Chapter 2

Differential Equations

The purpose of this chapter is to introduce several important classes of first and second-order ordinary differential equations (ODEs) and to develop the principal methods used to solve them.

2.1 Ordinary Differential Equations

We begin by introducing several definitions and general concepts related to differential equation.

Definition 2.1.1 *An ordinary differential equations (ODE) of order n it is a relation involving an unknown function $y(x)$ of a single real variable x, its derivatives $y'(x), y''(x), \cdots, y^{(n)}(x)$ and the independent variable x of the form :*

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0 \qquad (E)$$

(or, simply, $F(x, y, y', y'', \cdots, y^{(n)}) = 0$) where F is a continuous function.

- The order n of a differential equation is the order of the highest derivative that appears in the equation.

- A differential equation is called ordinary if it involves only a single independent variable.
- If the function y takes values in $\mathbb{R}$, the equation (E) is called scalar.
- If the function y takes values in $\mathbb{R}^p$, the equation (E) is called vectorial.

Example 2.1.1 *1.* $x^3y' + y^2 - 7 = 0$ *is a first-order ordinary differential equation.*

2. $y'' + (5x^6 + 8)y' - 5y = x$ *is a second-order ordinary differential equation.*

3. $y^{(4)} + (7\sin x)y' + x^3y^2 = 0$ *is a fourth-order ordinary differential equation.*

Definition 2.1.2 *A solution of* (E) *on* I*, also called an integral of* (E) *on* I*, is every function* φ *defined on the open integral* $I \subset \mathbb{R}$ *admitting derivatives up to order* n *and such that:*

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \cdots, \varphi^{(n)}(x)) = 0.$$

Solving (E) *on* I*, is to determine the full set of solutions on* I*. An ordinary differential equation may admit infinitely many solutions.*

Particular case : $y' = f(x) \Rightarrow y = \int f(x)dx$.
The solutions of this equation are the primitives of the function f.

Definition 2.1.3 *Any solution of an ordinary differential equation, whether in explicit or implicit form, is called an integral curve of the equation. The integral curves of a differential equation therefore represent the graphs of its solutions.*

Example 2.1.2 *1. Consider the following ordinary differential equation :* $y' = 2xy + 4x$*. The function*

$$y = k\exp(x^2) - 2$$

is indeed a solution on $\mathbb{R}$*, for each* $k \in \mathbb{R}$*. In fact, we have :*

$$y' = 2kx\exp(x^2)$$

On the other hand

$$2xy + 4x = 2x(k\exp(x^2) - 2) + 4x = 2kx\exp(x^2) = y'.$$

2. *The differential equation* $x^2y'' - 2y + 2x = 0$ *has the function* $y(x) = kx^2 + x$ *as a solution for all* $k \in \mathbb{R}$. *Indeed,* $y'(x) = 2kx + 1$ *and* $y''(x) = 2k$. *Substituting into the equation*

$$x^2y'' - 2y + 2x = 0,$$

we obtain :

$$2kx^2 - 2(kx^2 + x) + 2x = 0 \Leftrightarrow 0 = 0.$$

Definition 2.1.4 *Two differential equations :*

$$F_1(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0 \quad \text{and} \quad F_2(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0$$

are said to be equivalent on an open interval I *if every solution of one is also a solution of the other, and vice versa.*

It will often happen that, instead of studying a given differential equation directly, we study another that is equivalent to it.

Definition 2.1.5 *Consider the following equation* (E) *where the function* F *is continuous on the domain* $D \subset \mathbb{R}^{n+1}$ *and let* $(x_0, y_0, y_1 \cdots, y_{n-1})$ *be a point in* $D \cap \mathbb{R}^n$. *Solving the initial value problem for the differential equation* (E) *consists of finding a solution* $y = \varphi(x)$ *of this equation that satisfies the following conditions:*

$$y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1},$$

called the initial conditions.

It can be noted that one must get used to renaming functions and variables. For example:

$$x''(t) + (t+2)x' + (\cos t)x^3 = 0$$

it is a second order differential equation where the unknown is the function x, which depends on the variable t.

Remark 2.1.1 *The concept of an interval in solving a differential equation is fundamental. If the interval changes, the solution can change as well. For example, on* $I_1 =]0, +\infty[$ *the differential equation* $y' = \frac{1}{x}$ *has the solution* $y = \ln x + k$. *Whereas on the interval* $I_2 =]-\infty, 0[$, *the general solution is* $y = \ln(-x) + k$. *If the interval* I *is not specified, it is assumed to be* $\mathbb{R}$.

2.2 First-Order Differential Equations

First-Order Differential Equations is an equation of type:

$$F(x, y, y') = 0 \Leftrightarrow y' = f(x, y).$$

We have the following types :

1. Separable equations : $f(y)y' = g(x)$.
2. Homogenous equations $y' = f(\frac{y}{x})$.
3. Linear equations : $y' + f(x)y = g(x)$.
4. Bernoulli's differential equations : $y' + f(x)y = g(x)y^\alpha$, $\alpha \in \mathbb{R} - \{0, 1\}$.

2.2.1 Separable Equations

A separable differential equation is one that can be written in the form

$$f(y)y' = g(x) \qquad \text{or} \qquad f(y)\frac{dy}{dx} = g(x)\ .$$

To solve it we follow the following steps :

step 1. We separate the variables: move all terms involving y to one side and all terms involving x to the other:

$$f(y)\,dy = g(x)\,dx$$

step 2. Integrate both sides:

$$\int f(y)dy = \int g(x)dx.$$

Example 2.2.1 *Consider the following differential equation :* $\sqrt{1-x^2}y'y^2 = 1$.

This equation is equivalent to :

$$\begin{aligned}\sqrt{1-x^2}y'y^2 = 1 \Leftrightarrow\ y'y^2 &= \frac{1}{\sqrt{1-x^2}}\\ \Leftrightarrow y^2\frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}}\\ \Leftrightarrow y^2dy &= \frac{1}{\sqrt{1-x^2}}dx\end{aligned}$$

which is a separable equation. Integrate both side we find :

$$\begin{aligned}\int y^2dy &= \int \frac{1}{\sqrt{1-x^2}}dx\\ \frac{1}{3}y^3 &= \arcsin x + C\\ y &= \sqrt[3]{3\arcsin x + 3C}.\end{aligned}$$

Example 2.2.2 *Solve the following differential equation :*

$$y' = \frac{6x^2}{2y + \cos y} \tag{2.1}$$

Then find the solution which satisfies the following initial condition $y(1) = \pi$.

