Chapter 4

Fundamental Theorems on Holomorphic Functions

Part 1: Integration

4.1 Integration

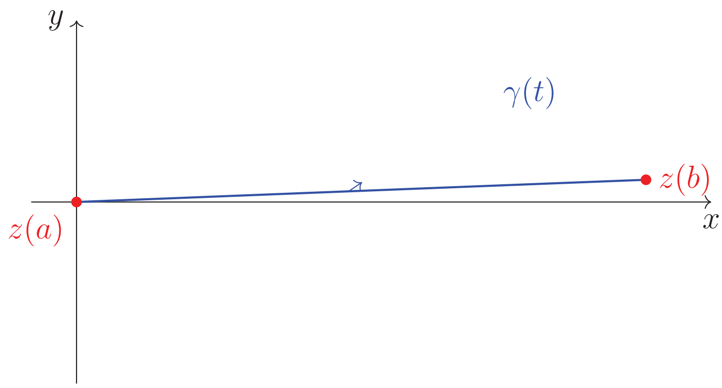
4.1.1 Paths and in the Complex Plane

Definition 4.1 (Path or Curve) ***A path (or curve) in the complex plane is a continuous function***

$$\gamma : [a, b] \to \mathbb{C}, \quad t \mapsto z(t) = x(t) + i\, y(t),$$

where $x(t)$ ***and*** $y(t)$ ***are real-valued functions of class*** $\mathcal{C}^1$ ***on*** $[a, b]$***. The set of points described by*** $\gamma(t)$ ***as*** t ***varies from*** a ***to*** b ***is called an arc of curve.***

The points $z(a)$ ***and*** $z(b)$ ***are called respectively the initial point and the terminal point of the path.***

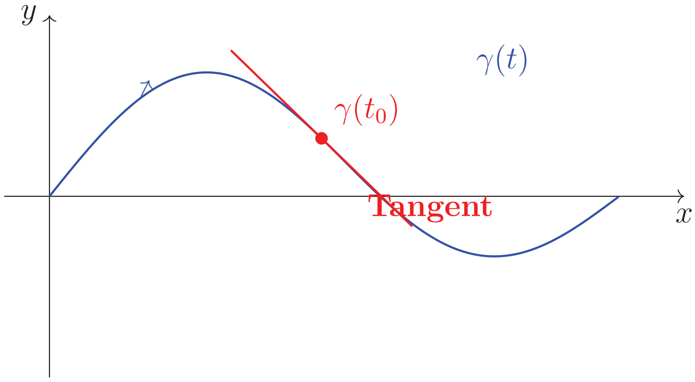


Definition 4.2 (Smooth Path) ***A path*** $\gamma(t)$ ***is said to be smooth if its derivative***

$$\gamma'(t) = x'(t) + i\, y'(t)$$

exists and is continuous on $[a, b]$***, and*** $\gamma'(t) \neq 0$ ***for all*** $t \in (a, b)$***.***

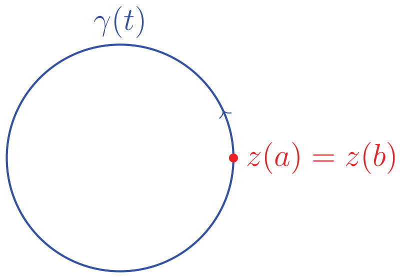
Geometrically, this means that the curve has a smooth ly varying tangent direction at every point, and the tangent vector is never zero, so the curve never stops or forms a sharp corner.



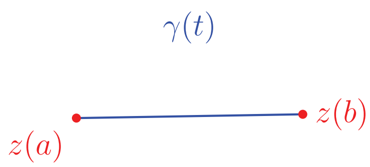
Definition 4.3 (Closed Path) ***A path is called closed (or a loop) if its initial and terminal points coincide:***

$$\gamma(a) = \gamma(b).$$

In this case, the path forms a closed curve in the complex plane.



Definition 4.4 (Simple Path) ***A path*** γ ***is called simple if it does not intersect itself, except possibly at the endpoints in the case of a closed path. That is, if*** $\gamma(t_1) = \gamma(t_2)$ ***implies either*** $t_1 = t_2$ ***or*** $t_1, t_2 \in \{a, b\}$***.***



4.2 Line Integrals

Line integrals are a fundamental tool in physics and complex analysis. They allow us to "sum" a function or a vector field along a curve in the plane or in space.

4.2.1 Line Integral of a of a Scalar Function

Definition 4.5 (Line integral of a scalar function) ***Let*** γ ***be a curve in the plane, parameterized by***

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

If $f(x,y)$ ***is a continuous function along*** γ***, the line integral of*** f ***along*** γ ***is defined by***

$$\int_\gamma f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\Big(x'(t)\Big)^2 + \Big(y'(t)\Big)^2}\,dt.$$

Here, $ds = \sqrt{(dx)^2 + (dy)^2}$ ***represents an infinitesimal segment of the curve.***

Remark 4.1 ***This integral "weights" the function*** f ***by the length of the path.***

4.2.2 Line Integral of a Vector Field

Definition 4.6 (Line integral of a vector field) ***Let*** $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ ***be a continuous vector field, and*** $\gamma(t) = (x(t), y(t)), t \in [a,b]$ ***an oriented curve. The line integral of*** $\mathbf{F}$ ***along*** γ ***is defined by***

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_a^b \Big[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\Big]\,dt.$$

Remark 4.2 (Physical interpretation:) ***- If*** $\mathbf{F}$ ***represents a force, this integral corresponds to the work done by the force along the path*** γ***. - The projection of the vector*** $\mathbf{F}$ ***onto the tangent vector*** $d\mathbf{r}$ ***of the curve is used.***

Example 4.1 ***Let*** $f(x,y) = x + y$ ***and*** γ ***the line from*** $(0,0)$ ***to*** $(1,1)$ ***parameterized by*** $\gamma(t) = (t,t), t \in [0,1]$***.***

$$x'(t) = 1, \quad y'(t) = 1, \quad ds = \sqrt{1^2 + 1^2}\,dt = \sqrt{2}\,dt.$$

The line integral is then:

$$\int_\gamma f(x,y)\,ds = \int_0^1 (t+t)\sqrt{2}\,dt = \sqrt{2}\int_0^1 2t\,dt = \sqrt{2}.$$

Example 4.2 ***Let*** $\mathbf{F}(x,y) = (y,-x)$ ***and*** $\gamma(t) = (\cos t, \sin t), t \in [0,\pi/2]$ ***(a quarter circle).***

$$x'(t) = -\sin t, \quad y'(t) = \cos t$$

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \Big[y(t)x'(t) + (-x(t))y'(t)\Big]dt = \int_0^{\pi/2} \Big[\sin t(-\sin t) + (-\cos t)(\cos t)\Big]dt$$

$$= \int_0^{\pi/2} -(\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi/2} -1\,dt = -\frac{\pi}{2}.$$

Remark 4.3

- ***The line integral depends on the orientation of the curve.***
- ***If*** $\mathbf{F}$ ***is a conservative field, the integral depends only on the start and end points.***
- ***In complex analysis, line integrals are essential for computing integrals of holomorphic functions along paths in the complex plane.***

4.3 Complex Line Integrals

Line integrals are fundamental in complex analysis and physics. They allow us to integrate a complex function along a curve in the complex plane.

