{12}, \ldots, a_{1p})$.

   For example: $A = \left(-7, 4, \sqrt{3}\right)$.

3. The matrix A is a column matrix (column vector) if $\dim A = n \times 1$ $(p = 1)$:

   $$A = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{n1} \end{pmatrix}. \text{ Par example: } A = \begin{pmatrix} i \\ 0 \\ \pi \end{pmatrix}.$$

4. The matrix A is a square matrix if $n = p$ (The number of rows is equal to the number of columns). In this case we denote $M_{n,n}(\mathbb{k})$ by $M_n(\mathbb{k})$. The elements of a_{11}, $a_{22}, \ldots, a_{nn}$ form the principal (main) diagonal.

5. The identity matrix, also called the unit matrix, denoted by I_n is a square matrix of order n where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$. For example : $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. The matrix A is diagonal matrix if A is square matrix and if $a_{ij} = 0$ when $i \neq j$. For example $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

7. A is an upper triangular matrix if A is square matrix and $a_{ij} = 0$ if $i > j$. Par example:

   $$A = \begin{pmatrix} 5 & -2 & 11 \\ 0 & \sqrt{2} & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

8. A is a lower triangular matrix if A is square matrix and $a_{ij} = 0$ if $i < j$. Par example:

$$A = \begin{pmatrix} \pi & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 8 & -1 \end{pmatrix}.$$

4.1.3 Equality of two matrices

Two matrices are equal if they have the same dimensions and if the coefficients (or elements) in the same positions are equal:

$$\dim A = \dim B = n \times p \ \text{ and } \ a_{ij} = b_{ij}, \quad \forall 1 \leq i \leq n, 1 \leq j \leq p.$$

Example 4.1.2 *Let* $A = \begin{pmatrix} -1 & 0 & 5 \\ -2 & 1 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & -2 \\ 0 & 1 \\ 5 & 3 \end{pmatrix}$.
We have $\dim A = 2 \times 3$ *and* $\dim B = 3 \times 2$. *Then* $A \neq B$ *because* $\dim A \neq \dim B$.

Example 4.1.3 *Find the value of* x *and* y *such that* $A = B$, *where :*

$$A = \begin{pmatrix} -2 & 7 \\ y^2 - 3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 2x - 1 \\ 6 & 0 \end{pmatrix}.$$

Since the two matrices have the same dimension, we have $A = B$ *if and only if:*

$$\begin{cases} y^2 - 3 = 6 \\ 2x - 1 = 7 \end{cases} \Rightarrow \begin{cases} y = \pm 3 \\ x = 4 \end{cases}.$$

4.1.4 Matrix operations

This section presents some fundamental operations on the set of matrices $M_{n,p}(\mathbb{k})$.

Addition of matrices

Definition 4.1.2 *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *two matrices of dimension* $n \times p$. *Their sum is the matrix* $C = (\ c_{ij})$ *of dimension* $n \times p$ *defined by :*

$$c_{ij} = a_{ij} + b_{ij}.$$

(Elements in the same position are added together). We note $C = A + B$.

Remark 4.1.2

- *It is not possible to add two matrices of different dimensions.*
- *Likewise, matrix subtraction can be defined.* $C = A - B$ *is the matrix of dimension* $n \times p$ *defined by* $c_{ij} = a_{ij} - b_{ij}$.

Product of a Matrix by a Scalar

Definition 4.1.3 *Let* $A = (a_{ij}) \in M_{n,p}(\mathbb{k})$ *a matrix and* $\alpha \in \mathbb{k}$ *a scalar. The product of the matrix* A *by the scalar* α *give a new matrix denoted* $\alpha A \in M_{n,p}(\mathbb{k})$ *and defined by :* $\alpha A = \alpha\,(a_{ij}) = (\alpha a_{ij})$ *(i.e. we multiply every coefficient of* A *by* α*).*

Example 4.1.4 *Let :* $A = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -5 & 0 \\ 7 & 0 & 8 \end{pmatrix}$.
Evaluate $A+2B$, $B+C$. *The matrices* A *and* B *have the same dimension* $(\dim A = 2 \times 2 = \dim B)$. *Thus* A *and* $2B$ *have also the same dimension then* $A + 2B$ *is well defined:*

$$\begin{aligned} A + 2B &= \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix} + 2 \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2-6 & 5+0 \\ -1+4 & 0+2 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 3 & 2 \end{pmatrix}. \end{aligned}$$

We remark that $\dim B = 2 \times 2 \neq \dim C = 2 \times 3$, *so* $B + C$ *isn't well defined it cannot be calculated).*

Proposition 4.1.1 *Let* $A, B, C \in M_{n,p}(\mathbb{k})$ *and* $\alpha, \beta \in \mathbb{k}$. *We have the following properties :*

1. *Commutativity :* $A + B = B + A$.
2. *Associativity :* $(A + B) + C = A + (B + C)$.
3. *The identity element for the addition :* $A + 0_{n,p} = 0_{n,p} + A = A$.
4. *The inverse element :* $A - A = A + (-A) = (-A) + A = 0_{n,p}$.
5. *Associativity of the product by a scalar :* $(\alpha\beta)A = \alpha(\beta A)$.
6. *Distributivity of scalar multiplication over addition :* $(\alpha + \beta)A = \alpha A + \beta A$.
7. *Identity element for scalar multiplication :* $1A = A$.

Remark 4.1.3 *One can show that* $M_{n,p}(\mathbb{k})$ *equipped with the two operations defined above is a* $\mathbb{k}$*-vector space.*

Matrix product (multiplication)

Let $A = (a_{ij}) \in M_{n,p}(\mathbb{k})$ and $B = (b_{ij}) \in M_{p,q}(\mathbb{k})$ be two matrices. The product of this two matrices $C = AB$ is a new matrix in $M_{n,q}(\mathbb{k})$ whose coefficients c_{ij} are defined by :

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

In other words, the element c_{ij} is the result of the product scalar of the row i of the matrix A with the column j of the matrix B.