The equation (2.1) *is equivalent to :*

$$\begin{aligned} (2y + cosy)dy &= 6x^2dx \\ \int (2y + cosy)dy &= \int 6x^2dx \\ y^2 + \sin y &= 2x^3 + C. \end{aligned}$$

The solution is given implicitly.

The initial condition is $y(1) = \pi$, *we replace* $x = 1$ *and* $y = \pi$ *in the above equation, we find:*

$$\pi^2 + \sin \pi = 2 + C \Rightarrow C = \pi^2 - 2.$$

Hence, the solution is implicitly given by:

$$y^2 + \sin y = 2x^3 + \pi^2 - 2.$$

2.2.2 Homogeneous Equations

Definition 2.2.1 *A differential equation is said to be homogeneous if it can be written in the following form*

$$y' = f(\frac{y}{x}).$$

In order to solve this type of equations, we use the substitution

$$z = \frac{y}{x}, \text{ so } y = zx \text{ and } y' = z'x + z.$$

Therefore :

$$y' = f(\frac{y}{x}) \Leftrightarrow xz' = f(z) - z$$
$$\Leftrightarrow \frac{1}{f(z) - z}dz = \frac{1}{x}dx.$$

This differential equation is a separable differential equation.

To solve it, we integrate :

$$\int \frac{dz}{f(z) - z} = \int \frac{1}{x}dx.$$

Example 2.2.3 *Consider the following homogeneous equation:*

$$y' = \exp(\frac{y}{x}) + \frac{y}{x} \qquad (E)$$

Let's move on to changes of variables :

$$z = \frac{y}{x} \Rightarrow y' = z + xz'.$$

We replace in (E) *we obtain:*

$$z + xz' = \exp(z) + z \Rightarrow xz' = \exp(z) \Rightarrow z' \exp(-z) = \frac{1}{x}.$$

By integrating we find :

$$\int \exp(-z)dz = \int \frac{1}{x}dx \Rightarrow -\exp(-z) = \ln|x| + C,$$

which is equivalent to $\exp(-z) = \ln \frac{K}{|x|}$. *Subsequently*

$$y = -x \ln(\ln \frac{K}{|x|}), \qquad K > 0.$$

2.2.3 Linear differential equations

We are interested here in linear ordinary differential equations.

Definition 2.2.2 *A first-order differential equation is said to be linear if it can be written in the form :*

$$y' + f(x)y = g(x),$$

(i.e. it is linear with respect to the unknown function y and its derivative y') where f and g are real functions defined and continuous on an interval I of $\mathbb{R}$. The function g is called the right-hand side of the equation.

Homogeneous linear differential equations

Definition 2.2.3 *Let f be a real continuous function on I of $\mathbb{R}$. A first-order homogeneous differential equation is a differential equation of the form:*

$$y' + f(x)y = 0. \qquad (E_H)$$

Remark 2.2.1 *The zero function is a solution of a first-order homogeneous differential equation (E_H).*

The set S_H of solutions of the differential equation (E_H) on I is given by:

$$S_H = \{y, \text{ such that } y(x) = C\exp(-F(x)),\ C \in \mathbb{R}\}$$

where F is a primitive of f on I i.e. $F(x) = \int f(x)dx$. So the solution can be written as :

$$y(x) = C\exp(-\int f(x)dx), \quad C \in \mathbb{R}.$$

Example 2.2.4 *Consider the first-order homogeneous differential equation :*

$$y' + \frac{5}{1+x^2}y = 0. \qquad (E_H)$$

Since $f(x) = \frac{5}{1+x^2}$, *we have :*

$$(E_H) \Leftrightarrow y' = -\frac{5}{1+x^2}y \Leftrightarrow y = C\exp(-\int \frac{5}{1+x^2}dx)$$
$$\Leftrightarrow y = C\exp(-5\arctan x).$$

Therefore :

$$S_H = \{y, \quad such\ that \quad y(x) = C\exp(-5\arctan x),\ C \in \mathbb{R}\}.$$

Non-homogeneous linear differential equations

Consider on the interval I the following non-homogeneous linear differential equation :

$$y' + f(x)y = g(x), \qquad (E).$$

which is a linear differential equation in which the right-hand side is not zero i.e. $g(x) \neq 0$ is not identically zero on the interval I. The associated homogeneous differential equation (E_H) to (E) is

$$y' + f(x)y = 0. \qquad (E_H).$$

Proposition 2.2.1 *Let* y_P *be a particular solution on* I *of the linear differential equation* (E) *and* (y_H) *the solutions of the differential equation* (E_H) *on* I. *The set* S *of solutions of the differential equation* (E) *on* I *is given by :*

$$S = \{y,\ y = y_H + y_P\}.$$

Example 2.2.5 *Consider the following equation* :

$$y' - xy = -x^3 \qquad (E').$$

Let verify that $y_p = x^2 + 2$ *is a particular solution of* (E') *then deduce the general solutions of* (E'). *We have* $y'_P = 2x$, *substituting in* (E') *we find :*

$$2x - x(x^2 + 2) = -x^3.$$

Then y_P *is indeed a particular solution of* (E').

The associated homogeneous differential equation to (E') *is :*

$$y' - xy = 0 \quad (E_H)$$

Its solutions are of the form $y_H = C\exp(\frac{1}{2}x^2), \quad C \in \mathbb{R}$.