Definition 4.7 (Complex Line Integral) ***Let*** γ ***be a smooth curve in the complex plane, parameterized by***

$$\gamma(t) = x(t) + iy(t), \quad t \in [a, b].$$

If $f : \mathbb{C} \to \mathbb{C}$ ***is continuous along*** γ***, the line integral of*** f ***along*** γ ***is defined by***

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt,$$

where $\gamma'(t) = x'(t) + iy'(t)$***.***

Remark 4.4 ***This is the complex analogue of a vector line integral. The integral combines the function values with the infinitesimal displacements along the curve.***

4.3.1 Decomposition into Real Components

Definition 4.8 (Real Component Form) ***If*** $f(z) = u(x, y) + iv(x, y)$ ***and*** $\gamma(t) = x(t) + iy(t), \ t \in [a, b]$, ***then***

$$\int_\gamma f(z)\, dz = \int_a^b \Big[u(x(t), y(t)) + iv(x(t), y(t))\Big]\Big[x'(t) + iy'(t)\Big]dt.$$

Expanding, we get

$$\int_\gamma f(z)\, dz = \int_a^b \Big(ux' - vy'\Big)dt + i\int_a^b \Big(vx' + uy'\Big)dt,$$

which expresses the integral in terms of real integrals.

Example 4.3 (Simple Complex Line Integral) ***Let*** $f(z) = z$ ***and*** $\gamma(t) = t + it$***,*** $t \in [0, 1]$***. Then***

$$\gamma'(t) = 1 + i, \quad f(\gamma(t)) = t + it.$$

The line integral is

$$\int_\gamma f(z)\, dz = \int_0^1 (t + it)(1 + i)\, dt = \int_0^1 \Big[(t - t) + i(t + t)\Big]dt = \int_0^1 2it\, dt = i.$$

Example 4.4 (Vector Field Analogy) ***Let*** $f(z) = f(x+iy) = u(x, y) + iv(x, y)$***, where*** $u(x, y) = x - y$ ***and*** $v(x, y) = x + y$***, and let*** $\gamma(t) = t + it$***,*** $t \in [0, 1]$***.***

$$\textbf{\textit{Compute}} \ \int_\gamma f(z)\, dz.$$

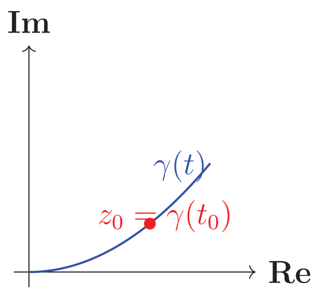
Solution:

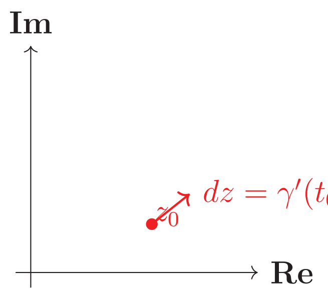
$$\gamma'(t) = 1 + i, \quad f(\gamma(t))f(t + it) = u(t, t) + iv(t, t) = 0 + i(2t) = 2it.$$

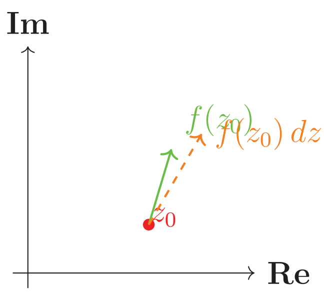
Hence,

$$\int_\gamma f(z)\, dz = \int_0^1 2it\, dt = i.$$

Remarks 4.1 *1- The line integral depends on the orientation of the curve. Reversing the path changes the sign of the integral.*







Remark 4.5

- *The quantity $\gamma'(t) = x'(t) + i\,y'(t)$ is the complex derivative of the path.*
- *The integral depends on the orientation of the curve. If the path is traversed in the opposite direction, then*

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

Definition 4.9 (Integral along a Path) *A path Γ may be composed of a finite number of smooth curves:*

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n,$$

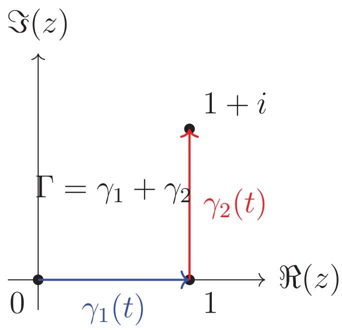
where each $\gamma_k : [a_k, b_k] \to \mathbb{C}$ is of class $\mathcal{C}^1$, and $\gamma_k(b_k) = \gamma_{k+1}(a_{k+1})$. If f is continuous on a domain containing Γ, we define

$$\boxed{\int_{\Gamma} f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.}$$

Example 4.5 (Integral along a Path) *Let the path* $\Gamma = \gamma_1 + \gamma_2$ *be composed of two line segments:*

$$\gamma_1(t) = t, \quad t \in [0,1], \qquad \gamma_2(t) = 1 + it, \quad t \in [0,1].$$

Hence, the path goes from 0 *to* 1*, then from* 1 *to* $1+i$*. Consider* $f(z) = z^2$*.*



By definition, the integral of f *along the path* Γ *is*

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz,$$

where, for each curve γ_k*,*

$$\int_{\gamma_k} f(z)\,dz = \int_{a_k}^{b_k} f(\gamma_k(t))\,\gamma_k'(t)\,dt.$$

For the first segment:

$$\gamma_1(t) = t, \quad \gamma_1'(t) = 1.$$

Thus,

$$\int_{\gamma_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}.$$

For the second segment:

$$\gamma_2(t) = 1 + it, \quad \gamma_2'(t) = i.$$

Hence,

$$\begin{aligned}
\int_{\gamma_2} f(z)\,dz &= \int_0^1 f(\gamma_2(t))\,\gamma_2'(t)\,dt \\
&= \int_0^1 (1+it)^2 \cdot i\,dt \\
&= i\int_0^1 (1 + 2it - t^2)\,dt \\
&= i\left[t + it^2 - \frac{t^3}{3}\right]_0^1 \\
&= i\left(1 + i - \frac{1}{3}\right) = i\left(\frac{2}{3} + i\right) = -1 + \frac{2i}{3}.
\end{aligned}$$

Therefore, the total integral is:

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$
$$= \frac{1}{3} + \left(-1 + \frac{2i}{3}\right)$$
$$= -\frac{2}{3} + \frac{2i}{3}.$$

$$\boxed{\int_{\Gamma} z^2\,dz = \frac{2}{3}(i-1).}$$

Remark 4.6 ***Choice of the Integration Limits on Each Curve***

To compute a complex integral along a path composed of several curves, we first need to parameterize each part of the path and determine its corresponding limits of integration.