Example 4.1.5 *Let:* $A = \begin{pmatrix} 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 0 \\ 2 & 1 \\ 4 & -7 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -5 \\ 7 & 0 \end{pmatrix}$.

Evaluate the matrix product AB and AC if it is possible.

We have : $\dim A = 2 \times 3$ *and* $\dim B = 3 \times 2$. *Then the number of columns of A is equal to the number of rows of B then the product $AB$is well defined with* $\dim AB = 2 \times 2$ *and we have :*

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 2 & 1 \\ 4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times (-3) + 5 \times 2 + 4 \times 0 & 2 \times 0 + 5 \times 1 + 0 \times (-7) \\ -1 \times (-3) + 0 \times 2 + 3 \times 4 & (-1) \times 0 + 0 \times 1 + 3 \times (-7) \end{pmatrix} \\ &== \begin{pmatrix} 4 & 5 \\ 15 & -21 \end{pmatrix}. \end{aligned}$$

he number of columns of A is not equal to the number of rows of C, since $\dim A = 2 \times 3$ *and* $\dim C = 2 \times 2$, *then the product AC is not defined.*

The matrix product has properties different from those of real-number multiplication :

- The product AB is not always well define: The number of columns of A must be equal to the number of rows of B.
- In general $AB \neq BA$: the matrix product is not commutative, even in the case of two square matrices. Consider

$$A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}.$$

We have

$$AB = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix} \neq BA = \begin{pmatrix} 10 & 2 \\ 29 & -2 \end{pmatrix}.$$

- $AB = 0 \not\Rightarrow A = 0$ or $B = 0$, *i.e.* There exist A and B non-zero matrices such that $AB = 0$. We have for example :

$$A = \begin{pmatrix} 0 & 4 \\ 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- $AB = AC \not\Rightarrow B = C$. For instance we have:

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 \\ 5 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$$

and $AB = AC = \begin{pmatrix} -5 & -4 \\ 15 & 12 \end{pmatrix}$.

Proposition 4.1.2 *Let A, B and C be matrices of dimension $n \times p$, $p \times q$ and $q \times m$ respectively α and β two scalars. We have the following properties :*

1. *Associativity of matrix product:* $A(BC) = (AB)C$.
2. *Left distributivity with respect to matrix addition:* $A(B + C) = AB + AC$.
3. *Right distributivity with respect to matrix addition:* $(A + B)C = AC + BC$.
4. *Associativity with respect to scalar multiplication:* $(\alpha A)(\beta B) = \alpha\beta(AB)$.
5. $I_n A = AI_p = A$, $A0_{p,r} = 0_{n,r}$ *and* $0_{r,n}A = 0_{r,p}$.

Definition 4.1.4 *Lett $A \in M_n(\mathbb{k})$ a square matrix. The successive powers of A are defined as :*

$$A^0 = I_n, \ A^1 = A, \ A^{k+1} = A^k A, \ k \in \mathbb{N}.$$

That is:

$$A^k = \underbrace{AA\cdots A}_{k\ times}.$$

Proposition 4.1.3 *Let $A, B \in M_n(\mathbb{k})$. We have*

1) $(A+B)^2 = A^2 + AB + BA + B^2$,

2) $(A+I_n)^2 = A^2 + 2A + I_n$.

Example 4.1.6 *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

- Calculate A^2, A^3, A^4.

- Deduce $A^p, p \geq 4$.

We find

$$A^2 = AA = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad A^3 = A^2A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

$$A^4 = A^3A = \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

From the previous calculations, we can deduce that

$$A^p = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix}.$$

This formula can be proved by mathematical induction.

Transposition de matrices

Definition 4.1.5 *Let $A = (a_{ij}) \in M_{n,p}(\mathbb{k})$. The transposed matrix of A, denoted A^t is obtained by transposing (swapping) the rows and columns of A :*

$A = (a_{ij}) \in M_{n,p}(\mathbb{k}) \Rightarrow A^t = (a_{ji}) \in M_{p,n}(\mathbb{k})$.

Example 4.1.7 *Let* $A = \begin{pmatrix} 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix}$. *The transposed matrix of* A *is* $A^t = \begin{pmatrix} 2 & -1 \\ 5 & 0 \\ 0 & 3 \end{pmatrix}$.

Moreover

$$\dim A = 2 \times 3 \Rightarrow \dim A^t = 3 \times 2.$$

Proposition 4.1.4 *Let* A *and* B *two matrices and* $\alpha \in \mathbb{k}$ *a scalar. We have :*

1. $(A^t)^t = A$,
2. $(A + B)^t = A^t + B^t$,
3. $(\alpha A)^t = \alpha A^t$,
4. $(AB)^t = B^t A^t$.

Definition 4.1.6 *Let* $A = (a_{ij}) \in M_n(\mathbb{k})$ *be a square matrix.*

1) A *is said to be symmetric matrix if and only if* $A^t = A$.

2) A *is said to be symmetric matrix if and only if skew-symmetric (antisymmetric) if and only if* $A^t = -A$.

Example 4.1.8 *1) Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{pmatrix}$. *We have* $A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{pmatrix} = A$, *so the matrix* A *is symmetric.*

2) Let $B = \begin{pmatrix} 0 & 5 & 7 \\ -5 & 0 & -6 \\ -7 & 6 & 0 \end{pmatrix}$. *We have* $B^t = \begin{pmatrix} 0 & -5 & -7 \\ 5 & 0 & 6 \\ 7 & -6 & 0 \end{pmatrix} = -B$, *so the matrix* B *is skew-symmetric (antisymmetric).*

Definition 4.1.7 *The trace of a square matrix $A \in M_n(\mathbb{k})$, denoted trA, is the sum of the diagonal elements of A. In other words:*

$$trA = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example 4.1.9 *Let* $A = \begin{pmatrix} 2 & -2 & -7 \\ 2 & 6 & 3 \\ 8 & -3 & 5 \end{pmatrix}$. *Then* $trA = a_{11} + a_{22} + a_{33} = 2 + 6 + 5 = 13$.

