Chapter 3

Functions of Several Variables

3.1 Basic concepts

We begin by introducing the concept of a norm.

Definition 3.1.1 *Let E be a vector space over the field $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$. A norm $\|.\|$ is a map $\|.\| : E \to \mathbb{R}^+$ with the following properties :*

i) $\|x\| = 0 \Leftrightarrow x = 0_E$ *(zero vector),*

ii) $\forall \lambda \in \Bbbk, \forall x \in E : \ \|\lambda x\| = |\lambda| \, \|x\|$ *(homogeneity),*

iii) $\forall x, y \in E : \|x + y\| \leq \|x\| + \|y\|$ *(triangular inequality).*

Example 3.1.1 *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. The space $\mathbb{R}^n$ may be equipped with one of the following norm :*

1. $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \, .$
2. $\|x\|_2 = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}.$
3. $\|x\|_\infty = \max(|x_1| \, , |x_2| \, , \cdots \, , |x_n|).$

Definition 3.1.2 *A multivariable function, also called a function of several variables, is a mapping that assigns to each ordered n-tuple of real numbers* $x = (x_1, x_2, \cdots, x_n)$ *in* $D \subset \mathbb{R}^n$ *a unique real denoted* $f(x) = f(x_1, x_2, \cdots, x_n)$ *:*

$$\begin{aligned} f: \quad & D \subset \mathbb{R}^n \to \mathbb{R} \\ & x = (x_1, x_2, \cdots, x_n) \mapsto f(x) = f(x_1, x_2, \cdots, x_n). \end{aligned}$$

Example 3.1.2 *The functions* f *and* g *defined respectively by :*

$$\begin{array}{lll} f & : \mathbb{R}^2 \to \mathbb{R} & \qquad g \quad : \mathbb{R}^3 \to \mathbb{R} \\ & (x, y) \mapsto \quad f(x, y) = 3x - 5y^2 + 11, & \qquad (x, y, z) \mapsto \quad g(x, y, z) = \frac{\cos(x+yz)}{x^2+y^2+(z-1)^2}, \end{array}$$

are multivariable functions.

Definition 3.1.3 *Let* $f : D \subset \mathbb{R}^n \to \mathbb{R}$ *a multivariable function.*

- D *is called the domain of definition of* f*, the set of all* x *where the function is defined.*
- *The set* $R_f = f(D) = \{f(x),\ x \in D\}$ *is called the range of* f*.*
- *If* $F \subset \mathbb{R}$*, the subset denoted* $f^{-1}(F) \subset D$ *is called the preimage of* F *under* f *and defined by*

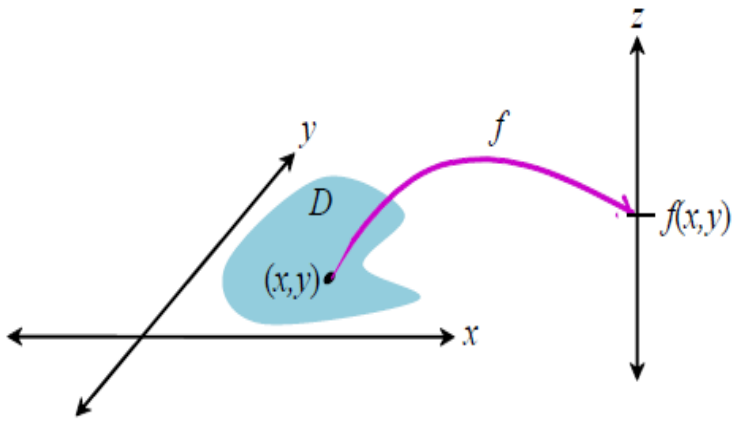
$$f^{-1}(F) = \{x \in D, \quad f(x) \in F\}$$

Example 3.1.3 *1. The domain of the function* f *defined by:*

$f(x, y) = 9x - 3y + 1$ *is* $D_f = \mathbb{R}^2$*. To determine the range, let* $z \in \mathbb{R}$ *and find a solution of the equation* $z = f(x, y)$ *so*

$$9x - 3y + 1 = z.$$

Since this equation has solution for any $z \in \mathbb{R}$*, for example* $(x, y) = (\frac{z-1}{9}, 0)$*. Hence the range is* $R_f = f(\mathbb{R}) = \mathbb{R}$*.*



2. *Let g be the function defined by $g(x,y) = \ln(5x - y)$. Then the domain of g is :*

$$\begin{aligned} D_g &= \{(x,y) \in \mathbb{R}^2,\ 5x - y > 0\} \\ &= \{(x,y) \in \mathbb{R}^2,\ 5x > y\} \\ &= \{(x,y) \in \mathbb{R}^2,\ y < 5x\}. \end{aligned}$$

D_g is the half-plane located below the line $y = 5x$. To find the range, let $z \in \mathbb{R}$ and solve the equation $z = g(x,y)$ i.e. $z = \ln(5x - y)$ which is equivalent to $e^z = 5x - y$. This equation has a solution (for instance $(x,y) = (\frac{e^z}{5}, 0) \in D_g$). Thus, the range is $Rg = g(D_g) = \mathbb{R}$.