Let the path be

$$\Gamma = \gamma_1 + \gamma_2,$$

where

$$\gamma_1(t) = t, \quad t \in [0,1], \qquad \gamma_2(t) = 1 + it, \quad t \in [0,1].$$

For the first curve γ_1***: It goes from the point*** $z = 0$ ***to*** $z = 1$ ***along the real axis. When*** $t = 0$***, we have*** $\gamma_1(0) = 0$***; when*** $t = 1$***, we have*** $\gamma_1(1) = 1$***. Hence, the integration limits are*** $t \in [0,1]$***.***

For the second curve γ_2***: It goes from the point*** $z = 1$ ***to*** $z = 1+i$ ***vertically along the imaginary direction. When*** $t = 0$***, we have*** $\gamma_2(0) = 1$***; when*** $t = 1$***, we have*** $\gamma_2(1) = 1 + i$***. Therefore, the integration limits are also*** $t \in [0,1]$***.***

In general, for a parameterized curve $\gamma : [a,b] \to \mathbb{C}$***, the complex integral is defined by***

$$\int_{\gamma} f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Thus, the integration limits correspond to the parameter values a ***and*** b ***such that*** $\gamma(a)$ ***is the starting point and*** $\gamma(b)$ ***is the endpoint of the curve.***

4.3.2 Properties of Complex Line Integrals

Definition 4.10 ***Properties of Complex Line Integrals***

Let $f(z)$ ***and*** $g(z)$ ***be continuous complex functions defined on a path*** Γ***, and let*** $\alpha, \beta \in \mathbb{C}$ ***be complex constants. The following properties hold:***

1. ***Linearity:***
$$\int_{\Gamma} \Big(\alpha f(z) + \beta g(z)\Big)\,dz = \alpha \int_{\Gamma} f(z)\,dz + \beta \int_{\Gamma} g(z)\,dz.$$

2. ***Constant factor:***
$$\int_C k\,f(z)\,dz = k \int_C f(z)\,dz, \qquad k \in \mathbb{C}.$$

3. ***Additivity (linearity) with respect to functions:***

$$\int_C \Big(f(z) + g(z)\Big)\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz.$$

4. ***Additivity with respect to the path: If the path*** Γ ***is composed of two consecutive curves,***

$$\Gamma = \Gamma_1 + \Gamma_2,$$

then

$$\int_\Gamma f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz.$$

5. ***Reversal of the path: If*** $-\Gamma$ ***denotes the same path traversed in the opposite direction, then***

$$\int_{-\Gamma} f(z)\, dz = -\int_\Gamma f(z)\, dz.$$

6. ***Integral of the conjugate: The complex conjugate satisfies***

$$\overline{\int_\Gamma f(z)\, dz} = \int_\Gamma \overline{f(z)}\, \overline{dz} = \int_\Gamma \overline{f(z)}\, d\overline{z}.$$

Remark 4.7 ***These properties allow us to manipulate complex line integrals in a manner similar to real integrals, but with careful attention to the orientation and parametrization of the path. The additivity property is particularly useful when a path is composed of several segments.***

Theorem 4.1 (Upper Bound Theorem) ***Let*** f ***be a continuous complex function defined on a smooth curve*** C ***parameterized by***

$$z = z(t) = x(t) + iy(t), \quad t \in [a, b].$$

If

$$|f(z)| \le M, \quad \forall z \in C,$$

then the following inequality holds:

$$\left| \int_C f(z)\, dz \right| \le ML,$$

where

$$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

is the length of the curve C***.***

Example 4.6 ***Upper bound for the integral***

$$\int_C \frac{e^z}{z^2 + 1}\, dz, \qquad C : |z| = 4.$$

Solution.

We apply the Upper Bound Theorem***:***

$$\left| \int_C f(z)\, dz \right| \le ML,$$

where

$$M = \max_{z \in C} |f(z)|, \qquad L = \textbf{\textit{length of }} C.$$

1. The function is

$$f(z) = \frac{e^z}{z^2+1}.$$

2. On the circle $|z| = 4$***:***

$$|e^z| = e^{\Re(z)} \le e^{|z|} = e^4.$$

Also,

$$|z^2+1| \ge ||z|^2 - 1| = |16-1| = 15.$$

Hence,

$$|f(z)| = \left| \frac{e^z}{z^2+1} \right| \le \frac{|e^z|}{|z^2+1|} \le \frac{e^4}{15}.$$

Thus $M = \dfrac{e^4}{15}$.

3. The length of the circle C ***is***

$$L = 2\pi r = 2\pi(4) = 8\pi.$$

4. By the upper bound theorem:

$$\left| \int_C \frac{e^z}{z^2+1}\, dz \right| \le ML = \frac{e^4}{15} \cdot 8\pi = \frac{8\pi e^4}{15}.$$

$$\boxed{\left| \int_C \frac{e^z}{z^2+1}\, dz \right| \le \frac{8\pi e^4}{15}.}$$

Remark 4.8 ***Theorem 4.2 (Reverse Triangle Inequality) For any complex numbers*** $a, b \in \mathbb{C}$***, we have:***

$$|\,|a| - |b|\,| \le |a+b|.$$

Equivalently,

$$|a+b| \ge \Big|\,|a| - |b|\,\Big|.$$

Proof. Starting from the standard triangle inequality:

$$|a| = |(a+b) - b| \le |a+b| + |b|.$$

Hence,

$$|a| - |b| \le |a+b|.$$

By interchanging the roles of a ***and*** b***, we also get***

$$|b| - |a| \le |a+b|.$$

Combining both inequalities gives

$$-|a+b| \leq |a| - |b| \leq |a+b|.$$

Therefore,

$$\Big| |a| - |b| \Big| \leq |a+b|.$$

■

4.3.3 Integral over a closed contour

Definition 4.11 ***Let*** $\gamma : [a,b] \to \mathbb{C}$ ***be a piecewise smooth and continuous curve given by*** $\gamma(t) = x(t) + i\,y(t)$***, such that*** $\gamma(a) = \gamma(b)$***. The curve*** γ ***is called a closed contour in the complex plane.***

If $f : D \to \mathbb{C}$ ***is a continuous function on a domain*** D ***containing*** γ***, the integral of*** f ***over the closed contour*** γ ***is defined by:***

$$\oint_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Remark 4.9

1. ***If the curve is not closed (that is,*** $\gamma(a) \neq \gamma(b)$***), the usual notation is used:***

$$\int_\gamma f(z)\,dz,$$

and the integral is called an integral along an open path.