Proposition 4.1.5 *Let $A, B \in M_n(\mathbb{k})$, then:*

1. $tr(A + B) = trA + trB$,
2. $tr(\alpha A) = \alpha tr(A)$,
3. $tr(A^t) = tr(A)$,
4. $tr(AB) = tr(BA)$.

4.2 Determinants

The determinant is a number associated with a square matrix $A \in M_n(\mathbb{k})$. The determinant serves as an important tool in matrix computations and the solution of linear systems. Before presenting its general formula, we start by giving its expression for matrices of small sizes.

The case of a 1×1 matrix

In this case $A = a_{11}$ and the determinant is defined by : $\det A = a_{11}$.

The case of a 2×2 matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The determinant of A is the number:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The case of a 3×3 matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. In this case the determinant is defined by :

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In the general case, the determinant is defined recursively.

Definition 4.2.1 *Let $A \in M_n(\mathbb{k})$ be a square matrix. We denote by A_{ij}, be the matrix obtained by deleting the $i-th$ row and $j-th$ column of A. Expanding along an arbitrary row the determinant is given by :*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \qquad 1 \leq i \leq n,$$

alternatively, the determinant can be computed by expanding along an arbitrary column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \qquad 1 \leq j \leq n.$$

The number $\det A_{ij}$ is called the minor of order $n-1$ of the matrix A and $(-1)^{i+j} \det A_{ij}$ is called cofactor of A relative to the coefficient a_{ij}. The term $(-1)^{i+j}$ determines the sign

of the cofactor and creates an alternating pattern of plus and minus signs, similar to the black-and-white pattern of squares on a checkerboard.

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

Example 4.2.1 *Let* $B = \begin{pmatrix} 5 & 9 \\ 3 & 7 \end{pmatrix}$. *Then* $\det(B) = \begin{vmatrix} 5 & 9 \\ 3 & 7 \end{vmatrix} = 5 \times 7 - 3 \times 9 = 8.$

Example 4.2.2 *Let* $A = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & -1 & -3 \end{pmatrix}$.

By expanding along the first row, we obtain the determinant :

$$\det(A) = \begin{vmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & -1 & -3 \end{vmatrix} = 3 \begin{vmatrix} 2 & 3 \\ -1 & -3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 3 \\ 4 & -3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}$$

$$= 3(-6+3) + (-3-12) + 0 = -24.$$

Expanding now along the third column we get :

$$\det(A) = \begin{vmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & -1 & -3 \end{vmatrix}$$

$$= 0 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} - 3 \begin{vmatrix} 3 & -1 \\ 4 & -1 \end{vmatrix} + (-3) \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = -24.$$

Sarrus's Rule

This rule applies only to matrices of order 3.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then we have

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - [a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}].$$

Example 4.2.3 *Evaluate the determinant using Sarrus's rule:* $\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$. *We have:*

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 3 & 2 & 1 \end{vmatrix} = [2 \times (-1) \times 1 + 1 \times 3 \times 3 + 0 \times 1 \times 2] - [0 \times (-1) \times 3 + 2 \times 3 \times 2 + 1 \times 1 \times 1]$$

$$= 6.$$

Proposition 4.2.1 *The determinant of diagonal or an upper (or lower) triangular matrix A is equal to the product of the diagonal elements*

$$\det(A) = a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

Example 4.2.4 *We have :* $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & 7 \\ 0 & 0 & 4 \end{vmatrix} = 1 \times (-8) \times 4 = -32.$

Remark 4.2.1 *It is easy to see*

$$\det(I_n) = 1 \text{ et } \det(0_n) = 0.$$

Theorem 4.2.1 *Let A, $B \in M_n(\mathbb{k})$ be two matrices and $\lambda \in \mathbb{k}$ be a scalar. Then :*

1. $\det(AB) = \det(A)det(B)$.
2. $\det(A^t) = \det(A)$.
3. $\det(\lambda A) = \lambda^n \det(A)$.
4. $\det A = 0$ *if A contains a zero column (respectively, a zero row).*
5. $\det A = 0$ *if A contains two equal colones (resp. rows) .*
6. $\det A = 0$ *if one of he columns(resp. rows) of A is a linear combination of the other columns (resp. rows).*
7. *The determinant of A does not change when we add to one column (or one row) a linear combination of the other columns (or rows).*
8. *The determinant changes sign if we make an odd number of row or column swaps, such as swapping two rows or two columns.*

Example 4.2.5 *1) The determinant* $\begin{vmatrix} 1 & 5 & 2 \\ -1 & 7 & 4 \\ 0 & 6 & 3 \end{vmatrix}$ *The determinant is zero because the second column C_2 is a linear combination of the others columns: $C_2 = C_1 + 2C_3$.*

2) $\begin{vmatrix} 3 & 1 & 5 \\ 4 & 2 & 0 \\ -1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 5 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{vmatrix} = 5 \times 2 \times (-1) = -10.$

We have multiplied by -1 because the first and the third columns were interchanged.

4.3 Inverse matrices

Definition 4.3.1 *Let $A \in M_n(\mathbb{k})$be a square. We say that A is invertible (or non-singular) if there is a square matrix $B \in M_n(\mathbb{k})$ such that:*

$$AB = I_n \qquad and \qquad BA = I_n,$$

where is I_n the identity matrix of order n. In this case, the matrix B is called the inverse of A denoted A^{-1}.