3. *Consider the function h given by $h(x,y) = \sqrt{49 - x^2 - y^2}$. Its domain is :*

$$\begin{aligned} D_h &= \{(x,y) \in \mathbb{R}^2,\ 49 - x^2 - y^2 \geq 0\} \\ &= \{(x,y) \in \mathbb{R}^2, x^2 + y^2 \leq 49\}. \end{aligned}$$

since $x^2 + y^2 = 49$ is the equation of a circle centred at $(0,0)$ and of radius $r = 7$. The domain D_h is the interior of this circle, including the circle itself. To find the

range we observe :

$$0 \leq z = h(x,y) = \sqrt{49 - x^2 - y^2} \leq 7 \Rightarrow R_h \subset [0,7].$$

4. *On other hand, for any there exists a solution* (x,y) *for the equation* $49 - z^2 = x^2 + y^2$ *(for instance* $(x,y) = (\sqrt{49 - z^2}, 0)$*) therefore* $R_h = [0,7]$.

3.2 Limits

In what follows, $\mathbb{R}^n$ is endowed with any of the three equivalent norms $\|\cdot\|_1$, $\|\cdot\|_2$ or $\|\cdot\|_\infty$. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ a function defined on a neighborhood of a point. a except possibly at a, and let $l \in \mathbb{R}$.

Definition 3.2.1 *We say that the function* f *admits* l *for limit at the point* a *if :*

$$\forall \varepsilon > 0, \exists \delta > 0 : \quad \|x - a\| < \delta, \quad \|f(x) - l\| < \varepsilon,$$

and we write $\lim\limits_{x \to a} f(x) = l$.

For functions of a single variable, when we let x approaches a, there are only two possible directions of approach, from the let or from the right. For functions of several variables the situation isn't as simple. Indeed, if we note that $x = (x_1, x_2, \cdots, x_n)$ and $a = (a_1, a_2, \cdots, a_n)$. By $x \to a$ we mean that all coordinates of x approach the corresponding coordinates of a simultaneously and independently. There are infinitely many paths along which x can tend to a.

For example in dimension 2, a point (x,y) may tend to $(0,0)$ along infinitely many paths; for example :

- along the horizontal axis, that is, $y = 0$ and $x \to 0$,
- along the vertical axis, that is, $x = 0$ and $y \to 0$,

- along any curve, for example, the parabola $y = x^2$.

Remark 3.2.1 *The standard algebraic properties of limits (sum, difference, product, quotient, and composition) that apply to single-variable functions remain valid for multivariable functions.*

Theorem 3.2.1 *If a function has a limit at a point a then this limit is unique.*

Proposition 3.2.1 *If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.*

Here are some remarks on limits for functions of two variables.

1. The limit of $f(x,y)$ exists when (x,y) tends to (a,b) if it is independent of the path taken. In other words, if the limit exists, then $f(x,y)$ must approaches the same limit no matter how (x,y) approaches (a,b).

2. To show that a limit does not exist, it is sufficient to find two different paths that yield two different values for the limit.

3. Using the substitution $y = mx$, where m is a real parameter, approaching $(0,0)$ becomes equivalent to letting $x \to 0$. If the resulting limit depends on m, then the limit does not exist. If the limit is independent of m, no definitive conclusion can be drawn.

4. $\lim\limits_{(x,y)\to(a,b)} f(x,y) \neq \lim\limits_{x\to a}(\lim\limits_{y\to b} f(x,y))$ and $\lim\limits_{(x,y)\to(a,b)} f(x,y) \neq \lim\limits_{y\to b}(\lim\limits_{x\to a} f(x,y))$ in general.

5. The change to polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$, $r > 0$, $\theta \in [0, 2\pi[$ is a technique that helps us determine whether a limit exists as $(x,y) \to (0,0)$. Indeed, $\sqrt{x^2+y^2} = r$, so $(x,y) \to (0,0)$ is equivalent to $r \to 0$. If the limit exists, it must be independent of θ.