2. ***The notation*** $\oint_\gamma f(z)\,dz$ ***is reserved for the case where the path*** γ ***is closed:***

$$\gamma(a) = \gamma(b).$$

This type of integral naturally appears in the Cauchy Integral Theorem and the Cauchy Integral Formula.

Example 4.7 ***Example: Integral over a closed contour***

Let γ ***be the unit circle centered at the origin, defined by the parametrization:***

$$\gamma(t) = e^{it}, \quad t \in [0, 2\pi].$$

Then, γ ***is a closed contour since*** $\gamma(0) = \gamma(2\pi) = 1$***.***

(a) Case 1: $f(z) = z$

$$\oint_\gamma z\,dz = \int_0^{2\pi} \gamma(t)\,\gamma'(t)\,dt = \int_0^{2\pi} e^{it} \cdot (ie^{it})\,dt = i\int_0^{2\pi} e^{2it}\,dt.$$

Since

$$\int_0^{2\pi} e^{2it}\,dt = \left[\frac{1}{2i}e^{2it}\right]_0^{2\pi} = 0,$$

we obtain:

$$\boxed{\oint_\gamma z\,dz = 0.}$$

(b) Case 2: $f(z) = \dfrac{1}{z}$

$$\oint_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\gamma(t)}\,\gamma'(t)\,dt = \int_0^{2\pi} e^{-it}\cdot(ie^{it})\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Thus,

$$\boxed{\oint_\gamma \frac{1}{z}\,dz = 2\pi i.}$$

Conclusion:

- ***For*** $f(z) = z$***, the integral over the closed contour is zero.***
- ***For*** $f(z) = \dfrac{1}{z}$***, the integral over the closed contour is nonzero and equals*** $2\pi i$***.***

This second case will play a fundamental role in the Cauchy Integral Formula.

4.4 Cauchy Integral Theorem and Cauchy Integral Formula

4.4.1 Simply Connected Domain

In physics, a system is said to be *connected* if all its parts are linked together, that is, if one can move from one point to another without leaving the system. This idea directly applies to complex analysis: a region in the complex plane is *connected* if it forms a continuous whole, with no separated pieces. Similarly, a *connected curve* is one that can be traced without any jump or interruption.

For instance, imagine a continuous metallic wire (fil métallique) forming a circle or a spiral. This wire represents a connected curve, since it can be followed entirely without lifting the pen. If the wire is cut into two separate pieces, we obtain two disjoint curves, which are no longer connected.

Mathematically, a curve

$$\gamma : [a, b] \to \mathbb{C}$$

is connected because its image is the set of points $\gamma(t)$ obtained as t varies continuously from a to b. In other words, it is a continuous curve without any break.

More generally, a subset $D \subset \mathbb{C}$ is said to be *connected* if, for any two points $z_1, z_2 \in D$, there exists at least one continuous path joining z_1 to z_2 while remaining entirely within D.

As a physical illustration, consider the electric field inside a solid conductor: it is defined on a connected region (the conductor is a single piece). If the conductor is split into two isolated parts, the region is no longer connected, since one cannot move from one point to another without leaving the domain.

4.4.2 Cauchy Integral Theorem

Theorem 4.3 (Cauchy Integral Theorem) ***Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be a holomorphic function on D.***

If γ is a closed contour lying entirely in D, then the integral of f around γ is zero:

$$\oint_\gamma f(z)\,dz = 0.$$

Remark 4.10

1. ***The domain D must be simply connected, meaning that it contains no holes (every closed curve in D can be continuously deformed to a point within D).***

2. ***The theorem expresses that the integral of a holomorphic function around any closed path in D is zero. This implies that the value of the integral between two points depends only on the endpoints, not on the path chosen.***

3. ***Geometrically, it means that the complex vector field defined by f is conservative.***

Example 4.8 ***For the function $f(z) = z^2$, and for the closed contour $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, we have:***

$$\oint_\gamma f(z)\,dz = \oint_\gamma z^2\,dz = 0,$$

which is consistent with Cauchy's Theorem.
Explicit calculation: Let $f(z) = z^2$ and let γ be the unit circle parametrized by $\gamma(t) = e^{it}$, for $t \in [0, 2\pi]$. We will compute the integral

$$\oint_\gamma z^2\,dz.$$

Using the parametrization $\gamma(t) = e^{it}$ and $\gamma'(t) = ie^{it}$, we have:

$$\oint_\gamma z^2\,dz = \int_0^{2\pi} \big(\gamma(t)\big)^2 \gamma'(t)\,dt = \int_0^{2\pi} e^{2it}\,(ie^{it})\,dt = i\int_0^{2\pi} e^{3it}\,dt.$$

Now integrate:

$$i\int_0^{2\pi} e^{3it}\,dt = i\left[\frac{1}{3i}e^{3it}\right]_0^{2\pi} = \frac{i}{3i}\Big(e^{6\pi i} - 1\Big) = \frac{1}{3}(1-1) = 0.$$

Therefore,

$$\boxed{\oint_\gamma z^2\,dz = 0,}$$

which explicitly confirms Cauchy's Theorem: the integral of a holomorphic function over any closed contour in a simply connected region is zero.

4.4.3 Cauchy Integral Formula

Theorem 4.4 (Cauchy Integral Formula) ***Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be holomorphic. Let γ be a positively oriented (counterclockwise: dans le sens contraire des aiguilles d?une montre) closed contour contained in D, and let z_0 be any point inside γ.***

Then,

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Remark 4.11

1. ***The formula gives the value of f inside the contour in terms of the values of f on the contour.***
2. ***The Cauchy Integral Formula is one of the most powerful tools in complex analysis; it leads to many fundamental results such as:***
   - ***the infinite differentiability of holomorphic functions,***
   - ***the Cauchy inequalities,***
   - ***the Liouville theorem,***
   - ***and the Maximum Modulus Principle.***

Example 4.9 ***Let $f(z) = \frac{1}{z+1}$, and let γ be the circle $|z| = 2$ oriented counterclockwise. We compute $f(z_0)$ for $z_0 = 1$, which lies inside γ:***

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Substituting $f(z) = \frac{1}{z+1}$ and $z_0 = 1$:

$$f(1) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{(z+1)(z-1)}\, dz.$$

We parametrize the contour γ by $z = 2e^{it}$ with $t \in [0, 2\pi]$, so that $dz = 2ie^{it}\, dt$. Then:

$$\oint_\gamma \frac{1}{(z+1)(z-1)}\, dz = \int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}+1)(2e^{it}-1)}\, dt.$$

Although the integral looks complicated, note that Cauchy's Integral Formula tells us directly that:

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-1}\, dz = f(1),$$

and by simple substitution $f(1) = \frac{1}{2}$.