Remark 4.3.1 *It is easy to verify that the matrix I_n is invertible with $I_n^{-1} = I_n$ and that the zero matrix 0_n is not invertible.*

Example 4.3.1 *Determine if the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$ is invertible. We will look for a matrix $B \in M_2(\mathbb{R})$ such that*

$$AB = I_2 \quad and \quad BA = I_2.$$

We have :

$$AB = I_2 \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} a + 2c = 1 \\ b + 2d = 0 \\ 5c = 0 \\ 5d = 1 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -\frac{2}{5} \\ c = 0 \\ d = \frac{1}{5} \end{cases}.$$

Then $B = \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{pmatrix}$. Moreover, we have

$$\begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is to say $BA = I_2$. *Then the matrix* A *is invertible and its inverse is*

$$A^{-1} = B = \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{pmatrix}.$$

Theorem 4.3.1 *The matrix* $A \in M_n(\mathbb{k})$ *is invertible if and only if* $\det A \neq 0$.

Example 4.3.2 *Let* $A = \begin{pmatrix} \alpha & 3 \\ 9 & \alpha^2 \end{pmatrix}$. *We have* $\det(A) = \alpha^3 - 27$. *The matrix* A *is invertible if*

$$\det(A) = \alpha^3 - 27 \neq 0 \Rightarrow \alpha \neq 3.$$

Proposition 4.3.1 *Let* $A, B, C \in M_n(\mathbb{k})$. *We have the following properties :*

1. *If* A *is invertible, then its inverse is unique and* $\det(A^{-1}) = \frac{1}{\det(A)}$.
2. *If* A *is invertible, then* A^{-1} *is also invertible and we have* $(A^{-1})^{-1} = A$.
3. *If* A *is invertible, then* A^t *is also invertible and we have* $(A^t)^{-1} = (A^{-1})^t$.
4. *If* A *and* B *are invertible, then* AB *is also invertible and we have* $(AB)^{-1} = B^{-1}A^{-1}$.
5. *If* C *is invertible, then* $AC = BC \Longrightarrow A = B$.

Let $A \in M_n(\mathbb{k})$ is an invertible matrix and $a_n, a_{n-1}, \cdots, a_1 \in \mathbb{R}$. If

$$a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A = I_n$$

then :

$$A\left(a_n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right) = I_n \Longrightarrow A^{-1} = \left(a_n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right).$$

Example 4.3.3 *Let* $A = \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Evaluate* $2A - A^2$ *and deduce* A^{-1}.

We have

$$A^2 = \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$2A - A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$2A - A^2 = I_3 \Rightarrow A(2I_3 - A) = I_3 \Rightarrow A^{-1} = (2I_3 - A).$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.3.1 Calculation of the inverse matrix using the cofactor method

We present here a practical method for calculating the inverse of a matrix.

Definition 4.3.2 *Let* $A \in M_n(\mathbb{k})$. *We call the cofactor matrix* $C = com(A)$ *(or the comatrix) the matrix of the coefficients* $c_{ij} = (-1)^{i+j} \det A_{ij}$ *where* A_{ij} *is the matrix obtained by deleting the* $i-th$ *row and the* $j-th$ *column of* A.

Theorem 4.3.2 *If* A *is invertible then* $A^{-1} = \frac{1}{\det(A)} (com(A))^t$.

Example 4.3.4 *Applying this theorem for a matrix of order* 2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Assume that $\det(A) = ad - bc \neq 0$, *then* A^{-1} *exists. We have*

$$com(A) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} +d & -c \\ -b & +a \end{pmatrix}, \ (com(A))^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Finally

$$A^{-1} = \frac{1}{\det(A)} (com(A))^t = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 4.3.5 *Let* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}$. *We have* $\det(A) = -2 \neq 0$, *then* A *is invertible.*
We start by evaluating the cofactor matrix $com(A)$ *:*

$$com(A) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \quad with \quad c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$c_{11} = + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \qquad c_{12} = - \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0 \qquad c_{13} = + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$c_{21} = - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad c_{22} = + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad c_{23} = - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$c_{31} = + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \qquad c_{32} = - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2 \qquad c_{33} = + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

Then:

$$com(A) = \begin{pmatrix} -3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix} \Rightarrow com(A)^t = \begin{pmatrix} -3 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore :

$$A^{-1} = \frac{1}{\det(A)} (com(A))^t = \frac{1}{-2} \begin{pmatrix} -3 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

4.4 Eigenvalues and eigenvectors

Let $A \in M_n(\mathbb{k})$ a square matrix of size $n \times n$.

Definition 4.4.1 *We say that the scalar* $\blacksquare$ *is an eigenvalue of* A *if there exists a column vector* $V \neq 0$ *such that* $AV = \lambda V$. *In this case,* V *is called an eigenvector of* A *corresponding to the eigenvalue* λ.

Example 4.4.1 *Lett* $A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ *and* $V = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

We have

$$AV = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Thus V is an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

Theorem 4.4.1 *Let* $\lambda_i, i = 1, \ldots n$ *distinct eigenvalues of the square matrix* A *and let* V_i *an eigenvector of* A *corresponding to the eigenvalue* λ_i. *So the vectors* $V_i, i = 1, \ldots n$ *are linearly independent.*

We now introduce a method to determine the eigenvalues.

Definition 4.4.2 *Let* $\lambda \in \mathbb{k}$ *be an eigenvalue of the matrix* A. *The subset* E *defined by*

$$E = \{V \in \mathbb{k}^n, \quad AV = \lambda V\},$$

is a vector subspace of $\mathbb{k}^n$, *It is the eigenspace associated with the eigenvalue* λ.