Therefore the solution of (E') *are :*

$$y = y_P + y_H = x^2 + 2 + C\exp(\frac{1}{2}x^2), \qquad C \in \mathbb{R}.$$

The method of variation of parameters

We seek for a particular solution of (E) :

$$y' + f(x)y = g(x) \qquad (E)$$

of the form $y_P(x) = \lambda(x))\exp(-F(x)$ were $F(x)$ is a primitive of $f(x)$. Then $\lambda(x)$ must verify :

$$\lambda'(x)\exp(-F(x)) + \lambda(x)(-F'(x))\exp(-F(x)) + f(x)\lambda(x)\exp(-F(x)) = g(x).$$

Using the fact that $F'(x) = f(x)$, the precedent equation is equivalent to :

$$\lambda'(x)\exp(-F(x)) = g(x) \Rightarrow \lambda'(x) = g(x)\exp(F(x))$$
$$\Rightarrow \lambda(x) = \int g(x)\exp(F(x))dx.$$

Therefore, a particular solution of (E) is given by :

$$y_P(x) = \exp(-F(x))\int g(x)\exp(F(x))dx.$$

The general solution of the differential equation (E) is given by :

$$y(x) = y_H(x) + y_P(x) = \lambda\exp(-F(x)) + \exp(-F(x))\int g(x)\exp(F(x))dx, \quad \lambda \in \mathbb{R},$$

or :

$$\begin{aligned} y(x) &= y_H(x) + y_P(x) \\ &= \lambda\exp(-\int f(x)dx) + \exp(-\int f(x)dx)\int g(x)\exp(\int f(x)dx)dx, \quad \lambda \in \mathbb{R}. \end{aligned}$$

Example 2.2.6 *Consider the equation :* $y' = 2y + x \qquad (1)$.

We mention here that $f(x) = -2$ *and* $g(x) = x$. *The associated homogeneous equation* $y' = 2y$ *whose general solution is :*

$$y_H(x) = \lambda\exp(-\int f(x)dx) = \lambda\exp(-\int(-2)dx) = \lambda\exp(2x), \quad \lambda \in \mathbb{R}.$$

Let us look for a particular solution in the form:

$$y_P(x) = \lambda(x)\exp(-\int f(x)dx) = \lambda(x)\exp(2x).$$

Then $y_p(x)$ is a solution of (E) on $\mathbb{R}$ if and only if :

$$\lambda'(x)\exp(-\int f(x)dx) = g(x)$$
$$\lambda'(x)\exp(2x) = x.$$

We must find an antiderivative $\lambda(x)$ of $x\exp(-2x)$.Integrating by parts we find :

$$\lambda(x) = \int x\exp(-2x)dx = -\frac{1}{4}e^{-2x}(2x+1).$$

Consequently

$$y_P(x) = -\frac{1}{4}(2x+1).$$

Thus, the general solution of (E) is :

$$\begin{aligned} y(x) &= y_P(x) + y_H(x) \\ &= -\frac{1}{4}(2x+1) + \lambda\exp(2x), \quad \lambda \in \mathbb{R}. \end{aligned}$$

Remark 2.2.2 *In the case of linear differential equation of the form :*

$$y' + ay = b \qquad a, b \in \mathbb{R} \text{ and } a \neq 0.$$

The solutions are :

$$y = C\exp(-ax) + \frac{b}{a}, \quad C \in \mathbb{R}.$$

Example 2.2.7 *Consider the equation :*

$$2y' + y = 5e^{2x} \qquad (1).$$

We note that $f(x) = \frac{1}{2}$ and $g(x) = \frac{5}{2}e^{2x}$.The associated homogeneous equation is $y' = 2y$

whose general solution is :

$$y(x) = \lambda \exp(-\int f(x)dx) = \lambda \exp(-\int \frac{1}{2}dx) = \lambda e^{\frac{-x}{2}}.$$

The particular solution $y_p(x)$ *of* (E) *on* $\mathbb{R}$ *is given by :*

$$y_P(x) = \lambda(x) \exp(-\int f(x)dx) = \lambda(x)e^{\frac{-x}{2}}.$$

and it satisfies that

$$\begin{aligned} \lambda'(x) \exp(-\int f(x)dx) &= g(x) \\ \lambda'(x)e^{\frac{-x}{2}} &= \frac{5}{2}e^{2x} \\ \lambda'(x) &= \frac{5}{2}e^{\frac{5}{2}x}. \end{aligned}$$

Integrating we find

$$\lambda(x) = e^{\frac{5}{2}x}.$$

So

$$y_P(x) = \lambda(x)e^{\frac{-x}{2}} = e^{\frac{5}{2}x}e^{\frac{-x}{2}} = e^{2x}.$$

Therefore, the general solution of (E) *is :*

$$y(x) = y_P(x) + y_H(x) = e^{2x} + \lambda e^{\frac{-x}{2}}, \quad \lambda \in \mathbb{R}.$$

2.2.4 Bernoulli differential equations

Definition 2.2.4 *A differential equation is called a Bernoulli equation if it can be written in the form:*

$$y' + f(x)y = g(x)y^{\alpha}, \quad \alpha \in \mathbb{R} - \{0, 1\},$$

where f and g are continuous functions.

Observe that when $\alpha = 0$, the Bernoulli equation is simply a non-homogeneous linear differential equation :

$$y' + f(x)y = g(x),$$

and if $\alpha = 1$ the equation is equivalent to the following homogeneous linear differential equation :

$$y' + (f(x) - g(x))y = 0.$$

In the cases $\alpha \neq 0$ and $\alpha \neq 1$ a Bernoulli equation is not linear, but it can be transformed into a linear differential equation by an appropriate change of variables.

We have :

$$y' + f(x)y = g(x)y^{\alpha} \Rightarrow y'y^{-\alpha} + f(x)y^{1-\alpha} = g(x). \tag{2.2}$$

Let us set : $z = y^{1-\alpha}$. Hence $z' = (1-\alpha)y'y^{-\alpha}$.

Substituting into the equation (2.2) we get :

$$z' + (1-\alpha)f(x)z = (1-\alpha)g(x).$$

We observe that the previous equation is linear and can be easily solved using the method of variation of parameters.

Example 2.2.8 *Solve the following Bernoulli equation :*

$$y' - xy = xy^3. \tag{2.3}$$

We remark that : $f(x) = -x$, $g(x) = x$ *and* $\alpha = 3$.