Proposition 3.2.2 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables defined in a neighborhood of a point (a, b), except possibly at (a, b), and let $L \in \mathbb{R}$. Assume that $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists and $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$. Assume, in addition, that for all $x \in \mathbb{R}$, $\lim\limits_{y\to b} f(x, y)$ exists and for all $y \in \mathbb{R}$, $\lim\limits_{x\to a} f(x, y)$ exists then :*

$$\lim_{x\to a}(\lim_{y\to b} f(x, y)) = L = \lim_{y\to b}(\lim_{x\to a} f(x, y)).$$

Example 3.2.1 *Evaluate the following limit :*

$$\lim_{(x,y)\to(0,0)} \frac{9x^3y}{x^2 + y^2}.$$

By converting to polar coordinates

$$(x, y) = (r\cos\theta, r\sin\theta), r > 0, \theta \in [0, 2\pi[,$$

we get:

$$\lim_{(x,y)\to(0,0)} \frac{9x^3y}{x^2 + y^2} = \lim_{r\to 0} \frac{9r^4\cos^3\theta\sin\theta}{r^2} = \lim_{r\to 0} 9r^2\cos^3\theta\sin\theta = 0.$$

Therefore, the limit exists and is equal to zero.

Example 3.2.2 *The limit $\lim\limits_{(x,y)\to(0,0)} \frac{6xy}{x^2+y^2}$ does not exist. Indeed, using polar coordinates :*

$$(x, y) = (r\cos\theta, r\sin\theta), r > 0, \theta \in [0, 2\pi[,$$

we get :

$$\lim_{(x,y)\to(0,0)} \frac{6xy}{x^2 + y^2} = \lim_{r\to 0} \frac{6r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} 6\cos\theta\sin\theta = 6\cos\theta\sin\theta.$$

It depends to θ, then the limit does not exist.

Example 3.2.3 *Calculate $\lim\limits_{(x,y)\to(0,0)} \frac{4x^2-y^2}{3x^2+y^2}$.*

Using the substitution $y = mx$ where m is a parameter ($(x, y) \to (0, 0)$ implies that $x \to 0$) we obtain that :

$$\frac{4x^2 - y^2}{3x^2 + y^2} = \frac{4x^2 - m^2x^2}{3x^2 + m^2x^2} = \frac{(4 - m^2)x^2}{(3 + m^2)x^2} = \frac{(4 - m^2)}{(3 + m^2)}.$$

Thus we have if $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{4x^2 - y^2}{3x^2 + y^2} = \lim_{x\to 0} \frac{4x^2 - m^2x^2}{3x^2 + m^2x^2} = \frac{(4 - m^2)}{(3 + m^2)},$$

It depends of m then the limit doesn't exist.

3.3 Continuity

Definition 3.3.1 *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function in a neighborhood of a point a. We say that the function f is continuous at the point a if and only if : $\lim_{x\to a} f(x) = f(a)$ and is continuous on D if it is continuous at every point of D.*

Elementary functions are continuous on their domains.

Example 3.3.1 *Let $f(x, y) = 3x^2 - 7y + 1$ and $(a, b) = (1, 2)$.*

We have :

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to(1,2)} 3x^2 - 7y + 1 = -10 = f(1, 2).$$

Then f is continuous at $(1, 2)$.

Theorem 3.3.1

1. *The sum and the product of two continuous functions on $D \subset \mathbb{R}^n$ to $\mathbb{R}$ are continuous on D and their quotient is continuous at any point on D for which the denominator is different to zero.*

2. *The composition of continuous function of $\mathbb{R}^n$ in $\mathbb{R}$ with a continuous function on $\mathbb{R}$ to $\mathbb{R}$ is continuous on $\mathbb{R}^n$.*

Example 3.3.2 *Study the continuity of the function f defined by :*

$$f(x) = \begin{cases} \frac{9x^3y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We remark that the domain of definition of f is $D_f = \mathbb{R}^2$.

1. *The function f is continuous if $(x,y) \neq (0,0)$ since the function $(x,y) \to 9x^3y$ is continuous and the function $(x,y) \to x^2+y^2$ is continuous, hence, their quotient is continuous.*

2. *Now we study the case $(x,y) = (0,0)$. The function f is continuous when $(x,y) = (0,0)$ if :*

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

We have $f(0,0) = 0$ and using the example 3.2.1 we have $\lim\limits_{(x,y)\to(0,0)} \frac{9x^3y}{x^2+y^2} = 0$, thus, the equality $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ is satisfied, then f is continuous at $(0,0)$. Consequently f is continuous on $\mathbb{R}^2$.

Example 3.3.3 *Study the continuity of f defined by :*

$$f(x) = \begin{cases} \frac{6xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The function f is defined on $\mathbb{R}^2$. It is continuous when $(x,y) \neq (0,0)$, since it is a rational function. When $(x,y) = (0,0)$ the function f is discontinuous $(0,0)$ since from example 3.2.2 the limit $\lim\limits_{(x,y)\to(0,0)} \frac{6xy}{x^2+y^2}$ does not exist. Then f is continuous on $\mathbb{R}^2 \backslash \{(0,0\}$.