Proposition 4.4.1 *Let $A \in M_n(\mathbb{k})$ be a square matrix, $I \in M_n(\mathbb{k})$ the identity matrix and let $\lambda \in \mathbb{k}$ be a scalar. Then*

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow \det(A - \lambda I) = 0.$$

Definition 4.4.3 *Let $A \in M_n(\mathbb{k})$ be a square matrix. The characteristic polynomial of A is defined as*

$$\chi_A(X) = \det(A - XI).$$

We mention that $\chi_A(X)$ is a polynomial of degree n. Therefore

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow \chi_A(\lambda) = 0.$$

The equation $\chi_A(\lambda) = 0$ is the characteristic equation of A. The roots of this equation are the eigenvalues of A.

Corollaire 4.4.1 *1. If $A \in M_n(\mathbb{R})$ then A has at most n eigenvalues.*

2. If $A \in M_n(\mathbb{C})$ then A has at least one eigenvalue.

Example 4.4.2 *Let* $A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ *then* $A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{pmatrix}$.

We have

$$\det(A - \lambda I) - \lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2.$$

So A has two eigenvalues $\lambda = 1$ and $\lambda = 2$, where is $\lambda = 2$ a repeated (double) root. For

$\lambda = 1$ *and* $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *we have*

$$AV - \lambda IV = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then $\begin{pmatrix} 2y+2z \\ y+z \\ 2y-x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

So we can take as first eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda = 2$ *we have :*

$$AV - \lambda IV = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then $\begin{pmatrix} 2y-x+2z \\ z \\ 2y-x+2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

So the second eigenvector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Proposition 4.4.2 *When a matrix $A \in M_n(\mathbb{k})$ has n eigenvalues, the sum of the eigenvalues is equal to the trace of A, that is, the sum of the diagonal elements of A :*

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = tr(A) = a_{11} + a_{22} + \ldots + a_{nn}.$$

4.5 Diagonalization

Definition 4.5.1 *Let $A \in M_n(\mathbb{k})$. We say that A is diagonalizable over $\mathbb{k}$ if there exists an invertible matrix P and a diafonal matrix D such that*

$$A = PDP^{-1}.$$

Theorem 4.5.1 ***(Diagonalization Theorem)***

A matrix $A \in M_n(\mathbb{k})$ is diagonalizable over $\mathbb{k}$ if and only if A Its eigenvectors are linearly independent.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the colones of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues of A, arranged in the same order as the eigenvectors in P.

A matrix $A \in M_n(\mathbb{k})$ has n distinct eigenvalues is diagonalizable.

Example 4.5.1 *Let us determine if* $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ *is diagonalizable.*

***1**st step* : *Finding the eigenvalues. We have*

$$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda)^2 (1-\lambda) = 0.$$

The eigenvalues are $\lambda = 1$ *and* $\lambda = 2$ *(repeated root).*

***2**nd **step** : Search for linearly independent eigenvectors. By solving* $(A - \lambda I)V = 0$, *for each eigenvalue* λ, *we find. For* $\lambda = 1$, $V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ *and for* $\lambda = 2$, $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ *and* $V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

***3**rd**step** Construction of* P :

$$P = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

***4**th**step** Construction of* D *from the corresponding eigenvalues*

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

One can check that $A = PDP^{-1}$, *indeed, it suffices to compare* AP *and* PD. *We have*

$$AP = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

$$PD = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

4.6 Systems of linear equations

4.6.1 Definitions and notations

Definition 4.6.1 *Let $n, p \geq 1$ be two integers. We call system of linear equations of n equations and p unknowns $x_1, x_2, \ldots, x_p$ the system :*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{np}x_p = b_n \end{cases} . \qquad (S)$$

The coefficients a_{ij} and the second members b_i are elements of $\mathbb{k}$. The unknowns $x_1, x_2, ..., x_p$ are to be found in $\mathbb{k}$.

Definition 4.6.2

1. *We call solution of the system (S) all vector $X = (x_j)_{1 \leq j \leq p}$, $x_j \in \mathbb{k}$ satisfying all the equations of the system simultaneously.*

2. *Solving (S) means determining all possible solutions.*

3. *Two systems are said to be equivalent if they have the same set of solutions.*

Example 4.6.1 *The vector* $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *is a solution of the system*

$$\begin{cases} 2x - y = 1 \\ 5x - 2y = 4 \end{cases},$$

since the vector X *satisfies the two equations of the system simultaneously :*

$$\begin{cases} 2 \times 2 - 3 = 1 \\ 5 \times 2 - 2 \times 3 = 4 \end{cases}.$$

Proposition 4.6.1 *A system of linear equations has either no solution, exactly one solution, or infinitely many solutions. A linear system that has no solution is called inconsistent or impossible*

4.6.2 Matrix form

The system (S) can be written in the following matrix form : $AX = B$ where :

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}.$$

$X = (x_j)_{1 \leq j \leq p}$: The vector of unknowns

$B = (b_j)_{1 \leq j \leq p}$: the vector of the right-hand side

$A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$: The matrix of coefficients of (S).

Example 4.6.2 *The matrix form of the following system :*

$$\begin{cases} 2x + 3y - 4z = 5 \\ 5x - y + 2z = -7 \\ x + 2y + 6x = 11 \end{cases}$$

is $AX = B$ *where*

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 5 & -1 & 2 \\ 1 & 2 & 6 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 \\ -7 \\ 11 \end{pmatrix}.$$

4.6.3 Methods for Solving Linear Systems

We are interested in solving the system (S) in the case where $n = p$. In this case the system (S) has the following matrix form : $AX = B$, where $A \in M_n(\mathbb{k})$ is a square matrix. There are two cases :

I) If $\det(A) = 0$ the system (S) has no solution or the system has infinitely many solutions.

II) If $\det(A) \neq 0$ the system (S) has a unique solution. In what follows, we present two different methods for computing this solution.