Remark 4.12 ***The Cauchy Integral Formula is not primarily used to compute $f(z_0)$ directly, since this value can be easily obtained by simple substitution in $f(z)$. Its real importance lies in the fact that it allows us to determine the numerical value of the contour integral without performing the integration.***

In practice, once we identify the function $f(z)$ and verify that the point z_0 lies inside the domain (enclosed by the contour γ), we can immediately conclude that

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz,$$

or equivalently,

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0),$$

without calculating the integral explicitly.

Corollary 4.1 ***If f is holomorphic in D, then for any integer $n \geq 1$,***

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.}$$

Interpretation. Cauchy's Integral Formula expresses the value of a holomorphic function inside a closed contour in terms of the values of the function on the contour itself. It is a cornerstone of complex analysis, from which many fundamental results follow (such as Cauchy's inequalities, Liouville's theorem, and Morera's theorem).

Theorem 4.5 (Generalized Cauchy Integral Formula) ***Let f be a holomorphic function on a simply connected domain D, and let γ be a positively oriented, closed, and piecewise smooth contour lying entirely in D. If z_0 is any point inside γ, then for every integer $n \geq 0$, we have:***

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Remark 4.13 ***This formula shows that all derivatives of a holomorphic function can be expressed as contour integrals. It is a direct consequence of the Cauchy Integral Formula and implies that a holomorphic function is infinitely differentiable inside its domain.***

Example 4.10 ***Let $f(z) = e^z$ and consider the contour $\gamma(t) = 2e^{it}$, with $t \in [0, 2\pi]$, which is a circle of radius 2 centered at the origin. We want to compute the following integral for $z_0 = 1$:***

$$\oint_\gamma \frac{e^z}{(z - 1)^3}\, dz.$$

According to the Cauchy Integral Formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

For $n = 2$***, we have***

$$\oint_\gamma \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!}\,f^{(2)}(z_0).$$

Considering the integrale

$$\oint_\gamma \frac{e^z}{(z-1)^3}\,dz.$$

we have $f(z) = e^z$***, then*** $f^{(2)}(z) = e^z$***. Therefore,***

$$\oint_\gamma \frac{e^z}{(z-1)^3}\,dz = \frac{2\pi i}{2}\,e^1 = \pi i\,e.$$

This confirms the Cauchy Integral Formula for higher-order derivatives.

4.4.4 Cauchy's Inequalities

Proposition 4.1 ***(Cauchy's Inequalities]***

Statement. Let f ***be a holomorphic function in a simply connected domain*** $D \subset \mathbb{C}$***. Let***

$$\Gamma = \{z \in \mathbb{C} : |z - z_0| = r\}$$

be a circle of center z_0 ***and radius*** r***, entirely contained in*** D***. Assume that***

$$|f(z)| \leq M, \quad \forall z \in \Gamma.$$

Then, for every nonnegative integer $n \in \mathbb{N}$***,***

$$\boxed{|f^{(n)}(z_0)| \leq \frac{n!\,M}{r^n}.}$$

Proof. From Cauchy?s integral formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Taking the modulus and using $|f(z)| \leq M$**, we obtain**

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\int_{|z-z_0|=r} \frac{M}{r^{n+1}}\,|dz| = \frac{n!M}{r^n}.$$

Interpretation. Cauchy's inequalities show that the size of the derivatives of a holomorphic function is controlled by its maximum modulus on a surrounding circle. This is a fundamental tool in analytic estimates and proofs of Liouville's theorem.

4.4.5 Integral of $\frac{1}{z-z_0}$ over a closed contour

Theorem 4.6 ***Let γ be a positively oriented closed contour and $z_0 \in \mathbb{C}$. Then***

$$\oint_\gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i, & \textbf{\textit{if } } z_0 \textbf{\textit{ is inside }} \gamma, \\ 0, & \textbf{\textit{if } } z_0 \textbf{\textit{ is outside }} \gamma. \end{cases}$$

Proof. We parametrize the contour γ as $z = z(t)$, $t \in [0, 2\pi]$.

Case 1: z_0 inside γ. Assume γ is a circle of radius r around z_0:

$$z(t) = z_0 + re^{it}, \quad dz = ire^{it}dt.$$

Then

$$\oint_\gamma \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it}dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Case 2: z_0 outside γ. In this case, $\frac{1}{z-z_0}$ is holomorphic on and inside the contour. By Cauchy's theorem,

$$\oint_\gamma \frac{1}{z-z_0}\,dz = 0.$$

■

Remark 4.14 ***This fundamental result allows us to evaluate integrals of the form $\oint_\gamma \frac{f(z)}{z-z_0}\,dz$ when f is holomorphic: we only need to check whether z_0 is inside or outside the contour. It is also the key step in proving the Cauchy integral formula. Integral is zero if z_0 is outside the contour.Indeed,***

if the point z_0 lies outside the closed contour γ, the function

$$f(z) = \frac{1}{z-z_0}$$

is holomorphic on the entire domain containing γ and its interior.

According to Cauchy's theorem, the integral of a holomorphic function over a closed contour is zero:

$$\oint_\gamma f(z)\,dz = 0.$$

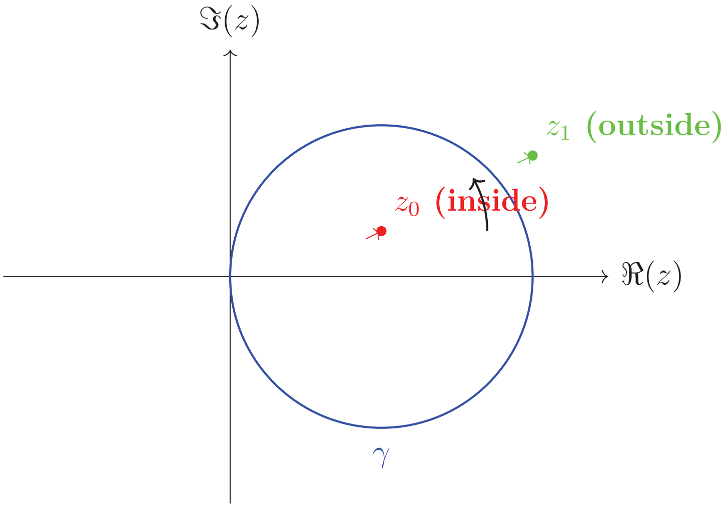
Therefore, when z_0 is outside the contour, the integral $\oint_\gamma \frac{1}{z-z_0}\,dz = 0$.