3.4 Partial Derivatives

Definition 3.4.1 *Let f be a multivariable function defined on D of $\mathbb{R}^n$. We say that f admits a first-order partial derivative with respect to the variable x_i at the point $a = (a_1, a_2, \cdots, a_n) \in D$ if the single-variable function $x_i \mapsto f(a_1, a_2, \cdots, x_i, a_{i+1,} \cdots a_n)$ is differentiable at a_i. In other words, the partial derivative of f with respect to x_i at the point $a = (a_1, a_2, \cdots, a_n)$, denoted $\frac{\partial f}{\partial x_i}(a_1, a_2, \cdots, a_n)$, is defined by :*

$$\frac{\partial f}{\partial x_i}(a_1, a_2, \cdots, a_n) = \lim_{h \to 0} \frac{f(a_1, a_2, \cdots, a_i + h, a_{i+1,} \cdots a_n) - f(a_1, a_2, \cdots, a_i, a_{i+1,} \cdots a_n)}{h}$$

if this limit exists.

In the case of a function of two variables $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, we have two partial derivatives (α, β) :

$$\frac{\partial f}{\partial x}(\alpha, \beta) = \lim_{h \to 0} \frac{f(\alpha + h, \beta) - f(\alpha, \beta)}{h} \qquad \text{(partial derivative with respect of } x\text{)},$$
$$\frac{\partial f}{\partial y}(\alpha, \beta) = \lim_{h \to 0} \frac{f(\alpha, \beta + h) - f(\alpha, \beta)}{h} \qquad \text{(partial derivative with respect of } y\text{)}.$$

They can also be denoted by :

$$\frac{\partial f}{\partial x} = f'_x = \partial_x f \quad \text{and} \quad \frac{\partial f}{\partial y} = f'_y = \partial_y f.$$

Remark 3.4.1 *To evaluate the partial derivative with respect to x_i, differentiate f with respect to x_i, treating all other variables as constants.*

Example 3.4.1 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = 2xy^3 + y\sin(z) + z\cos(4x)$.*

Then :

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= 2y^3 - 4z\sin(4x),\\
\frac{\partial f}{\partial y}(x,y,z) &= 6xy^2 + \sin(z),\\
\frac{\partial f}{\partial z}(x,y,z) &= y\cos(z) + \cos(4x).
\end{aligned}$$

Example 3.4.2 *Consider the following function:*

$$f(x) = \begin{cases} \frac{6xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0),\\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- *If* $(x,y) \neq (0,0)$ *then :*

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left(\frac{6xy}{x^2+y^2}\right) = -6y\frac{x^2-y^2}{(x^2+y^2)^2},\\
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left(\frac{6xy}{x^2+y^2}\right) = 6x\frac{x^2-y^2}{(x^2+y^2)^2}.
\end{aligned}$$

- *If* $(x,y) = (0,0)$. *To determine whether partial derivatives exist, we use their definition in terms of limits. That is*

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0,\\
\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.
\end{aligned}$$

Then the partial derivatives exist at $(0,0)$ *(although* f *is not continuous at* $(0,0)$*, as illustrated in example 3.3.3).*

Remark 3.4.2 *It is well known that if a real function of a single variable is differentiable at a point, then it is continuous at that point. However, this is not the case for functions of several variables: the existence of partial derivatives at a point does not imply continuity*

at that point.

Definition 3.4.2 *Let U be an open set of $\mathbb{R}^n$. The function $f : U \to \mathbb{R}$ is of class $C^1(U)$ if all its first partial derivatives exist and continuous on U.*

Proposition 3.4.1 *Let U be an open set of $\mathbb{R}^n$. If the function $f : U \to \mathbb{R}$ is of class $C^1(U)$ then it is continuous on U.*

3.5 Higher-order partial derivatives

Definition 3.5.1 *Let f be a function defined on a domain D of $\mathbb{R}^n$. Assume that f has a partial derivative with respect of the variable x_i at the point $a = (a_1, a_2, \cdots, a_n) \in D$. We say that f admits a second partial derivative, denoted $\frac{\partial^2 f}{\partial x_j \partial x_i}$, at a point a with respect to x_j and x_i if the partial derivative function $\frac{\partial f}{\partial x_i} : D \to \mathbb{R}$ it itself has a partial derivative $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})$ with respect to x_j :*