The Matrix Inversion Method

This method makes use of the inverse of the coefficient matrix. Since $\det(A) \neq 0$ implies that A is invertible (A^{-1} exists), then we can compute the solution as follow :

$$\begin{aligned} AX = B &\Leftrightarrow A^{-1}(AX) = A^{-1}B \\ &\Leftrightarrow (A^{-1}A)X = A^{-1}B \\ &\Leftrightarrow I_n X = A^{-1}B \\ &\Leftrightarrow X = A^{-1}B. \end{aligned}$$

Example 4.6.3 *Consider the following system:*

$$\begin{cases} x+3y+4z=50 \\ 3x+5y-4z=2 \\ 4x+7y-2z=31. \end{cases} .$$

The matrix form of the system is :

$$AX=B \text{ with } A=\begin{pmatrix} 1 & 3 & 4 \\ 3 & 5 & -4 \\ 4 & 7 & -2 \end{pmatrix}, \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 50 \\ 2 \\ 31 \end{pmatrix}.$$

We have $\det A=-8\neq 0$ *so the solution admits a unique solution* $X=A^{-1}B$. *Let us evaluate the inverse matrix* A^{-1}. *We begin by evaluating the cofactor matrix:*

$$com(A)=\begin{pmatrix} +\begin{vmatrix} 5 & -4 \\ 7 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 4 & -2 \end{vmatrix} & +\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ 7 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 4 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 4 \\ 5 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 3 & -4 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} 18 & -10 & 1 \\ 34 & -18 & 5 \\ -32 & 16 & -4 \end{pmatrix}$$

Then

$$A^{-1} = \frac{1}{\det(A)} (com(A))^t = \frac{1}{-8} \begin{pmatrix} 18 & 34 & 1 \\ -10 & -18 & 16 \\ 1 & 5 & -4 \end{pmatrix}.$$

The solution is given by

$$X = A^{-1}B = \frac{1}{-8} \begin{pmatrix} 18 & 34 & 1 \\ -10 & -18 & 16 \\ 1 & 5 & -4 \end{pmatrix} \begin{pmatrix} 50 \\ 2 \\ 31 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

Cramer's Method

If $\det(A) \neq 0$ the system (S) which is equivalent to : $AX = B$, $A \in M_n(\mathbb{k})$ admits a unique solution given by :

$$x_i = \frac{\det A_i}{\det A} = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}{\det A}, \quad \text{for } 1 \leq i \leq n.$$

The matrix A_i is obtained by replacing the $i-th$ column of A with the vector of constants B.

Example 4.6.4 *Consider the system :*

$$\begin{cases} 2x + y - 3z = 5 \\ 3x - 2y + 2z = 5 \\ 5x - 3y - z = 16. \end{cases} . \qquad (S_1)$$

The matrix form of the system (S_1) *is :*

$$AX = B \quad \text{where } A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & 2 \\ 5 & -3 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ x \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 \\ 5 \\ 16 \end{pmatrix}.$$

We have $\det(A) = 26 \neq 0$ *then* (S_1) *admits a unique solution. We will evaluate this solution by Cramer's method :*

$$x = \frac{\det A_x}{\det A} = \frac{\begin{vmatrix} 5 & 1 & -3 \\ 5 & -2 & 2 \\ 16 & -3 & -1 \end{vmatrix}}{26} = 1,$$

$$y = \frac{\det A_y}{\det A} = \frac{\begin{vmatrix} 2 & 5 & -3 \\ 3 & 5 & 2 \\ 5 & 16 & -1 \end{vmatrix}}{26} = -3,$$

$$z = \frac{\det A_z}{\det A} = \frac{\begin{vmatrix} 2 & 1 & 5 \\ 3 & -2 & 5 \\ 5 & -3 & 16 \end{vmatrix}}{26} = -2.$$

Therefore the solution is $X = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$.

Example 4.6.5 *Solve* $\begin{cases} 2x - y = 3 \\ 4x - 2y = 5 \end{cases}$.

The coefficients matrix is $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. *We have* $\det(A) = 0$. *Consequently, the*

system either admits infinitely many solutions or has no solution. From the first equation of the system, we deduce that:

$$y = 2x - 3.$$

By substituting the value of y into the second equation, we obtain

$$4x - 2(2x - 3) = 5 \Leftrightarrow 6 = 5,$$

contradiction. Then the system has no solution.

Example 4.6.6 *Let :* $\begin{cases} x - 3y = 1 \\ 3x - 9y = 3 \end{cases}$.

The coefficients matrix is $A = \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix}$. *We have* $\det(A) = 0$. *Using the first row of the system we get :*

$$x = 3y + 1.$$

Replacing the value xin the second row we get:

$$3(3y + 1) - 9y = 3 \Leftrightarrow 3 = 3,$$

which is always true $\forall y \in \mathbb{R}$. *As a result, this system admits infinitely many solutions, given by*

$$\left\{ \begin{pmatrix} 3y + 1 \\ y \end{pmatrix}, \quad y \in \mathbb{R} \right\}.$$

4.7 Exercises

Exercise 4.7.1 *Find the real numbers x, y, z,and w such that :*

$$3\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}.$$

Solution.

The initial equation is equivalent to

$$3\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{pmatrix}.$$

Identifying the element of same position we get :

$$\begin{array}{rcl} 3x &=& x+4 \\ 3y &=& 6+x+y \\ 3z &=& -1+z+w \\ 3w &=& 2w+x \end{array} \Rightarrow \begin{array}{rcl} 2x &=& 4 \\ 2y &=& 6+x \\ 2z &=& -1+w \\ w &=& 3 \end{array}$$

Therefore, the solution is : $x = 2$, $y = 4$, $z = 1, w = 3$.

Exercise 4.7.2 *Let the matrices :*

$$A = \begin{pmatrix} -3 & 4 \\ 0 & 5 \\ 1 & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 3 \end{pmatrix},\ D = \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}.$$

1. *Find a matrix C such that $A - 3B + C = 0_{3\times 2}$.*

2. *Evaluate AB, A^tB and B^tA.*

3. *Evaluate the matrix D.*

Solution.

