Chapter 1

Taylor Expansions (Limited Development)

Taylor expansions are a fundamental method for approximating functions locally. In this chapter, we introduce practical methods for computing these expansions and illustrate their use.

1.1 Taylor's and Maclaurin's polynomial

Let $I \subset \mathbb{R}$ be an open interval. We recall that f is of class $\mathcal{C}^0$ if f is continuous on I. For $n \in \mathbb{N}^*$, f is of class $\mathcal{C}^n$ if f is n times differentiable on I and $f^{(n)}$ is continuous on I and f is of class $\mathcal{C}^\infty$ if f is of class $\mathcal{C}^n$ for all $n \in \mathbb{N}$.

Definition 1.1.1 *Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^n$ and let $a \in I$. Then, for all $x \in I$, the Taylor's polynomial of order n at a of f, the polynomial :*

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor's rest of order n at a of f, is defined by :

$$R_n(x) = f(x) - T_n(x).$$

Example 1.1.1 *Give the Taylor's polynomial of order 4 at 1 of $f(x) = \ln x$.*

We have :

$f(x) = \ln x,$ $f(1) = 0,$

$f'(x) = \frac{1}{x},$ $f'(1) = 1,$

$f''(x) = -\frac{1}{x^2},$ $f''(1) = -1,$

$f'''(x) = \frac{2}{x^3},$ $f'''(1) = 2,$

$f^{(4)}(x) = -\frac{6}{x^4}$ $f^{(4)}(x) = -6.$

Then

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4.$$

$$T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

Definition 1.1.2 *Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^n$ and let $a \in I$. Then, for all $x \in I$, the Maclaurin's polynomial of order n of f, is Taylor's polynomial of order n at 0 of f:*

$$M_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Example 1.1.2 *Give the Maclaurin's polynomial of order 4 of $f(x) = e^{2x}$.*

We have

$f(x) = e^{2x},$ $f(0) = 1,$

$f'(x) = 2e^{2x},$ $f'(0) = 2,$

$f''(x) = 4e^{2x},$ $f''(0) = 4,$

$f'''(x) = 8e^{2x},$ $f'''(0) = 8,$

$f^{(4)}(x) = 16e^{2x},$ $f^{(4)}(x) = 16.$

Then

$$M_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4.$$
$$M_4(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

1.2 Taylor's Formulas

Theorem 1.2.1 *(Taylor's formula with integral remainder)*

Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^n$ and let $a \in I$. Then for all $x \in I$, we have :

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt. \tag{1.1}$$

This is Taylor's formula with integral remainder

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

Remark 1.2.1 *Set $x = a+h$ i.e. $h = x-a$ the Taylor's formula with integral remainder becomes (for all a and $a+h$ of I)*

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n)}(a)}{n!}h^n + \int_0^h \frac{f^{(n+1)}(a+t)}{n!}(h-t)^n dt.$$

Theorem 1.2.2 *(Taylor's formula with Lagrange remainder)*

Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^n$ and let $a, x \in I$. There exists a real number c between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{n!}(x-a)^{n+1}.$$

This is Taylor's formula with Lagrange remainder

$$R(x) = \frac{f^{(n+1)}(c)}{n!}(x-a)^{n+1}.$$

In most cases, c is unknown, but this formula gives us an estimate of the remainder. Indeed, if $\left|f^{(n+1)}(x)\right| \leq M$, for all $x \in I$. Then :

$$|f(x) - T_n(x)| \leq M\frac{|x-a|^{n+1}}{n!}.$$

Theorem 1.2.3 ***(Taylor-Young's Formula)*** *Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^n$ and let $a \in I$. Then for all $x \in I$, we have*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + (x-a)^n\xi(x),$$

where ξ is a function defined on I such that $\xi(x) \to 0$ when $x \to a$.

The polynomial part is denoted by $T_n(x)$ i.e.