Example 4.11 ***Let $\gamma : |z| = 1$ and $z_0 = 0.5$. Then***

$$\oint_\gamma \frac{1}{z-0.5}\,dz = 2\pi i.$$

If $z_0 = 2$ (outside the circle), then

$$\oint_\gamma \frac{1}{z-2}\,dz = 0.$$



4.4.6 Cauchy-Goursat Theorem for Domains with Holes

Definition 4.12 (Multiply Connected Domain) ***A domain $D \subset \mathbb{C}$ is called multiply connected if it contains one or more "holes", i.e., subregions removed from the domain. Let $\gamma_0, \gamma_1, \dots, \gamma_n$ be positively oriented closed contours such that:***

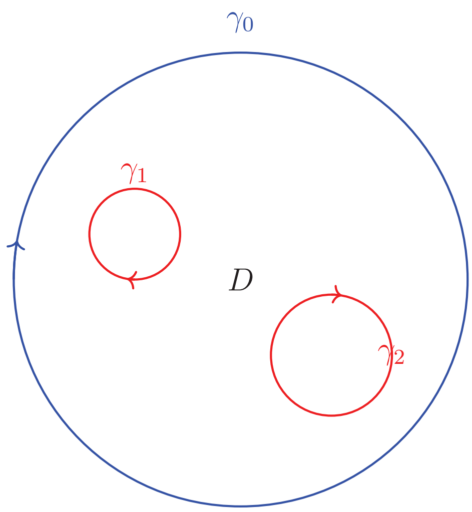
- ***γ_0 is the outer boundary of D,***
- ***$\gamma_1, \dots, \gamma_n$ are boundaries of holes in D.***

Example 4.12 ***Consider the annulus $D = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$, which has one hole (Exclusion region:the disc $|z| < 1$). Let $f(z) = z^2$, $\gamma_0 : |z| = 2$ (outer boundary) and $\gamma_1 : |z| = 1$ (inner boundary), both counterclockwise. Then***

$$\oint_{\gamma_0} z^2\,dz - \oint_{\gamma_1} z^2\,dz = 0.$$

Example 4.13 ***Let D be the domain with two holes as in the figure below. Let $f(z) = e^z$. Then f is holomorphic on D, and***

$$\oint_{\gamma_0} e^z\,dz - \oint_{\gamma_1} e^z\,dz - \oint_{\gamma_2} e^z\,dz = 0.$$

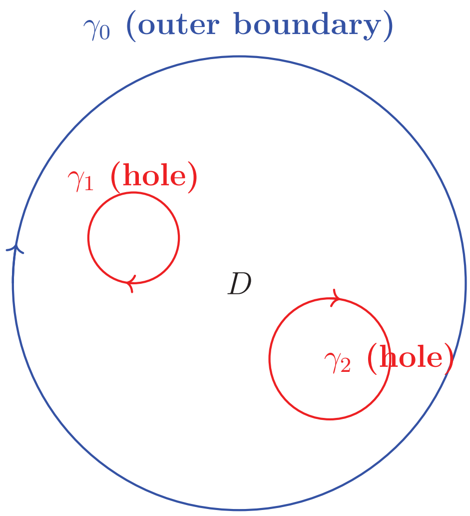


Theorem 4.7 (Cauchy-Goursat for Multiply Connected Domains) *Let $D \subset \mathbb{C}$ be a multiply connected domain and let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be positively oriented closed contours such that:*

- *γ_0 is the outer boundary of D,*
- *$\gamma_1, \ldots, \gamma_n$ are boundaries of holes in D,*

and all contours are oriented positively with respect to the domain. If f is holomorphic on D and continuous on its closure, then

$$\oint_{\gamma_0} f(z)\, dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\, dz.$$

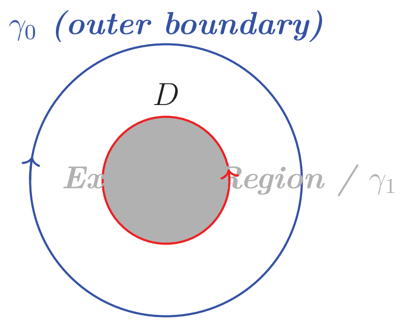


Remark 4.15 *This theorem generalizes the classical Cauchy-Goursat theorem to domains with holes. It states that the integral of a holomorphic function over the outer boundary is equal to the sum of integrals over the inner boundaries (holes), taking orientation into account.*

Example 4.14 (Annulus with one hole) *Consider the annulus*

$$D = \{z \in \mathbb{C} \mid 1 < |z| < 2\},$$

which has one inner excluded region (the disc $|z| < 1$).



Let $f(z) = z^2$, and let $\gamma_0 : |z| = 2$ be the outer boundary, and $\gamma_1 : |z| = 1$ be the inner boundary, both oriented counterclockwise.

1. ***$f(z) = z^2$ is holomorphic on the entire domain D because it is a polynomial (no singularities).***

2. ***According to the Cauchy-Goursat theorem for multiply connected domains, the integral along the outer boundary minus the integral along inner boundaries is zero:***
$$\oint_{\gamma_0} z^2\,dz - \oint_{\gamma_1} z^2\,dz = 0.$$

3. ***Intuitively, the "effect" of the inner hole γ_1 cancels part of the integral around γ_0, making the sum zero. This is why the presence of holes changes the classical Cauchy theorem.***

4. ***If you compute explicitly using parameterization:***
$$\gamma_0(t) = 2e^{it}, \quad t \in [0, 2\pi] \quad \Rightarrow \quad dz = 2ie^{it}dt,$$
$$\oint_{\gamma_0} z^2 dz = \int_0^{2\pi} (2e^{it})^2 \cdot 2ie^{it}dt = 8i\int_0^{2\pi} e^{3it}dt = 0$$
and similarly
$$\oint_{\gamma_1} z^2 dz = \int_0^{2\pi} (1e^{it})^2 \cdot ie^{it}dt = i\int_0^{2\pi} e^{3it}dt = 0.$$
So indeed the formula is verified.

4.5 Primitives and Morera's Theorem

4.5.1 Primitives

Definition 4.13 ***Let f be a function holomorphic on a domain $D \subset \mathbb{C}$. A function $F : D \to \mathbb{C}$ is called a primitive (or antiderivative) of f on D if***
$$F'(z) = f(z), \quad \forall z \in D.$$

Remark 4.16 ***If F is a primitive of f in D, then for any contour γ in D from z_0 to z_1, we have***
$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$
In particular, if γ is a closed contour, the integral vanishes:
$$\oint_\gamma f(z)\,dz = 0.$$

Example 4.15 ***Let $f(z) = e^z$. Then a primitive of f is $F(z) = e^z$, since $F'(z) = e^z = f(z)$. Hence, for any contour γ from z_0 to z_1,***
$$\int_\gamma e^z dz = e^{z_1} - e^{z_0}.$$

4.5.2 Morera's Theorem

Theorem 4.8 (Morera) ***Let f be a continuous function on a domain $D \subset \mathbb{C}$. If for every closed contour γ in D we have***

$$\oint_\gamma f(z)\,dz = 0,$$

then f is holomorphic on D.