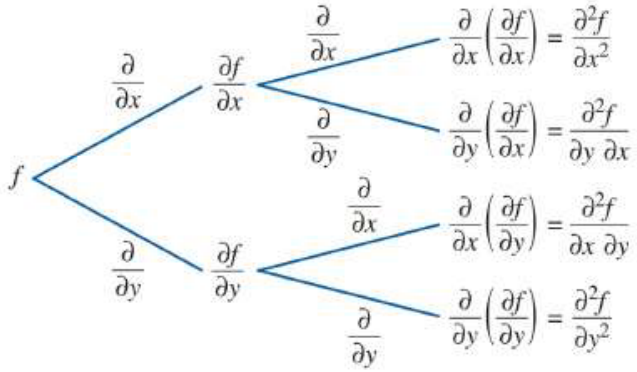
$$\frac{\partial^2 f}{\partial x_j \partial x_i} := \frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i}).$$

In the case of a function of two variables we have :

- $\frac{\partial^2 f}{\partial x \partial x}(x, y) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}(x, y))$ we differentiate twice with respect to x,
- $\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}(x, y))$ we take the partial derivative of the function with respect to x first, and then differentiate the result with respect to y.
- $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(x, y))$ we take the partial derivative of the function with respect to y first, and then differentiate the result with respect to x.
- $\frac{\partial^2 f}{\partial y \partial y}(x, y) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}(x, y))$ we differentiate twice with respect to y.

Second-order partial derivatives can be expressed using other notations

- $\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 f}{\partial x^2} = \partial_{xx} f = f''_{xx}$,



- $\frac{\partial^2 f}{\partial y \partial x} = \partial_{yx} f = f''_{xy}$,
- $\frac{\partial^2 f}{\partial x \partial y} = \partial_{xy} f = f''_{yx}$,
- $\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial^2 f}{\partial y^2} = \partial_{yy} f = f''_{yy}$.

Remark 3.5.1 *We mention that the order in which the derivatives are taken, when we use the notation* $\frac{\partial^2 f}{\partial y \partial x}$ *or* $\partial_{xy} f$*, we differentiate first with respect to* x *then with the respect of* y*. In the notation* f''_{xy} *we differentiate first with respect to* x *then with the respect of* y*.*

Example 3.5.1 *Calculate the second partial derivatives of the following function :*

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = \sin(x^2 + xy).$$

We have :

$$\frac{\partial f}{\partial x}(x, y) = (2x + y)\cos(x^2 + xy) \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x\cos(x^2 + xy).$$

Then :

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x, y)\right) = \frac{\partial}{\partial x}\left((2x + y)\cos(x^2 + xy)\right) \\ &= 2\cos(x^2 + xy) - (2x + y)^2 \sin(x^2 + xy), \end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}(x,y)) = \frac{\partial}{\partial y}((2x+y)\cos(x^2+xy))$$
$$= \cos(x^2+xy) - x(2x+y)\sin(x^2+xy),$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(x,y)) = \frac{\partial}{\partial x}(x\cos(x^2+xy))$$
$$= \cos(x^2+xy) - x(2x+y)\sin(x^2+y),$$

and

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}(x,y)) = \frac{\partial}{\partial y}(x\cos(x^2+xy))$$
$$= -x^2\sin(x^2+xy).$$

Definition 3.5.2 *Let U be an open set in $\mathbb{R}^n$. The function $f : U \to \mathbb{R}$ is of class of $C^2(U)$ if all second-order partial derivative functions exist and are continuous on U.*

Theorem 3.5.1 *(of Schwarz) Let U be an open set in $\mathbb{R}^n$. The function $f : U \to \mathbb{R}$ is of class of $C^2(U)$ then :*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \quad \text{for all } i = 1, \cdots, n \text{ and } j = 1, \cdots, n \text{ such that } i \neq j.$$

3.6 Differentiability

Definition 3.6.1 *1. Let U be an open set, $f : U \subset \mathbb{R}^n \to \mathbb{R}$ a function and $a = (a_1, a_2, \cdots, a_n) \in U$. We say that f is differentiable at a if and only if :*

$$\lim_{\|(h_1,\cdots,h_n)\|\to 0} \frac{f(a_1+h_1,\cdots,a_n+h_n) - f(a_1,\cdots,a_n) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1,\cdots,a_n)h_i}{\|(h_1,\cdots,h_n)\|} = 0.$$

In the case of $n = 2$, the previous equation is written as :

$$\lim_{\|(h_1,h_2)\|\to 0} \frac{f(a_1+h_1, a_2+h_2) - f(a_1,a_2) - \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}(a_1,a_2)h_i}{\|(h_1,h_2)\|} = 0.$$

2. f is differentiable on U if it is differentiable at every point of U.