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

and is called the Taylor polynomial of order n. The term $R_n(x) = (x-a)^n\xi(x)$ is called the Young remainder of order n. Therefore $f(x) = T_n(x) + R_n(x)$.

Special Case (Taylor-Young Formula in the Neighborhood of 0)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n + x^n\xi(x), \quad \text{with } \xi(x) \to 0 \text{ when } x \to 0.$$

Sometimes we set $o(x^n) = x^n\xi(x)$.

Example 1.2.1 *Let $f(x) = \ln(1+x)$ with $x > -1$. It is known that f is infinitely differentiable. We aim to derive the Taylor–Young formula in a neighborhood of 0 for*

the first orders. We obtain

$$\begin{aligned} f(x) &= \ln(1+x) \quad \textit{then} \quad f(0) = 0 \\ f'(x) &= \frac{1}{(1+x)} \quad \textit{then} \quad f'(0) = 1 \\ f''(x) &= \frac{-1}{(1+x)^2} \quad \textit{then} \quad f''(0) = -1 \\ f^{(3)}(x) &= \frac{2}{(1+x)^3} \quad \textit{then} \quad f^{(3)}(0) = 2. \end{aligned}$$

By induction, we can prove that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{1}{(1+x)^n} \quad \textit{then} \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Therefore

$$T_0(x) = 0, \quad T_1(x) = x, \quad T_2(x) = x - \frac{x^2}{2}, \quad T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3},$$

and for all $n > 0$

$$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n}.$$

and

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n} + x^n \xi(x), \quad \textit{with } \xi(x) \to 0 \textit{ when } x \to 0.$$

Taylor's formulas indicate that, in a neighborhood of a, the remainder terms become progressively smaller as the order n increases.

1.3 Taylor expansions about a point

Let $f : I \to \mathbb{R}$ be a function where I is an open interval.

Definition 1.3.1 *For $a \in I$, we say that f admits a Taylor expansion of order n, also called a limited development, abbreviated (DL) at the point a, if there exist real numbers $c_0, c_1, \ldots, c_n$ and a function $\xi : I \to \mathbb{R}$ such that $\lim_{x\to a}\xi(x) = 0$ and*

$$f(x) = c_0 + c_1(x-a) + \ldots c_n(x-a)^n + (x-a)^n\xi(x).$$

i) The previous expression is called a Taylor expansion of f in a neighborhood of a of order n. The Taylor–Young formula gives Taylor expansions by setting : $c_k = \frac{f^k(a)}{k!}$.

ii) The polynomial part of the Taylor expansion is given by $c_0 + c_1(x-a) + \ldots c_n(x-a)^n$.

iii) The term $(x-a)^n\xi(x)$ denotes the remainder of the Taylor expansion.

Proposition 1.3.1 *If f is of $\mathcal{C}^n$ in a neighborhood of a point a then f admits a Taylor expansion at the point a of order n, which follows from the Taylor–Young formula given by*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + (x-a)^n\xi(x),$$

with $\lim_{x\to a}\xi(x) = 0$.

Remark 1.3.1 *When f is of class $\mathcal{C}^n$ in a neighborhood of 0 then f admits a Taylor expansion at 0 of order n given by*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n + x^n\xi(x), \qquad \xi(x) \to 0 \text{ when } x \to a.$$

Corollaire 1.3.1 *1. If f has a Taylor expansion in a neighborhood of a then this expansion is unique.*

2. *If f is even (rep. odd) then the polynomial part of its Taylor expansion at 0 contains only even (rep. odd) monomials.*

Example 1.3.1 *a) We know that $\sinh x$ is an odd function. Its Taylor expansion in a neighborhood of 0 is given by*

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\xi(x).$$

b) Then function $\cosh x$ is even. Its Taylor expansion in a neighborhood of 0 is given by

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + x^{2n+1}\xi(x).$$

1.3.1 Taylor expansions of usual functions about 0

$$\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + x^n\xi(x).\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n+1}\xi(x).\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\xi(x).\\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + x^{2n+1}\xi(x).\\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\xi(x).\\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^n x^n}{n} + x^n\xi(x).
\end{aligned}$$