Remark 4.17 ***Morera's theorem is essentially the converse of Cauchy's theorem: - Cauchy's theorem says: If f is holomorphic, then its integral around any closed contour is zero. - Morera's theorem says: If a continuous function has zero integral around every closed contour, then it is holomorphic.***

Example 4.16 ***Let $f(z) = \overline{z}$ on $D = \mathbb{C}$. For the closed contour $\gamma(t) = e^{it}, t \in [0, 2\pi]$, we have***

$$\oint_\gamma f(z)dz = \oint_0^{2\pi} \overline{e^{it}} ie^{it}dt = i\int_0^{2\pi} e^{-it}e^{it}dt = i\int_0^{2\pi} 1dt = 2\pi i \neq 0.$$

Hence, by Morera's theorem, f is not holomorphic on $\mathbb{C}$.

Interpretation. Morera's theorem provides a converse to Cauchy's theorem: while Cauchy's theorem states that the integral of a holomorphic function over a closed path is zero, Morera's theorem asserts that if a continuous function has zero integral over every closed contour, then it must be holomorphic.

Remark 4.18 ***Morera's theorem is particularly useful for proving that limits of sequences of holomorphic functions are holomorphic, provided the convergence is uniform on compact subsets.***

Example 4.17 ***Using Morera's Theorem show that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is holomorphic.***

Solution 4.1 ***Let $f_N(z) = \sum_{n=0}^{N} \frac{z^n}{n!}$ denote the N-th partial sum. Each finite sum f_N is a polynomial, hence entire (holomorphic on $\mathbb{C}$). We will show that for every closed, piecewise-smooth contour Γ contained in $\mathbb{C}$,***

$$\int_\Gamma f(z)\,dz = 0,$$

and then invoke Morera's theorem.

Fix a closed contour Γ. For each N,

$$\int_\Gamma f_N(z)\,dz = \int_\Gamma \sum_{n=0}^{N} \frac{z^n}{n!}\,dz = \sum_{n=0}^{N} \frac{1}{n!}\int_\Gamma z^n\,dz.$$

But $\int_\Gamma z^n\,dz = 0$ ***for every integer*** $n \geq 0$ ***because*** z^n ***is entire (Cauchy's theorem). Hence***

$$\int_\Gamma f_N(z)\,dz = 0 \quad \textbf{\textit{for all}}\ N.$$

Next we pass to the limit $N \to \infty$***. The power series for*** f ***has infinite radius of convergence, so it converges uniformly on every compact subset of*** $\mathbb{C}$***. In particular, the sequence*** (f_N) ***converges uniformly to*** f ***on the compact set given by the image of*** Γ***. Therefore we may interchange limit and integral:***

$$\int_\Gamma f(z)\,dz = \lim_{N\to\infty} \int_\Gamma f_N(z)\,dz = \lim_{N\to\infty} 0 = 0.$$

By Morera's theorem (continuous function whose integral over every closed contour is zero is holomorphic), f ***is holomorphic on*** $\mathbb{C}$***. That is, the power series defines an entire function.*** □

4.6 Identity Principle, Mean Value, Maximum Principle, and Classical Theorems

4.6.1 Identity Principle

Definition 4.14 (Accumulation Point) ***Let*** $S \subset D \subset \mathbb{C}$***. A point*** $z_0 \in D$ ***is called an accumulation point of*** S ***if, for every*** $\varepsilon > 0$***, the disk***

$$B(z_0, \varepsilon) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$

contains at least one point of S ***different from*** z_0***.***

In simple terms: no matter how small a circle you draw around z_0***, there is always at least one point of*** S ***inside that circle.***

Example 4.18 ***Let***

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \subset \mathbb{C}.$$

The point 0 ***is an accumulation point of*** S ***because the elements*** $1/n$ ***get arbitrarily close to*** 0***. For any small circle around*** 0***, there is always a point*** $1/n \in S$ ***inside it.***

On the other hand, the point 1 ***is not an accumulation point of*** S***, because we can find a small circle around*** 1 ***that contains no other points of*** S ***besides*** 1 ***itself.***

Theorem 4.9 (Identity Principle) ***Let*** f ***and*** g ***be holomorphic on a connected domain*** $D \subset \mathbb{C}$***. If*** $f(z) = g(z)$ ***on a set*** $S \subset D$ ***that has an accumulation point in*** D***, then*** $f \equiv g$ ***on*** D***.***

Example 4.19 ***Consider the functions***

$$f(z) = e^z, \quad g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \textbf{\textit{on}}\ \mathbb{C}.$$

Define the set of points

$$S = \{0, 0.1, 0.2, 0.3, \dots\}.$$

- $f(z) = g(z)$ ***for all*** $z \in S$***, since the power series for*** $g(z)$ ***converges to*** e^z***.***
- S ***has an accumulation point at*** 0***.***

By the Identity Principle, since both f ***and*** g ***are holomorphic and agree on a set with an accumulation point, we conclude that***

$$f(z) \equiv g(z) \quad \textbf{\textit{on}}\ \mathbb{C}.$$

Remark 4.19 ***This principle shows the*** **rigidity** ***of holomorphic functions: knowing the values on even a tiny set with an accumulation point determines the function everywhere.***

4.6.2 Mean Value Property for Holomorphic Functions

Theorem 4.10 (Mean Value Property) ***Let*** f ***be holomorphic on a domain*** $D \subset \mathbb{C}$***. For any closed disk*** $\overline{B(z_0, r)} \subset D$***:***

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

where z_0 ***is the center and*** r ***the radius.***

Remarks 4.2

1. ***If*** f ***is holomorphic (complex-differentiable) on a domain*** D***, then the value of*** f ***at the center of any disk entirely contained in*** D ***is equal to the*** **average of** f **on the boundary of the disk*****.***
2. - z_0 ***is the center of the disk.***
   - $r > 0$ ***is the radius.***
   - $\overline{B(z_0, r)} = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ ***is the closed disk.***
   - ***The integral***

     $$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

     is the average of f ***along the circle of radius*** r ***centered at*** z_0***.***
3. ***The property follows directly from Cauchy's Integral Formula:***

   $$f(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz.$$

   Parametrize the circle by $z = z_0 + re^{i\theta}$***, so that*** $dz = ire^{i\theta}\, d\theta$***. Substituting gives:***

   $$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta,$$

   which is exactly the Mean Value Property.