If we divide by y^3 *we get :*

$$\frac{y'}{y^3} - x\frac{1}{y^2} = x$$

Let $z = \frac{1}{y^2}$ *, then* $z' = -\frac{2y'}{y^3}$*. Therefore, the equation* (2.3) *becomes :*

$$z' + 2xz = 2x. \tag{2.4}$$

The resolution of (2.4) *gives :*

$$z = 1 + Ce^{-x^2}, \qquad C \in \mathbb{R}.$$

The general solution of (2.3) *is given by :*

$$y = \pm\sqrt{\frac{1}{z}} = \sqrt{\frac{1}{1 + Ce^{-x^2}}}, \qquad C \in \mathbb{R}.$$

Example 2.2.9 *Consider the following equation*

$$y' + \frac{1}{3}y = e^x y^4.$$

It is a Bernoulli equation with $\alpha = 4$*. First we divide by* y^α*, we get*

$$y^{-4}y' + \frac{1}{3}y^{-3} = e^x.$$

Then we multiply by (-3) *we find*

$$-3y^{-4}y' - y^{-3} = -3e^x.$$

Set $z = y^{1-\alpha} = y^{-3}$ *then* $dz = -3y^{-4}y'$*. The previous differential equation becomes*

$$z' - z = -3e^x.$$

The solution of this last equation is

$$z = (k - 3x)e^x, \quad k \in \mathbb{R}.$$

Which implies that

$$y^3 = (k - 3x)e^x, \quad k \in \mathbb{R} \quad or \quad y = \sqrt[3]{\frac{1}{(k - 3x)e^x}}.$$

2.3 Second-order linear differential equations with constant coefficients

Definition 2.3.1 *A second-order linear differential equation with constant coefficients is an equation of the form :*

$$ay'' + by' + cy = g(x) \qquad (E)$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$ *and* g *is a continuous function on the interval* I.
The equation :

$$ay'' + by' + cy = 0 \qquad (E_0)$$

is called the associated homogeneous equation of (E).

Theorem 2.3.1 *The set of solutions of the homogeneous equation* (E_0) *is a* $\mathbb{R}$*-vector space of dimension* 2.

2.3.1 Solving the homogeneous equation

We are looking for a solution of (E_0) in the form $y(x) = e^{rx}$ where $r \in \mathbb{C}$ is a constant to be determined. We have

$$ay'' + by' + cy = 0 \Rightarrow (ar^2 + br + c)e^{rx} = 0 \Rightarrow ar^2 + br + c = 0.$$

Definition 2.3.2 *The equation $ar^2 + br + c = 0$ is called the characteristic equation associated with (E_0) and $\triangle = b^2 - 4ac$ is called the characteristic discriminant associated with (E_0).*

Theorem 2.3.2 *1. If $\triangle > 0$, the characteristic equation has two distinct real roots.$r_1 \neq r_2$ and the solutions of (E_0) are :*

$$y(x) = \lambda \exp(r_1 x) + \mu \exp(r_2 x), \qquad \text{where} \quad \lambda, \mu \in \mathbb{R}.$$

2. If $\triangle = 0$, the characteristic equation has a double root r_0 and the solutions are :

$$y(x) = (\lambda + \mu x) \exp(r_0 x) \qquad \text{where} \quad \lambda, \mu \in \mathbb{R}.$$

3. If $\triangle < 0$, The characteristic equation has two complex conjugate roots.

$r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ and the solution of (E_0) are :

$$y(x) = \exp(\alpha x)(\lambda \cos(\beta x) + \mu \sin(\beta x)), \qquad \text{where } \lambda, \mu \in \mathbb{R}.$$

Example 2.3.1 *1. Consider the following differential equation :*

$$y'' - 5y + 6y = 0.$$

The characteristic equation is $r^2-5r+6=0$ *and it has two distinct real roots* $r_1=2$, $r_2=3$. *Then, for all* $x\in\mathbb{R}$, *the solution is:*

$$y(x)=\lambda\exp(2x)+\mu\exp(3x),\qquad \textit{where}\quad \lambda,\mu\in\mathbb{R}.$$

2. *The characteristic equation of :*

$$y''-6y+9y=0$$

is $r^2-6r+9=0$. *The discriminant is zero* $\triangle=0$, *then we have a double root* r_0 *and the solutions are* $r_0=3$ *and subsequently the solutions are :*

$$y(x)=(\lambda+\mu x)\exp(3x)\qquad \textit{where}\quad \lambda,\mu\in\mathbb{R}.$$

3. *The characteristic equation of :*

$$y''-2y+5y=0$$

is $r^2-2r+5=0$. *We have* $\triangle<0$, *it has two complex conjugate roots* $r_1=1+2i$, $r_2=1-2i$ *and subsequently the solutions are :*

$$y(x)=\exp(x)(\lambda\cos(2x)+\mu\sin(2x)),\qquad \textit{where}\quad \lambda,\mu\in\mathbb{R}.$$

2.3.2 Equations with right-hand side (nonhomogeneous)

We now consider the case where there is a second member g, which is a continuous function on an open interval $I\subset\mathbb{R}$ *:*

$$ay''+by'+cy=g(x)\qquad (E).$$

Proposition 2.3.1 *The general solutions of the equation* (E_0) *are obtained by adding the general solutions of the homogeneous equation to a particular solution of* (E)*. Thus, it remains to find a particular solution. We now consider two important special cases, providing a general method:*

Second member of type $\exp(\alpha x)P(x)$ **:**

If $g(x) = \exp(\alpha x)P(x)$, where $\alpha \in \mathbb{R}$ and P is polynomial, then we look for a particular solution in the form

$$y_p(x) = x^m \exp(\alpha x)Q(x),$$

where Q is a polynomial of the same degree as P with :

- $y_p(x) = \exp(\alpha x)Q(x)$, $(m = 0)$ if α is not a root of the characteristic equation.
- $y_p(x) = x\exp(\alpha x)Q(x)$, $(m = 1)$ if α is a simple root of the characteristic equation.
- $y_p(x) = x^2\exp(\alpha x)Q(x)$, $(m = 2)$ if α is a double root of the characteristic equation."