Definition 3.6.2 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^1(U)$ and $a = (a_1, a_2, \cdots, a_n) \in U$. The differential of f at a point a is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ denoted Df_a such that :*

$$Df_a : h = (h_1, h_2, \cdots, h_n) \mapsto Df_a(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, a_2, \cdots, a_n)h_i.$$

Theorem 3.6.1 *If the function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point in U then it is continuous at this point.*

Remark 3.6.1 *The function f can be continuous at a but not differentiable at a.*

3.6.1 Higher-order differential

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^m(U)$, we have :

$$D^{(m)}f = \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right]^{(m)} = \sum_{p=0}^{m} C_m^p h^p k^{m-p} \frac{\partial^m f}{\partial x^p \partial y^{m-p}}.$$

For example :

$$Df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}.$$
$$D^2 f = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}.$$
$$D^3 f = h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}.$$

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a function of $C^2(U)$, we have :

$$Df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z},$$

and

$$D^2 f = h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + l^2\frac{\partial^2 f}{\partial z^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + 2hl\frac{\partial^2 f}{\partial x \partial z} + 2kl\frac{\partial^2 f}{\partial y \partial z}.$$

3.7 Gradient, Divergence and Rotational

The gradient is a vector whose components are the partial derivatives. It plays a fundamental role in physics and has numerous geometric applications, as it points in the direction normal to curves and surfaces.

Definition 3.7.1 *If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a function of n variables then the gradient of f at $x = (x_1, x_2, \cdots, x_n)$, denoted* $\operatorname{grad} f$ *or* ∇f*, is the vector function whose components are the partial derivatives of f i.e. :*

$$\operatorname{grad} f(x) = \nabla f(x) = (f_{x_1}(x), f_{x_2}(x), \ldots, , f_{x_n}(x)).$$

Remark 3.7.1 *1. If f is a function of two variables x and y, then the gradient of f is*

$$\operatorname{grad} f(x, y) = \nabla f(x, y) = \left(f_x(x, y), f_y(x, y)\right).$$

2. If f is a function of three variables x, y and z, then the gradient of f is

$$\operatorname{grad} f(x, y, z) = \nabla f(x, y, z) = \left(f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\right).$$

Example 3.7.1 *If f is defined by $f(x,y) = x\cos yz$ then*

$$\begin{aligned}\operatorname{grad} f(x,y,z) = \nabla f(x,y,z) &= (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)) \\ &= (\cos yz, -xz\sin yz, -xy\sin yz)\,.\end{aligned}$$

At the point $(3,\pi,\frac{1}{2})$ we have

$$\begin{aligned}\operatorname{grad} f(3,\pi,\frac{1}{2}) = \nabla f(3,\pi,\frac{1}{2}) &= \left(f_x(3,\pi,\frac{1}{2}), f_y(3,\pi,\frac{1}{2}), f_z(3,\pi,\frac{1}{2})\right) \\ &= \left(0, -\frac{3}{2}, -3\pi\right).\end{aligned}$$

Remark 3.7.2 *The gradient provides an alternative expression of the differential. If f is differentiable at $x \in \mathbb{R}^n$ and if $h \in \mathbb{R}^n$ then*

$$Df_a(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, a_2, \cdots, a_n)h_i = \operatorname{grad} f(x) \cdot (h_1, h_2, \cdots, h_n),$$

where $\cdot$ is the symbol of dot product.

Definition 3.7.2 *The divergence computes a scalar quantity from a vector multivariable function by differentiation.*

Definition 3.7.3 *If $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function with $f = (f_1, f_2, f_3)$, then its divergence at any point is*

$$\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

which can be written $\operatorname{div} f = \nabla \cdot f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_1, f_2, f_3)$.

Example 3.7.2 1) *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (x,y,z)$ then*

$$\begin{aligned}\operatorname{div} f(x,y,z) &= (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})(x,y,z) \\ &= 1 + 1 + 1 = 3.\end{aligned}$$

2) *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $f(x, y, z) = (3x + z^2, \sin(xy), yz^4)$ *then*

$$\begin{aligned} \operatorname{div} f(x, y, z) &= (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})(x, y, z) \\ &= 3 + x\cos(xy) + 4yz^3. \end{aligned}$$

Definition 3.7.4 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *be a function with* $f = (f_1, f_2, f_3)$. *The curl (or rotational) is a vector operator defined by*

$$\begin{aligned} \nabla \times f \equiv \operatorname{curl}(f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right). \end{aligned}$$