In addition, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + x^n\zeta(x).$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + (-1)^n x^n + x^n\zeta(x).$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + ... + (-1)^{n-1}\frac{1\times 3\times 5\times \ldots \times (2n-1)}{2^n n!}x^n + x^n\zeta(x).$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + x^n\zeta(x).$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + x^{2n}\zeta(x).$$

Remark 1.3.2 *To obtain the Taylor expansion of a function at a point a ($a \neq 0$), we introduce the change of variable $h = x - x_0$, which reduces the problem about 0.*

1.3.2 Operations on Taylor expansions

Let f, g be two functions that admit Taylor expansions at 0 of order n given by

$$f(x) = P_n(x) + x^n\xi_f(x), \qquad \text{with} \quad \lim_{x\to 0}\xi_f(x) = 0$$
$$g(x) = P'_n(x) + x^n\xi_g(x), \qquad \text{with} \quad \lim_{x\to 0}\xi_g(x) = 0.$$

Sum and product

Proposition 1.3.2 *1. The function $f + g$ admits Taylor expansions at 0 of order n given by*

$$[f+g](x) = P_n(x) + P'_n(x) + x^n\xi(x), \qquad \textit{with} \quad \lim_{x\to 0}\xi(x) = 0.$$

2. The function fg admits Taylor expansions at 0 of order n given by

$$[fg](x) = T_n(x) + x^n\xi(x), \qquad \textit{with} \quad \lim_{x\to 0}\xi(x) = 0,$$

where $T_n(x)$ is the polynomial obtained by truncating $P_n(x)P_n'(x)$ of order n.

Remark 1.3.3 *Truncating a polynomial at order n means keeping only the monomials of degree less than or equal to n.*

Example 1.3.2 *1. Let's find the Taylor expansion (limited development LD) of $e^x + \cos x$ at order 3 at 0.*

We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3\xi_1(x) \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + x^3\xi_2(x).$$

Then

$$\begin{aligned} e^x + \cos x &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + \left(1 - \frac{x^2}{2!}\right) + x^3\xi(x) \\ &= 2 + x + \frac{x^3}{6} + x^3\xi(x). \end{aligned}$$

2. Let's find the Taylor expansion of $e^x \cos x$ at order 3 at 0. We have

$$\begin{aligned} e^x \cos x &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2!}\right) + x^3\xi(x) \\ &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2!}\right) + x^3\xi(x) \\ &= 1 + x - \frac{1}{3}x^3 + x^3\xi(x). \end{aligned}$$

Quotient and composition

Proposition 1.3.3 *1. If $g(0) = 0$ then $f \circ g$ admits a Taylor expansion at 0 whose polynomial part is the truncated polynomial of order n in the composition.*

2. If $g(0) \neq 0$ the quotient $\frac{f}{g}$ admits Taylor expansion (limited development LD) at 0

of order n given by

$$\frac{f(x)}{g(x)} = R(x) + x^n \xi(x),$$

where R is obtained by dividing P by P' according to increasing powers at order n.

Example 1.3.3 *1. Find the Taylor expansion (limited development LD) of $\frac{2+x+2x^3}{1+x^2}$ of order 2.*

We put

$$C(x) = 2 + x + 2x^3, \qquad D(x) = 1 + x^2.$$

Thus, using division in order of increasing powers we get

$$C(x) = D(x)\left(2 + x - 2x^2\right) + x^3(1 + 2x).$$

So

$$\frac{C(x)}{D(x)} = \left(2 + x - 2x^2\right) + x^2\xi(x).$$

2. Find the Taylor expansion (limited development LD) of $\frac{1+x}{2+x}$ of order 4 at 0. We are going to use a different method

$$\begin{aligned}
\frac{1+x}{2+x} &= \frac{1}{2}(1+x)\frac{1}{1+\frac{x}{2}} \\
&= \frac{1}{2}(1+x)\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + x^4\xi(x)\right) \\
&= \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{32} + x^4\xi(x).
\end{aligned}$$

Example 1.3.4 *Find the Taylor expansion (limited development LD) of $h(x) = \sin(\ln(1+x))$ of order 3 at 0.*

We remark that h is the composition of two functions and $h(0) = 0$.