Second member of type $\exp(\alpha x)(P_1(x)\cos(\beta x) + P_2(x)\sin(\beta x))$ **:**

If

$$g(x) = \exp(\alpha x)(P_1(x)\cos(\beta x) + P_2(x)\sin(\beta x)), \alpha, \beta \in \mathbb{R}$$

where P_1, P_2 are two polynomials, then we look for a particular solution y_p in the form:

- If $\alpha + i\beta$ is not a root of the characteristic equation then:

$$y_p(x) = \exp(\alpha x)(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)),$$

- If $\alpha + i\beta$ is a root of the characteristic equation then:

$$y_p(x) = x\exp(\alpha x)(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)).$$

In the both cases Q_1, Q_2 are two polynomials of degree $n = \max\{\deg P_1, \deg P_2\}$.

Method of variation of parameters

If $\{y_1, y_2\}$ is a basis of solutions of the homogeneous equation (E_0), we look for a particular solution y_p in the form $y_p = \lambda y_1 + \mu y_2$ where λ and μ are two functions such that :

$$\begin{cases} \lambda' y_1 + \mu' y_2 = 0 \\ \lambda' y_1' + \mu' y_2' = \frac{g(x)}{a}. \end{cases} \qquad (S)$$

Indeed, if $y_p = \lambda y_1 + \mu y_2$ is a function which satisfies (S), we have :

$$y_p' = \lambda' y_1 + \mu' y_2 + \lambda y_1' + \mu y_2' = \lambda y_1' + \mu y_2'.$$

Then :

$$y_p'' = \lambda' y_1' + \mu' y_2' + \lambda y_1'' + \mu y_2'' = \frac{g(x)}{a} + \lambda y_1'' + \mu y_2''.$$

Substituting into equation (E) and using the fact that y_1 and y_2 are solutions of the homogeneous equation (E_0), we find :

$$\begin{aligned} ay_p'' + by_p' + cy_p &= a(\frac{g(x)}{a} + \lambda y_1'' + \mu y_2'') + b(\lambda y_1' + \mu y_2') + c(\lambda y_1 + \mu y_2) \\ &= g(x) + \lambda(ay_1'' + by_1' + cy_1) + \mu(ay_2'' + by_2' + cy_2) \\ &= g(x). \end{aligned}$$

Then y_p is indeed a particular solution of (E). The system (S) is easily solved, in fact we find λ', μ' then λ and μ by integration.

Example 2.3.2 *Solve the following differential equation on the interval* $]-\frac{\pi}{2}, \frac{\pi}{2}[$:

$$y'' + y = \frac{1}{\cos x}.$$

The solution of the associated homogeneous equation $y'' + y = 0$ *are :*

$$y = \lambda \cos x + \mu \sin x, \quad \lambda, \mu \in \mathbb{R}.$$

We now look for a particular solution of the original equation in the form:

$$y_P = \lambda(x) \cos x + \mu(x) \sin x,$$

where are two functions to be determined and which satisfy the following system

$$\begin{cases} \lambda' y_1 + \mu' y_2 = 0 \\ \lambda' y_1' + \mu' y_2' = \frac{g(x)}{a}. \end{cases}$$

Then :

$$\begin{cases} \lambda' \cos x + \mu' \sin x = 0 \\ -\lambda' \sin x + \mu' \cos x = \frac{1}{\cos x}. \end{cases}$$

Multiplying the first equation by $\sin x$ *and the second one* $\cos x$*, we obtain :*

$$\begin{cases} \lambda' \sin x \cos x + \mu' \sin^2 x = 0 \\ -\lambda' \sin x \cos x + \mu' \cos^2 x = 1. \end{cases}$$

By addition, we obtain $\mu'(x) = 1$*. Then* $\mu(x) = x$*. From the first row of the system we get*

$$\lambda'(x) = -\frac{\sin x}{\cos x} \Rightarrow \lambda(x) = \ln(\cos x).$$

Thus the particular solution is :

$$y_p(x) = \ln(\cos x) \cos x + x \sin x.$$

Therefore the general solution is :

$$y(x) = \lambda \cos x + \mu \sin x + \ln(\cos x)\cos x + x \sin x, \quad \lambda, \mu \in \mathbb{R}.$$

2.4 Incomplete second-order differential equations

2.4.1 Equations in which the function appears only through its derivatives

In this case the differential equation can be written as $F(x, y', y'') = 0$ (The absence of y). The method used here is to set $z = y'$, which is equivalent to $F(x, z, z') = 0$, which is a first order differential equation. We first start by solving $F(x, z, z') = 0$, we get $z = g(x, c_1)$ then we solve $y' = z$, which allows us to find $y = G(x, C_1) + C_2$.

Example 2.4.1 *Let us consider the equation :*

$$(x^2 + 1)y'' + 2xy' = 0.$$

We notice the absence of the function y. We then set $z = y'$.

Therefore we get

$$\begin{aligned}
(x^2 + 1)y'' + 2xy' = 0 &\Leftrightarrow (x^2 + 1)z' + 2xz = 0 \\
&\Leftrightarrow \frac{dz}{z} = -\frac{2x}{x^2 + 1} \\
&\Leftrightarrow \ln(z) = -\ln(x^2 + 1) + C \\
&\Leftrightarrow z = \frac{C_1}{x^2 + 1} \\
&\Leftrightarrow y' = \frac{C_1}{x^2 + 1} \\
&\Leftrightarrow y = C_1 \arctan(x) + C_2.
\end{aligned}$$

2.4.2 Equations where only the second derivative of the function appears

Under these conditions, the equations can be written as $F(x, y'') = 0$ (absence of y and y').

To solve them, we first find y'', and then integrate twice to obtain y.

Example 2.4.2 *If we have the following differential equation :*

$$(1 + x^2)y'' - 7 = 0.$$

Then :

$$y'' = \frac{7}{1 + x^2} \Rightarrow y' = 7 \arctan x + C.$$

By integration by parts we obtain : $y = 7(x \arctan x - \frac{1}{2} \ln(1 + x^2)) + Cx + C_1$.

Example 2.4.3 *Consider the following equation :*

$$xy'' - 3 = 0, \qquad x > 0.$$

Which is equivalent to :

$$y'' = \frac{3}{x},$$

Then :

$$y' = 3 \ln x.$$

By integration by parts we find :

$$y = 3x \ln x - 3x + C, \qquad C \in \mathbb{R}.$$

2.5 Exercises

Exercise 2.5.1 *Verify that each given function is a solution of the corresponding differential equation*

1. $xy' - y = x^2$, $y(x) = 3x + x^2$.
2. $2x^2y'' + 3xy' - y = 0$, $y(x) = \sqrt{x}$, $x > 0$.
3. $y' - 2ty = 1$, $y(t) = e^{t^2} + e^{t^2}\int_0^t e^{-s^2}ds$.