1. $A-3B+C=0_{3\times 2} \Leftrightarrow C=3B-A=3\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 3 \end{pmatrix}-\begin{pmatrix} -3 & 4 \\ 0 & 5 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 6 & -7 \\ 0 & 1 \\ 8 & 10 \end{pmatrix}$.

2. The number of columns of A is different to the number of rows of B so we can't evaluate AB.

   We have $A^t=\begin{pmatrix} -3 & 0 & 1 \\ 4 & 5 & -1 \end{pmatrix}$. The number of columns of A^t is equal to the number of rows of B, so we can perform the product A^tB and we have :

$$A^tB=\begin{pmatrix} -3 & 0 & 1 \\ 4 & 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 3 \end{pmatrix}=\begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix}.$$

   We have also :

$$B^tA=\begin{pmatrix} -3 & 0 & 1 \\ 4 & 5 & -1 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 0 & 5 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 10 & -13 \\ -13 & 42 \end{pmatrix}.$$

3. The matrix D is equal to

$$\begin{aligned} D &= \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix} \\ &= \begin{pmatrix} a+b+c & a^2+b^2+c^2 & b^2+2ac \\ a+b+c & b^2+2ac & a^2+b^2+c^2 \\ 3 & a+b+c & a+b+c \end{pmatrix}. \end{aligned}$$

Exercise 4.7.3 *1. Find for which values of t the following matrix is invertible*

$$A = \begin{pmatrix} t+3 & -1 & 1 \\ 5 & t-3 & 1 \\ 6 & -6 & t+4 \end{pmatrix}.$$

2. If $t = 0$, find the inverse of A using the cofactor method.

Solution. We have :

$$\det A = \begin{vmatrix} t+3 & -1 & 1 \\ 5 & t-3 & 1 \\ 6 & -6 & t+4 \end{vmatrix}$$
$$= t^3 + 4t^2 - 4t - 16 = (t+4)(t-2)(t+2).$$

A is invertible if $\det A \neq 0$, so if $t \in \mathbb{R} - \{-4, 2, -2\}$.

2. If $t = 0$ then $A = \begin{pmatrix} 3 & -1 & 1 \\ 5 & -3 & 1 \\ 6 & -6 & 4 \end{pmatrix}$ and $\det(A) = -16 \neq 0$, so A is invertible. The comatrix of A is: $\begin{pmatrix} -6 & -14 & -12 \\ -2 & 6 & 12 \\ 2 & 2 & -4 \end{pmatrix}$. Then we have :

$$A^{-1} = \frac{1}{\det A}(com(A))^t = \frac{-1}{16}\begin{pmatrix} -6 & -14 & -12 \\ -2 & 6 & 12 \\ 2 & 2 & -4 \end{pmatrix}^t$$

$$= \frac{-1}{16}\begin{pmatrix} -6 & -2 & 2 \\ -14 & 6 & 2 \\ -12 & 12 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{7}{8} & -\frac{3}{8} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

Exercise 4.7.4 *Find for which value of m the following matrix is invertible :*

$$D = \begin{pmatrix} m & 0 & 1 & 2m \\ 1 & m & 0 & 0 \\ 0 & 2m+2 & m & 1 \\ m & 0 & 0 & m \end{pmatrix}.$$

Solution. We have :

$$\det D = \begin{vmatrix} m & 0 & 1 & 2m \\ 1 & m & 0 & 0 \\ 0 & 2m+2 & m & 1 \\ m & 0 & 0 & m \end{vmatrix}.$$

Replacing C_4 by $C_4 - C_1$ we get :

$$\det D = \begin{vmatrix} m & 0 & 1 & m \\ 1 & m & 0 & -1 \\ 0 & 2m+2 & m & 1 \\ m & 0 & 0 & 0 \end{vmatrix}.$$

The determinant is computed by expanding along the 4th row:

$$\begin{aligned} \det D &= -m \begin{vmatrix} 0 & 1 & m \\ m & 0 & -1 \\ 2m+2 & m & 1 \end{vmatrix} \\ &= -m(m^3 - 3m - 2) = -m(m+1)^2(m-2), \end{aligned}$$

The matrix D is invertible if $\det D \neq 0$, then $m \in \mathbb{R} - \{0, -1, 2\}$.

Exercise 4.7.5 *Consider the following square matrix :*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

1. *Calculate $A^3 - A$.*

2. *Deduce that A is invertible and determine A^{-1}.*

Solution. 1. We find :

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$

$$A^3 - A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 4I_3.$$

2. We have :

$$A^3 - A = 4I_3 \Leftrightarrow A[A^2 - I_3] = 4I_3$$

Therefore :

$$A \times \frac{1}{4}[A^2 - I_3] = I_3.$$

This show that A is invertible and its inverse is equal to

$$A^{-1} = \frac{1}{4}[A^2 - I_3] = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

Exercise 4.7.6 *Find the inverse of the following matrix if it exists:*

$$A = \begin{pmatrix} 5 & 2 \\ 9 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & -1 \\ -1 & 2 & -1 \\ 5 & 3 & -4 \end{pmatrix}, C = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & -3 \\ 5 & 3 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution.

1. We have $\det A = -3 \neq 0$ then A is invertible. Its inverse is equal to $A^{-1} = \begin{pmatrix} -1 & \frac{2}{3} \\ 3 & -\frac{5}{3} \end{pmatrix}$.