Example 3.7.3 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $f(x, y, z) = (0, 0, x^2y^2)$ *then*

$$\nabla \times f \equiv \operatorname{curl}(f) = \left(2x^2y, 2xy^2, 0\right)$$

3.8 Exercises

Exercise 3.8.1 *Determine the domain of the following functions :*

a) $f(x, y) = \frac{\sqrt{-y+x^2}}{\sqrt{y}}$, *b)* $g(x, y) = \frac{\ln(y)}{\sqrt{x-y}}$ *c)* $h(x, y) = \sqrt{x^2 - 2y}$

Solution.

a) $D_f = \{(x, y) \in \mathbb{R}^2,\ -y + x^2 \geq 0 \text{ and } y > 0\} = \{(x, y) \in \mathbb{R}^2,\ y \leq x^2 \text{ and } y > 0\}$.

b) $Dg = \{(x, y) \in \mathbb{R}^2,\ x > y \text{ and } y > 0\}$.

a) $Dh = \{(x, y) \in \mathbb{R}^2,\ x^2 - 2y \geq 0\} = \left\{(x, y) \in \mathbb{R}^2,\ y \leq \frac{x^2}{2}\right\}$

Exercise 3.8.2 *Let f be the function defined on $\mathbb{R}^2\setminus\{(0,0)\}$ by :*

$$f(x,y)=\frac{x\ln(1+x^3)}{y(x^2+y^2)}.$$

Calculate the limit of f when (x,y) approaches $(0,0)$.

Solution. For $m\neq 0$,we have $f(x,mx)=\frac{x\ln(1+x^3)}{mx^3(1+m^2)}=\frac{\ln(1+x^3)}{x^3}\frac{x}{m(1+m^{2)}}$ approaches 0 when x approaches 0.

But

$$f(x,x^2)=\frac{x\ln(1+x^3)}{x^2(x^2+x^4)}=\frac{\ln(1+x^3)}{x^3}\frac{1}{(1+x^2)}$$

and it approaches 1 when x approaches 0. So the limit does not exist.

Exercise 3.8.3 *Show that the $\lim\limits_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$ does not exist.*

Solution. Set $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$. First let's approach along the x-axis. Then $y=0$ gives $f(x,0)=\frac{x^2}{x^2}=1$, $\forall x\neq 0$. So $f(x,y)\to 1$ as $(x,y)\to(0,0)$ along the x-axis. We now approach along the yy-axis by putting $x=0$. Then $f(0,y)=\frac{-y^2}{y^2}=-1$, $\forall y\neq 0$. So $f(x,y)\to -1$ as $(x,y)\to(0,0)$ along the y-axis. since f has two different limits along two different lines, the given limit does not exist.

Exercise 3.8.4 *If $f(x,y)=7xy/(x^2+y^2)$. Does $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ exist?*

Solution. If $y=0$, then $f(x,0)=0/x^2=0$. Therefore, $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis.

if $x=0$, then $f(0,y)=0/y^2=0$. Therefore, $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the y-axis. We remark that we obtained identical limits along the axes, but that does not show that the given limit exists and is 0. Let's approach $(0,0)$ along another line, say $y=x$. For all $x\neq 0$, $f(x,y)=\frac{x^2}{x^2+x^2}=\frac{1}{2}$. Therefore $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along $x=y$. Then $\frac{1}{2}\neq 0$.

so the limit doesn't exists.

Exercise 3.8.5 *Consider the following function*

$$f(x,y)=\begin{cases}\frac{xy}{(x^2+y^2)^{\alpha}} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

For what values of α the function f will be continuous at the origin.

Solution. Note that $D(f)=\mathbb{R}^2$. Suppose that $\alpha\geq 1$. Then, we have

$$\lim_{(x,x)\to(0,0)} f(x,x)=\frac{1}{x^{2(\alpha-1)}}=+\infty.$$

Therefore, f is discontinuous at $(0,0)$ when $\alpha\geq 1$. Suppose that $\alpha\in\,]0,1[$ and using the fact that

$$-\left(x^2+y^2\right)\leq xy\leq\left(x^2+y^2\right),\quad \forall x,y\in\mathbb{R},$$

we get that :

$$-\left(x^2+y^2\right)^{1-\alpha}\leq\frac{xy}{(x^2+y^2)^{\alpha}}\leq\left(x^2+y^2\right)^{1-\alpha}.$$

Thus, by the squeeze theorem we obtain :

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{xy}{\left(x^2+y^2\right)^{\alpha}}&=\lim_{(x,y)\to(0,0)}\left(-\left(x^2+y^2\right)^{1-\alpha}\right)=\lim_{(x,y)\to(0,0)}\left(x^2+y^2\right)^{1-\alpha}=0\\&=f(0,0).\end{aligned}$$

So f is continuous at $(0,0)$ when $\alpha\in\,]0,1[$. Assume $\alpha\leq 0$, it follows that f is continuous at $(0,0)$ since

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\left(x^2+y^2\right)^{\alpha}}=\lim_{(x,y)\to(0,0)}xy\left(x^2+y^2\right)^{-\alpha}=0=f(0,0).$$

Consequently, f is continuous at $(0,0)$ when $\alpha<1$.