Solution.

1. We have

$$y(x) = 3x + x^2 \Rightarrow y'(x) = 3 + 2x.$$

By substituting into the equation $xy' - y = x^2$ we obtain :

$$x(3 + 2x) - (3x + x^2) = x^2 \Rightarrow x^2 = x^2.$$

2. Since

$$y(x) = \sqrt{x} \Rightarrow y'(x) = \frac{1}{2}x^{\frac{-1}{2}} \Rightarrow y''(x) = \frac{-1}{4}x^{\frac{-3}{2}}.$$

By substituting into the equation: $2x^2y'' + 3xy' - y = 0$, we find :

$$2x^2(\frac{-1}{4}x^{\frac{-3}{2}}) + 3x(\frac{1}{2}x^{\frac{-1}{2}}) - \sqrt{x} = \frac{-1}{2}\sqrt{x} + \frac{3}{2}\sqrt{x} - \sqrt{x} = 0 \Rightarrow 0 = 0.$$

3. If $y(t) = e^{t^2} + e^{t^2}\int_0^t e^{-s^2}ds$ then :

$$\begin{aligned} y'(t) &= 2te^{t^2} + 2te^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}e^{-t^2} \\ &= 2te^{t^2} + 2te^{t^2}\int_0^t e^{-s^2}ds + 1. \end{aligned}$$

By substituting into the equation: $y' - 2ty = 1$, we get :

$$(2te^{t^2} + 2te^{t^2}\int_0^t e^{-s^2}ds + 1) - 2t(e^{t^2} + e^{t^2}\int_0^t e^{-s^2}ds) = 1 \Rightarrow 1 = 1,$$

Therefore, the given function is indeed a solution of the differential equation $y' - 2ty = 1$.

Exercise 2.5.2 *Solve the following differential equations :*

1. $y' - y = (x+1)e^x$, *for* $x \in \mathbb{R}$.
2. $y' - (\tan x)y = \sin x$, *for* $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.
3. $(1+e^x)y' = ye^x$ *for* $x \in \mathbb{R}$.
4. $2x^2y' = x^2 + y^2$ *for* $x \in \mathbb{R}$.

Solution.

1. It is a first order equation with $f(x) = -1$ and $g(x) = (x+1)e^x$. The solution of the associated homogenous equation : $y' - y = 0$ are:

$$y_H(x) = \lambda e^{-\int f(x)dx} = \lambda e^{-\int (-1)dx} = \lambda e^x, \quad \lambda \in \mathbb{R}.$$

The right-hand side is the function $g(x) = (x+1)e^x$.

Since e^x appears in the solutions of the homogeneous equation, we seek a particular solution of the form:

$$y_P(x) = (ax^2 + bx + c)e^x.$$

We have :

$$y_P'(x) - \ y_P(x) = (x+1)e^x \Rightarrow (2ax + b)e^x = (x+1)e^x.$$

By comparing coefficients, we obtain:

$$y_P(x) = (\frac{1}{2}x^2 + x)e^x.$$

The solutions of our initial differential equations are :

$$y = y_H + y_P = (\frac{1}{2}x^2 + x + \lambda)e^x, \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{R}.$$

2. The associated homogenous equation is : $y' - \tan x \; y = 0$.

A Primitive of :

$$f(x) = -\tan x = -\frac{\sin x}{\cos x}.$$

is :

$$F(x) = \int f(x)dx = \ln|\cos x| = \ln(\cos x) \quad \text{since} \quad x \in]-\frac{\pi}{2}, \frac{\pi}{2}[.$$

So a general solution of the homogenous equation is :

$$y(x) = C\exp(-F(x)) = C\exp(-\ln(\cos x)) = \frac{C}{\cos x}.$$

We now look for a particular solution y_P which verifies :

$$y'_P(x) - \tan x y_P(x) = \sin x.$$

Using the method of variation of parameters we have:

$$y_P(x) = C(x)\exp(-F(x)) \text{ with } C(x) \text{ primitive of } \sin x \exp(F(x)) = \cos x \sin x.$$

Therefore we find:

$$C(x) = \frac{1}{2}\sin^2 x \qquad \text{and} \quad y_P(x) = C(x)\frac{1}{\cos x} = \frac{\sin^2 x}{2\cos x}.$$

The general solution is:

$$y(x) = \frac{C}{\cos x} + \frac{\sin^2 x}{2\cos x} = \frac{\sin^2 x + C'}{2\cos x}.$$

3. The zero function ($y = 0$) is a solution of $(1 + e^x)y' = ye^x$ for $x \in \mathbb{R}$. Suppose that $y \neq 0$, thus, the previous equation is equivalent to a separable differential equation

$$\frac{y'}{y} = \frac{e^x}{1 + e^x},$$

where :

$$\int \frac{1}{y} dy = \int \frac{e^x}{1 + e^x} dx.$$

Then :

$$\ln |y| = \ln(1 + e^x) + K \Rightarrow y = C(1 + e^x), \quad C \in \mathbb{R}^*.$$

The general solution is :

$$y = C(1 + e^x), \ C \in \mathbb{R}.$$

4. For $x \neq 0$, the equation

$$2x^2y' = x^2 + y^2 \qquad (E)$$

is equivalent to :

$$2y' = 1 + \left(\frac{y}{x}\right)^2,$$

which is homogenous equation. Set $z = \frac{y}{x}$ then :

$$y = zx \Rightarrow y' = 2(z + xz').$$

Substituting in (E), we find :

$$1 + z^2 = 2(z + xz')$$

$$1 + z^2 - 2z = 2xz' \Rightarrow (1 - z)^2 = 2xz'.$$

Then we find :

$$\frac{2z'}{(1-z)^2} = \frac{1}{x} \Rightarrow \int \frac{2}{(1-z)^2}dz = \int \frac{1}{x}dx \Rightarrow \frac{2}{(1-z)} = \ln Cx.$$

It then follows that :

$$z = 1 - \frac{2}{\ln Cx} \Rightarrow y = x - \frac{2x}{\ln Cx}, \quad \text{with } \ Cx \in \mathbb{R}^+\backslash\{1\}.$$

Exercise 2.5.3 *Solve the following differential equations by finding a particular solution using the method of variation of the parameters :*

1. $x(1+\ln^2 x)y' + 2y\ln x = 1, \quad on\ ]0,+\infty[.$

2. $(x+1)y' - xy + 1 = 0, \quad y(0) = 2, \quad on \quad]-1,+\infty[.$

Solution.