Set $f(u) = \sin u$ *and* $g(x) = \ln(1+x)$. *We have*

$$f(u) = u - \frac{u^3}{3!} + u^3\xi_1(u).$$
$$u = g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\xi_2(x).$$

We obtain

$$u^2 = \left(x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\xi_2(x)\right)^2$$
$$= x^2 - x^3 + x^3\xi_3(x)$$
$$u^3 = u \cdot u^2 = x^3 + x^3\xi_3(x).$$

Then

$$h(x) = f \circ g(x) = f(u) = u - \frac{u^3}{3!} + u^3\xi_1(u)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{6}x^3 + x^3\xi_3(x).$$
$$= x - \frac{x^2}{2} + \frac{1}{6}x^3 + x^3\xi_3(x).$$

Integration

Theorem 1.3.1 *Let* $f : I \to \mathbb{R}$ *be a function which admits Taylor expansion (limited development LD) about* a *of order* n *given by*

$$f(x) = c_0 + c_1(x-a) + \ldots + c_n(x-a)^n + (x-a)^n\xi(x).$$

Assume that f *possesses the function* F *as a primitive. Then* F *admits a Taylor expansion at* a *of order* $n+1$ *given by*

$$F(x) = F(a) + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \ldots + c_n\frac{(x-a)^{n+1}}{n+1} + (x-a)^{n+1}\eta(x) \qquad \text{with} \quad \lim_{x\to 0}\eta(x) = 0.$$

Example 1.3.5 *In order to give the LD of* $\arctan x$ *at* 0*, we use the fact that*

$$\begin{aligned}(\arctan x)' &= \frac{1}{1+x^2} \\ &= 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + x^{2n}\zeta(x) \qquad \textit{with} \ \lim_{x\to 0}\zeta(x) = 0\end{aligned}$$

and since $\arctan 0 = 0$*, by integrating, we obtain*

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + x^{2n}\eta(x) \qquad \textit{with} \lim_{x\to 0}\eta(x) = 0.$$

Derivation

Proposition 1.3.4 *Let* $f : I \to \mathbb{R}$ *be a differentiable function which admits a Taylor expansion about* a *of order* n *given by*

$$f(x) = P_n(x) + x^n\zeta(x) \qquad \textit{with} \ \lim_{x\to 0}\zeta(x) = 0.$$

Then f' *the derivative of* f *admits a Taylor expansion about* a *of order* n *given by*

$$f'(x) = P_{n-1}(x) + x^n\eta(x) \qquad \textit{with} \ \ P_{n-1}(x) = P_n'(x) \qquad \textit{and} \ \lim_{x\to 0}\eta(x) = 0.$$

Example 1.3.6 *The function* $\sin x$ *is differentiable near* 0*. We have*

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^6\xi(x). \\ \cos x &= (\sin x)' \\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)' + x^5\eta(x) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\eta(x).\end{aligned}$$

1.4 Applications of Taylor expansions

1.4.1 Limit calculations

Taylor expansions are very effective for calculating limits with indeterminate forms.

Example 1.4.1 *If* $f(x) = c_0+c_1(x-a)+\ldots+c_n(x-a)^n+(x-a)^n\xi(x)$ *then* $\lim\limits_{x\to a} f(x) = c_0$.

Example 1.4.2 *Evaluate* $\lim\limits_{x\to 0}\frac{\sqrt{1-2x}-\sqrt{1+x}}{x}$.