Exercise 3.8.6 *Evaluate the second-order partial derivatives of the following function :*

$$f(x,y) = \cos(3x) - x^2e^{5y} + 7y^5.$$

Solution. *We begin by computing the first-order partial derivatives :*

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= -3\sin(3x) - 2xe^{5y},\\ \frac{\partial f}{\partial y}(x,y) &= -5x^2e^{5y} + 35y^4.\end{aligned}$$

We now proceed to compute the second-order partial derivatives :

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x,y) &= -9\cos(3x) - 2e^{5y},\\ \frac{\partial^2 f}{\partial y^2}(x,y) &= -25x^2e^{5y} + 140y^3,\\ \frac{\partial^2 f}{\partial x\partial y}(x,y) &= \frac{\partial^2 f}{\partial y\partial x}(x,y) = -10xe^{5y}.\end{aligned}$$

Exercise 3.8.7 *Evaluate the second-order partial derivatives of the following function :*

$$f(x,y) = y\arctan x - x^2e^{3y}.$$

Solution. First, we compute the first-order partial derivatives :

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{y}{x^2+1} - 2xe^{3y},\\ \frac{\partial f}{\partial y}(x,y) &= \arctan x - 3x^2e^{3y}.\end{aligned}$$

We now proceed to compute the second-order partial derivatives :

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-2xy}{(x^2+1)^2} - 2e^{3y},$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = -9x^2e^{3y},$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{x^2+1} - 6xe^{3y}.$$

Exercise 3.8.8 *Consider the following function f :*

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

1) *Prove that f has first-order partial derivatives everywhere on $\mathbb{R}^2$.*

2) *Find $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$. What can you deduce ?*

Solution. If $(x,y) \neq (0,0)$ the function f is the composition of functions that are infinitely differentiable, hence f has a first-order partial derivatives at (x,y). Then if $(x,y) \neq (0,0)$ we have :

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}; \qquad \frac{\partial f}{\partial y}(x,y) = xy^2\frac{(3x^2+y^2)}{(x^2+y^2)^2}.$$

and If $(x,y) = (0,0)$ then

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

f has first-order partial derivatives everywhere on $\mathbb{R}^2$.

2) We have

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

and

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

We remark that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$. Therefore, Schwarz's theorem implies that the mixed second-order partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are not continuous at $(0,0)$.

Exercise 3.8.9 *Calculate the gradient of the following functions :*

$a) f(x,y) = x^4 y + y^2 e^x$, $\quad b)\ g(x,y,z) = z\sin x + y^2 - xyz,$, $\quad c)\ h(x,y,z) = \cos xy + z^3$.

Solution. $a)$ $\operatorname{grad} f = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)) = (4x^3 y + y^2 e^x, 2ye^x + x^4)$,

$b)$ $\operatorname{grad} g(x,y,z) = (\frac{\partial g}{\partial x}(x,y,z), \frac{\partial g}{\partial y}(x,y,z), \frac{\partial g}{\partial z}(x,y,z)) = (z\cos x - yz, 2y - xz, \sin x - xy)$,

$c)$ $\operatorname{grad} h(x,y,z) = (\frac{\partial h}{\partial x}(x,y,z), \frac{\partial h}{\partial y}(x,y,z), \frac{\partial h}{\partial z}(x,y,z)) = (-y\sin xy, -x\sin xy, 3z^2)$.

Exercise 3.8.10 *Calculate the divergence and the curl (rotational) of the following functions :*

$a)\ f(x,y,z) = (xy, x+z, zy)$, $\qquad b)\ g(x,y,z) = (x+y, y^2, 3zx)$.

Solution. $a)$ We have $f = (f_1, f_2, f_3)$ then

$$\begin{aligned} \operatorname{div} f &= (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})(x,y,z) \\ &= y + 0 + y = 2y. \end{aligned}$$

and

$$\nabla \times f \equiv \operatorname{curl}(f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x+z & zy \end{vmatrix}$$

$$= (z-1, 0, 1-x).$$

b) We have $g(x, y, z) = (g_1, g_2, g_3)$ then

$$\begin{aligned} \operatorname{div} g &= (\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z})(x, y, z) \\ &= 1 + 2y + 3x. \end{aligned}$$

and

$$\nabla \times f \equiv \operatorname{curl}(f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y^2 & 3zx \end{vmatrix}$$

$$= (0, -3z, -1).$$