1) Since the coefficient y' does not vanish on $]0,+\infty[$, the equation can be written :

$$y' + \frac{2\ln x}{x(1+\ln^2 x)}y = \frac{1}{x(1+\ln^2 x)}.$$

The general solution of the associated homogeneous equation :

$$y' + \frac{2\ln x}{x(1+\ln^2 x)}y = 0$$

is : $y = C\exp(-F(x))$ where $F(x)$ is primitive of $f(x) = \frac{2\ln x}{x(1+\ln^2 x)}$ and $C \in \mathbb{R}$. We can take $F(x) = \ln(1+\ln^2 x)$. Then :

$$y_H = C\exp(-\ln(1+\ln^2 x)) = \frac{C}{1+\ln^2 x}.$$

We use the method of variation of constants to find a particular solution of the non-homogeneous equation. Let us define $y_p = \frac{C(x)}{1+\ln^2 x}$ with C a differential function. Let $z(x) = \frac{1}{1+\ln^2 x}$, then : $y_p = C(x)z(x)$.

Differentiating and substituting into the original equation, and noting that $z(x)$ satisfies the homogeneous equation, we obtain:

$$C'(x)z(x) = \frac{1}{x(1+\ln^2 x)} \Rightarrow \frac{C'(x)}{1+\ln^2 x} = \frac{1}{x(1+\ln^2 x)} \Rightarrow C'(x) = \frac{1}{x}.$$

Thus we an take $C(x) = \ln x$ *i.e.* $y_p = \frac{\ln x}{1+\ln^2 x}$.

The general solution of our problem is:

$$y = y_H + y_p = \frac{C}{1+\ln^2 x} + \frac{\ln x}{1+\ln^2 x}, \qquad C \in \mathbb{R}, \quad x \in]0, +\infty[.$$

2) The differential equation $(x+1)y' - xy + 1 = 0$ is equivalent to :

$$y' - \frac{x}{x+1}y = -\frac{1}{x+1}$$

This is a first-order linear non-homogeneous differential equation with :

$$f(x) = -\frac{x}{x+1} = \frac{1}{x+1} - 1 \quad \text{and} \quad g(x) = -\frac{1}{x+1}.$$

a primitive of f is:

$$\int f(x)dx = \ln|x+1| - x = \ln(x+1) - x.$$

The general solution of the associated homogeneous equation $y' - \frac{x}{x+1}y = 0$ is:

$$y_H(x) = \lambda \exp(-\int f(x)dx) = \lambda \exp(-\ln(x+1) + x) = \lambda \frac{\exp(x)}{x+1}.$$

This time we will directly use the formula for the particular solution given in the lecture:

$$y_P(x) = \exp(-\int f(x)dx)(\int g(x)\exp(\int f(x)dx)dx).$$

Since :

$$\int g(x)\exp(\int f(x)dx)dx = \int -\frac{1}{x+1}\frac{x+1}{\exp(x)}dx = \int -\exp(-x)dx = \exp(-x).$$

So :

$$y_P(x) = \frac{\exp(x)}{x+1}\exp(-x) = \frac{1}{x+1}.$$

The general solution :

$$y(x) = y_H(x) + y_P(x) = \lambda\frac{\exp(x)}{x+1} + \frac{1}{x+1}.$$

Since :

$$y(0) = 2 \Rightarrow \lambda + 1 = 2 \Rightarrow \lambda = 1.$$

The solution is :

$$y(x) = \frac{\exp(x)+1}{x+1}.$$

Exercise 2.5.4 *Solve the following differential equation with initial condition :*

$$y' = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1.$$

Solution.

The equation is equivalent to :

$$2(y-1)dy = \left(3x^2+4x+2\right)dx$$

which is an equation with separable variables. Integrating :

$$\int 2(y-1)dy = \int \left(3x^2+4x+2\right)dx,$$

we get :

$$y^2 - 2y = x^3 + 2x^2 + 2x + C \quad (S).$$

To determine the solution that satisfies the initial condition, $y(0) = -1$. We substitute $x = 0$ and $y = -1$ in (S) we get $C = 3$.

Thus, the solution to the problem is given implicitly by :

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

To find the solution in an explicit form, we proceed as follows:

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \Rightarrow \ y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4.$$

Then we obtain:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

But since $y(0) = -1$ the solution is :

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

which is well defined if $x > -2$ (since -2 is the only real root of $x^3 + 2x^2 + 2x + 4$).

Exercise 2.5.5 *Solve on* $\mathbb{R}$ *the following differential equations :*

1. $y''(x) - y(x) = 0.$
2. $y''(x) + 6y'(x) + 9y(x) = 0.$
3. $y''(x) + 4y'(x) + 13y(x) = 0.$

Solution.

1. The characteristic equation of $y''(x) - y(x) = 0$ is $r^2 - 1 = 0$, then $\Delta = 4 > 0$, so $r_1 = -1$ and $r_2 = +1$.

   Thus, we have two distinct real roots. Consequently, the general solution is given by:

   $$y(x) = \lambda \exp(-x) + \mu \exp(x), \qquad x \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

2. The characteristic equation of $y''(x) + 6y'(x) + 9y(x) = 0$ is $r^2 + 6r + 9 = 0$, then $\Delta = 0$, so $r_0 = -3$.

   Since we have a double real root, the general solution is in the form

   $$y(x) = (\lambda + x\mu) \exp(-3x), \quad x \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

3. The characteristic equation of $y''(x)+4y'(x)+13y(x) = 0$ is given by $r^2+4r+13 = 0$. Then $\Delta = -36 < 0$, so $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$.