We know that

$$\sqrt{1+x} = 1 + \frac{x}{2} + x\xi_1(x).$$

By composition we get

$$\sqrt{1-2x} = 1 + \frac{(-2x)}{2} + x\xi_2(x) = 1 - x + x\xi_2(x).$$

Then

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1-2x}-\sqrt{1+x}}{x} &= \lim_{x\to 0}\frac{\left(1+\frac{x}{2}+x\xi_1(x)\right)-(1-x+x\xi_2(x))}{x}\\ &= \lim_{x\to 0}\frac{\frac{-3}{2}x+x\xi(x)}{x} = \frac{-3}{2}.\end{aligned}$$

Example 1.4.3 *Calculate*

$$\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}.$$

We have

$$\sin x = x - \frac{x^3}{6} + x^3\xi_1(x) \quad \textit{and} \quad \cos x = 1 - \frac{x^2}{2} + x^2\xi_2(x).$$

By substituting, we find

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3} &= \lim_{x\to 0}\frac{\frac{x^3}{3}+x^3\xi(x)}{x^3}\\ &= \frac{1}{3}.\end{aligned}$$

1.4.2 Position of a curve relative to its tangent

Let f be a function which its Taylor expansion about a is given by

$$f(x) = c_0 + c_1(x-a) + c_k(x-a)^k + x^k\xi(x),$$

where k is the smallest integer greater than or equal to 2 such that $c_k \neq 0$. The equation of the tangent (T) at the curve of f, denoted C_f, at a is

$$y = c_0 + c_1(x-a).$$

The sign of $f(x) - y$, i.e. the sign of $c_k(x-a)^k$, gives us the position of the curve relative to its tangent at a, indeed

1. If $f(x) - y \geq 0$ then the curve is above the tangent.
2. If $f(x) - y \leq 0$ then the curve is below the tangent.
3. If the sign of $f(x) - y$ change changes when we go from $x < a$ to $x > a$ (i.e. $f''(a) = 0$) then the curve crosses its tangent at the point $(a, f(a))$ which is called the inflection point.

1.5 Exercises

Exercise 1.5.1 *Determine the Taylor expansions of the following functions*

1. $\frac{1}{1+x} - e^x$ *of order* 3 *at* 0,
2. $\sqrt{1-x} - \sqrt{1+x}$ *of order* 4 *at* 0,
3. $\cos x \ln(1+x)$ *of order* 4 *at* 0,
4. $\frac{\ln(1+x)}{\sin x}$ *of order* 3 *at* 0.

Solution.

1. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^3\zeta_1(x)$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\zeta_2(x).$$

Then

$$\frac{1}{1-x} - e^x = \frac{x^2}{2} + \frac{5x^3}{6} + x^3\zeta(x).$$

2. We have

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{6}x^3 - \frac{5}{128}x^4 + x^4\zeta_1(x),$$
$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{1}{8}x^2 - \frac{1}{6}x^3 - \frac{5}{128}x^4 + x^4\zeta_2(x).$$

Then

$$\sqrt{1+x} + \sqrt{1-x} = 2 - \frac{1}{4}x^2 - \frac{5}{64}x^4 + x^4\zeta(x).$$

3. We have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\zeta_1(x),$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{1}{6}x^3 + x^4\zeta_2(x).$$

Then

$$\cos x \ln(1+x) = (1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\zeta_1(x))(x - \frac{x^2}{2} + \frac{1}{6}x^3 + x^4\zeta_2(x))$$