2. We have $\det B = 19 \neq 0$ then A is invertible. We have :

$$B^{-1} = \frac{1}{\det A}(com(B))^t.$$

Since

$$com(B) = \begin{pmatrix} + \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} & - \begin{vmatrix} -1 & -1 \\ 5 & -4 \end{vmatrix} & + \begin{vmatrix} -1 & 2 \\ 5 & 3 \end{vmatrix} \\ - \begin{vmatrix} 3 & -1 \\ 3 & -4 \end{vmatrix} & + \begin{vmatrix} 1 & -1 \\ 5 & -4 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix} \\ + \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} & - \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -5 & -9 & -13 \\ 9 & 1 & 12 \\ -1 & 2 & 5 \end{pmatrix},$$

then

$$B^{-1} = \frac{1}{-19} \begin{pmatrix} -5 & 9 & -1 \\ -9 & 1 & 2 \\ -13 & 12 & 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{19} & -\frac{9}{19} & \frac{1}{19} \\ \frac{9}{19} & -\frac{1}{19} & -\frac{2}{19} \\ \frac{13}{19} & -\frac{12}{19} & -\frac{5}{19} \end{pmatrix}.$$

3. Since $\det C = 0$ (because the third column is equal to the difference between the second column and the first column). Then C is not invertible.

4. We have $\det D = -2$, so D is invertible. We know that :

$$D^{-1} = \frac{1}{\det A}(Com(D))^t.$$

The cofactor matrix of D is given by :

$$Com(D) = \begin{pmatrix} -3 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Then :

$$D^{-1} = -\frac{1}{2}\begin{pmatrix} -3 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Exercise 4.7.7 *Let A be a matrix in $M_3(\mathbb{R})$ given by*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}.$$

Find the eigenvalues of A and the corresponding eigenspaces.

Solution. We have

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \lambda^3 - \lambda^2 - 6\lambda.$$

Thus A has three eigenvalues $\lambda_1 = -2$, $\lambda_2 = 0$, $\lambda_3 = 3$.

Let us now find the eigenspaces. We have

$$V = (x, y, z)^t \quad \text{and } V \in E_{\lambda_i} \Leftrightarrow AV = \lambda_i V, \quad 1 \leq i \leq 3.$$

For $\lambda_1 = -2$, we have :

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}.$$

this is equivalent to $x = 0$, $y = -z$.

$$\begin{aligned} E_{\lambda_1} &= \left\{(x, y, z)^t \in \mathbb{R}^3,\ AV = \lambda_1 V\right\} \\ &= \left\{(x, y, z)^t \in \mathbb{R}^3,\ x = 0,\ y = z\right\} \\ &= \left\{(0, -z, z)^t,\ z \in \mathbb{R}\right\} \\ &= \left\langle (0, -1, 1)^t \right\rangle. \end{aligned}$$

Using the same method we find

$$E_{\lambda_2} = \left\langle (-2, 3, 1)^t \right\rangle, \quad \text{and} \quad E_{\lambda_3} = \left\langle (-5, 3, 2)^t \right\rangle.$$

Exercise 4.7.8 *Consider the following system (S), where α is a real parameter:*

$$\begin{cases} x + y + \alpha z = 0 \\ \alpha x + (\alpha + 1)y + \alpha z = -1 \\ \alpha x + y + z = -2. \end{cases} \qquad (S)$$

1. *Give the matrix form of the system (S).*

2. *Determine the value of α for which the system (S) has a unique solution.*

3. *Solve the system (S) for $\alpha = -1$.*

Solution.

1. The matrix form is $AX = B$ with :

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ \alpha & \alpha+1 & \alpha \\ \alpha & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}.$$

2. The system has a unique solution if $\det A \neq 0$.

   We have $\det A = -\alpha^3 + \alpha^2 - \alpha + 1 = (1-\alpha)(\alpha^2+1)$ then $\det A \neq 0$ if $\alpha \neq 1$.

3. For $\alpha = -1$,the system becomes :

$$\begin{cases} x + y - z = 0 \\ -x - z = -1 \\ -x + y + z = -2. \end{cases} \qquad (S)$$

4. We can use Cramer's Method to find the solution

$$x = \frac{\det Ax}{\det A} = \frac{\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix}}{4} = \frac{4}{4} = 1,$$

$$y = \frac{\det A_y}{\det A} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ -1 & -2 & 1 \end{vmatrix}}{4} = \frac{-4}{4} = -1$$

$$z = \frac{\det A_z}{\det A} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -2 \end{vmatrix}}{4} = \frac{0}{4} = 0.$$

So the solution is $x = 1, y = -1, z = 0$.

Exercise 4.7.9 *Solve the following system using Cramer's method*

$$\begin{cases} 2x + 2y + z = 1 \\ 2x + y - z = 2 \\ 3x + y + z = 3. \end{cases}$$

Solution. The matrix form is :

$$AX = B \text{ with } A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We have $\det A = -7 \neq 0$, then the system admits a unique solution, given by Cramer's method a follows :

$$x = \frac{\det Ax}{\det A} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}{-7} = \frac{-9}{-7} = \frac{9}{7}.$$

$$y = \frac{\det A_y}{\det A} = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix}}{-7} = -\frac{5}{7}.$$

$$z = \frac{\det A_z}{\det A} = \frac{\begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix}}{-7} = -\frac{1}{7}.$$

Exercise 4.7.10 *Solve the following system using the inverse method :*

$$\begin{cases} x - z = m \\ -2x + 3y + 4z = 1 \\ y + z = 2m. \end{cases} \qquad (S)$$

Solution. The matrix form of the system is $AX = B$ with :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} m \\ 1 \\ 2m \end{pmatrix}.$$

We have: $\det A = 1 \neq 0$, then our system has a unique solution given by

$$X = A^{-1}B.$$

The inverse of A is given by :

$$A^{-1} = \frac{1}{\det A}(comA)^t = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix}^t = \begin{pmatrix} -1 & -1 & 3 \\ 2 & 1 & -2 \\ -2 & -1 & 3 \end{pmatrix}.$$

Then :

$$X = \begin{pmatrix} -1 & -1 & 3 \\ 2 & 1 & -2 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} m \\ 1 \\ 2m \end{pmatrix} = \begin{pmatrix} 5m-1 \\ 1-2m \\ 4m-1 \end{pmatrix}.$$