   We have two complex conjugate roots. The general solution in this case is

   $$y(x) = (\lambda \cos 3x + \mu \sin 3x) \exp(-2x), \quad x \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

Exercise 2.5.6 *Solve the following differential equation:*

$$y''(x) + y(x) = \cos x. \qquad (E)$$

Solution.

The given equation is a second-order non-homogeneous equation. Its associated homogeneous equation is:

$$y''(x) + y(x) = 0.$$

The characteristic equation is $r^2 - 1 = 0$. Then $\Delta = -4 < 0$, $r_1 = i$ and $r_2 = -i$.
We have two complex conjugate roots. Thus the general solution is:

$$y_H(x) = (\lambda \cos x + \mu \sin x), \quad x \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

The second member is $g(x) = \cos x$. We use the method of variation of constants and look for a particular solution of the form

$$y_p(x) = C_1(x)y_1 + C_2(x)y_2 \qquad \text{with} \quad y_1 = \cos x \quad \text{and} \quad y_2 = \sin x,$$

which verify :

$$C_1'(x)y_1 + C_2'(x)y_2 = 0 \quad \text{and} \quad C_1'(x)y_1' + C_2'(x)y_2' = g(x). \qquad (S)$$

Replacing with :

$$y_p(x) = C_1(x)\cos x + C_2(x)\sin x$$

in (S) and solving the system we find :

$$C_1(x) = \frac{\cos 2x}{4} \quad \text{and} \quad C_2(x) = \frac{\sin 2x}{4} + \frac{x}{2}.$$

A particular solution is :

$$y_p(x) = \frac{\cos 2x}{4}\cos x + (\frac{\sin 2x}{4} + \frac{x}{2})\sin x = \frac{\cos x}{4} + \frac{x}{2}\sin x.$$

Then, the general solution is given by :

$$\begin{aligned} y(x) = \ & y_H(x) + y_P(x) \\ & = \lambda \cos x + (\mu + \frac{x}{2})\sin x, \qquad x \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}. \end{aligned}$$

Exercise 2.5.7 *Using the change of variable $x = e^t$, solve the following differential equations :*

1. $x^2y'' + xy' + y = 0$ *on* $]0, +\infty[$.

2. $x^2y'' + y = 0$ *on* $]0, +\infty[$.

Solution.

Since we are looking for a function of $x \in]0, +\infty[$, we can take $x = e^t$ and $z(t) = y(e^t)$. Then we have :

$$t = \ln x, \quad y(x) = z(lnx), \quad y'(x) = \frac{1}{x}z'(\ln x) = e^{-t}z'(t)$$

and

$$y''(x) = -\frac{1}{x^2}z'(\ln x) + \frac{1}{x^2}z''(\ln x) = -e^{-2t}z'(t) + e^{-2t}z''(t).$$

1. In the first case, we find :

$$x^2y'' + xy' + y = 0 \quad \text{on} \quad]0, +\infty[\Leftrightarrow z''(t) + z(t) = 0, \quad \text{on} \quad \mathbb{R}.$$

The characteristic equation is:

$$r^2 + 1 = 0 \Rightarrow r = \pm i.$$

Therefore, this last differential equation has the following general solution:

$$z(t) = \lambda \cos t + \mu \sin t, \quad t \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

We conclude that the solution to our problem is :

$$y(x) = z(\ln x) = \lambda \cos(\ln x) + \mu \sin(\ln x), \quad x \in]0, +\infty[, \quad \lambda, \mu \in \mathbb{R}.$$

2. The equation $x^2y'' + y = 0$ becomes :

$$x^2y'' + y = 0 \qquad \text{on} \quad]0, +\infty[\Leftrightarrow z''(t) - z'(t) + z(t) = 0, \quad \text{on} \quad \mathbb{R}.$$

The characteristic equation is: :

$$r^2 - r + 1 = 0 \Rightarrow r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

The solution of $z''(t) - z'(t) + z(t) = 0$ are :

$$z(t) = \exp(\frac{t}{2})(\lambda \cos \frac{\sqrt{3}}{2}t + \mu \sin \frac{\sqrt{3}}{2}t), \quad t \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R}.$$

Subsequently, the solution to our initial problem :

$$y(x) = z(\ln x) = (\sqrt{x}\lambda \cos(\frac{\sqrt{3}}{2} \ln x) + \mu \sin(\frac{\sqrt{3}}{2} \ln x)), \quad x \in]0, +\infty[, \quad \lambda, \mu \in \mathbb{R}.$$

Exercise 2.5.8 *Determine the value of the real constant r for which the given differential equation has a solution of the form: $y = t^r$ pour $t > 0$:*

1. $t^2y'' + 4ty' + 2y = 0.$
2. $t^2y'' - 4ty' + 4y = 0.$

Solution.

1. We have :

$$y = t^r \Rightarrow y' = rt^{r-1} \Rightarrow y'' = r(r-1)t^{r-2}.$$

We substitute into the equation $t^2y'' + 4ty' + 2y = 0$ we get :

$$r(r-1)t^{r-2}t^2 + 4rt^{r-1}t + 2t^r = 0 \Rightarrow r(r-1)t^r + 4rt^r + 2t^r = 0$$

which is equivalent to $(r^2 + 3r + 2)t^r = 0$.

Consequently $r^2 + 3r + 2 = 0$ i.e. $r = -1$ or $r = -2$. Then $y_1(t) = \frac{1}{t}$ and $y_2(t) = \frac{1}{t^2}$ are the solutions of the given equation.

2. Using the same method as in the previous question, we obtain

$$y = t^r \Rightarrow y' = rt^{r-1} \Rightarrow y'' = r(r-1)t^{r-2}.$$

We substitute into $t^2y'' - 4ty' + 4y = 0$ we obtain :

$$r(r-1)t^{r-2}t^2 - 4rt^{r-1}t + 4t^r = 0 \Rightarrow r(r-1)t^r - 4rt^r + 4t^r = 0.$$

Which is equivalent to : $(r^2 - 5r + 4)t^r = 0$.

Subsequently : $r^2 - 5r + 4 = 0$ *i.e.* $r = 1$ or $r = 4$. Then $y_1(t) = t$ and $y_2(t) = t^4$.