$$\cos x \ln(1+x) = x - \frac{x^2}{2} - \frac{1}{6}x^3 + x^4\zeta(x).$$

4. We have

$$\begin{aligned}\frac{\ln(1+x)}{\sin x} &= \ln(1+x)\cdot\frac{1}{\sin x}\\ &= \left(x-\frac{x^2}{2}+\frac{1}{3}x^3+x^3\zeta_1(x)\right)\cdot\frac{1}{\left(x-\frac{1}{6}x^3+x^3\zeta_2(x)\right)}\\ &= \left(1-\frac{x}{2}+\frac{1}{3}x^2+x^3\zeta_1(x)\right)\cdot\frac{1}{\left(1-\frac{1}{6}x^2+x^2\zeta_2(x)\right)}\\ &= \left(1-\frac{x}{2}+\frac{1}{3}x^2+x^3\zeta_1(x)\right)\left(1+\frac{x^2}{6}+x^3\zeta_3(x)\right)\\ &= \left(1-\frac{x}{2}+\frac{1}{2}x^2-\frac{1}{3}x^3+x^3\zeta(x)\right).\end{aligned}$$

Exercise 1.5.2 *Give the limited development of the following functions*

1. $\int_0^x e^{t^2}dt$ *of order* 5 *at* 0.
2. $\frac{1}{4+3x}$ *of order* 2 *at* 0.
3. $\sin(\ln(1+x))$ *of order* 3 *at* 0.

Solution.

1. We have

$$\begin{aligned}e^x &= 1+x+\frac{x^2}{2}+o(x^2),\\ e^{x^2} &= 1+x^2+\frac{x^4}{2}+o(x^4).\end{aligned}$$

By integrating we find

$$\int_0^x e^{t^2}dt = x+\frac{x^3}{3}+\frac{x^5}{10}+o(x^4).$$

2. We have

$$\frac{1}{4+3x}=\frac{1}{4(1+\frac{3}{4}x)}=\frac{1}{4}\cdot\frac{1}{1+u}\quad\text{where}\quad u=\frac{3x}{4}\ ,\ u=0\ \text{ if } x=0.$$

But

$$\frac{1}{1+u} = 1 - u + u^2 + u^2\xi(u).$$

By substituting u with its value, we obtain

$$\frac{1}{(1+\frac{3}{4}x)} = 1 - \frac{3x}{4} + \frac{9x^2}{16} + \frac{9}{16}x^2\xi(\frac{3x}{4}).$$

Thus

$$\frac{1}{4+3x} = \frac{1}{4} - \frac{3x}{16} + \frac{9x^2}{64} + x^2\xi_1(x).$$

3. We have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3).$$

Let $u = \ln(1+x)$, we obtain $u = 0$ if $x = 0$, then

$$\sin u = u - \frac{1}{6}u^3 + o\left(u^3\right)$$

By substitution, we obtain

$$\sin(\ln(1+x)) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) - \frac{1}{6}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)^3 + x^3o(x^3).$$

By neglecting all terms of degree greater than 3 and grouping them in the remainder term, we obtain

$$\sin(\ln(1+x)) = x - \frac{x^2}{2} - \frac{x^3}{6} + o(x^3).$$

Exercise 1.5.3 *Find the following limits*

1. $\lim\limits_{x\to 0}\frac{\ln(1+x)-\sin x}{x}$.

2. $\lim\limits_{x\to 0}\frac{\cos x-\sqrt{1-x^2}}{x^4}$.

Solution. 1. A second-order Taylor expansion is sufficient because Taylor expansions are different from order 2

$$\ln(1+x)-\sin x=\left(x-\frac{x^2}{2}+o(x^2)\right)-\left(x+o(x^2)\right).$$

Then

$$\frac{\ln(1+x)-\sin x}{x}=-\frac{x}{2}+o(x^2).$$

As a result

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+x)-\sin x}{x}&=\lim_{x\to 0}-\frac{x}{2}+o(x^2).\\&=0.\end{aligned}$$

2. Since

$$\begin{aligned}\cos x&=1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4),\\\sqrt{1-x^2}&=1-\frac{x^2}{2}+\frac{x^4}{8}+o(x^4).\end{aligned}$$

Then

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-\sqrt{1-x^2}}{x^4}&=\lim_{x\to 0}\frac{\frac{1}{6}x^4+o(x^4)}{x^4}\\&=\frac{1}{6}.\end{aligned}$$