4. ***Holomorphic functions are very*** rigid***: the value at the center of a disk is completely determined by the average along any surrounding circle. This property is special to holomorphic functions and does not generally hold for real differentiable functions.***

5. - ***If two holomorphic functions agree on the boundary of a disk, they must agree at the center.***

Example 4.20 Let $f(z) = z^2$ ***and consider the disk centered at*** 0 ***with radius*** r***:***

$$\frac{1}{2\pi}\int_0^{2\pi} (re^{i\theta})^2\, d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} e^{2i\theta}\, d\theta = 0,$$

which equals $f(0) = 0$ ***as expected.***

Theorem 4.11 (Gauss Mean Value Theorem) ***Let*** f ***be holomorphic in a domain*** $U \subset \mathbb{C}$***, and let*** $z_0 \in U$***. Then*** $f(z_0)$ ***is equal to the average of*** f ***on the boundary of any disk centered at*** z_0 ***and contained in*** U***. That is, for any disk*** $D(z_0, r) \subset U$***:***

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Example 4.21 ***Let***

$$f(z) = z^2$$

which is holomorphic on $\mathbb{C}$***. Let*** $z_0 = 1 + i$ ***and consider a disk of radius*** $r > 0$ ***centered at*** z_0***,*** $D(z_0, r)$***.***

According to the Gauss mean value theorem, we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Computation:

$$f(z_0 + re^{i\theta}) = (z_0 + re^{i\theta})^2 = z_0^2 + 2z_0 re^{i\theta} + r^2 e^{2i\theta}.$$

Now integrate over $\theta \in [0, 2\pi]$***:***

$$\frac{1}{2\pi}\int_0^{2\pi} \left(z_0^2 + 2z_0 re^{i\theta} + r^2 e^{2i\theta}\right) d\theta = z_0^2 + \frac{2z_0 r}{2\pi}\int_0^{2\pi} e^{i\theta} d\theta + \frac{r^2}{2\pi}\int_0^{2\pi} e^{2i\theta} d\theta.$$

Since

$$\int_0^{2\pi} e^{i\theta} d\theta = 0 \quad \textbf{\textit{and}} \quad \int_0^{2\pi} e^{2i\theta} d\theta = 0,$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = z_0^2 = f(z_0).$$

Thus, the value at the center equals the average on the circle, as stated by the theorem.

4.6.3 Maximum Principle

Theorem 4.12 (Maximum Principle) ***Let f be holomorphic and non-constant on a connected domain $D \subset \mathbb{C}$. Then $|f(z)|$ cannot attain a maximum value inside D. Any maximum occurs on the boundary of D.***

Remarks 4.3 ***1. If f is holomorphic and not constant on D, the largest value of $|f(z)|$ occurs on the boundary, not inside the domain.***

2. Suppose $|f(z)|$ attains a maximum at some $z_0 \in D$. Consider a small disk centered at z_0, that is to suppose, that $|f(z)|$ attains a maximum at some interior point $z_0 \in D$. Consider a small disk centered at z_0 entirely contained in D. By the Mean Value Property:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Since $|f(z)| \leq |f(z_0)|$ on the circle of radius r, the average of f on the circle cannot exceed $|f(z_0)|$. This forces f to be constant on the circle.

Repeating this argument for overlapping disks covering the connected domain D shows that f must be constant on D.

Contradiction: This is impossible because we assumed at the start that f is non-constant.

Therefore, the assumption that $|f(z)|$ attains a maximum inside D must be false, proving that any maximum occurs on the boundary of D.

3. Holomorphic functions are "rigid"; the modulus cannot have a peak in the interior without the function being constant. The maximum occurs at the boundary.

4.

5. The minimum of $|f|$ for non-zero functions occurs on the boundary.

Example 4.22 Let $f(z) = z$ and $D = \{z \in \mathbb{C} : |z| < 1\}$. Then:

$$|f(z)| = |z| < 1 \text{ for all } z \in D,$$

and the maximum $|f(z)| = 1$ occurs only on the boundary $|z| = 1$.

4.6.4 Maximum Modulus Theorem

Definition 4.15 (Local Maximum of a Complex Function) ***Let f be a complex-valued function. We say that $|f|$ has a*** local maximum ***at $z = z_0$ if there exists a neighborhood U of z_0 such that***

$$\forall z \in U : \quad |f(z)| \leq |f(z_0)|.$$

If the inequality is strict, i.e.,

$$\forall z \in U \setminus \{z_0\} : \quad |f(z)| < |f(z_0)|,$$

then the local maximum is said to be strict***.***

Theorem 4.13 (Maximum Modulus Theorem) ***Let f be a non-constant holomorphic function on a domain $U \subset \mathbb{C}$. Then the modulus $|f(z)|$ cannot attain a local maximum at any point inside U. In other words, if $z_0 \in U$ and there exists a neighborhood V of z_0 such that***

$$|f(z)| \leq |f(z_0)| \quad \textit{for all } z \in V,$$

then f must be constant.

4.6.5 Liouville's Theorem

Theorem 4.14 ***Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (that is, holomorphic everywhere on) $\mathbb{C}$. If there exists a constant $M > 0$ such that***

$$|f(z)| \leq M, \quad \forall z \in \mathbb{C},$$

then f is constant.

Proof. By Cauchy's integral formula, for any $R > 0$ and any z such that $|z| < R$,

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^2}\, dw.$$

Taking the modulus, we get

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|w|=R} \frac{|f(w)|}{|w-z|^2}\, |dw| \leq \frac{M}{R-|z|}.$$

Letting $R \to \infty$, we obtain $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence f is constant.

Corollary 4.2 (Fundamental result) ***Every bounded entire function is constant.***

Intuitive interpretation. Liouville's theorem shows that holomorphic functions cannot remain bounded over the entire complex plane unless they are constant. This result plays a key role in proving the Fundamental Theorem of Algebra.

Remarks 4.4

1. ***If a function f is holomorphic everywhere on $\mathbb{C}$ (entire) and there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f must be constant.***

2. ***Using Cauchy's estimate for the derivative on a disk of radius R around z_0:***

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} dz.$$

Taking absolute values and using $|f(z)| \leq M$:

$$|f'(z_0)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^2} = \frac{M}{R}.$$

Letting $R \to \infty$ gives $|f'(z_0)| \to 0$, so $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) \equiv 0$, hence f is constant.

3. ***Holomorphic functions are rigid. If bounded everywhere, they cannot oscillate or grow, so they must be flat (constant).***

4. ***Any non-constant entire function must grow unbounded somewhere in $\mathbb{C}$.***

   ***Example 4.23** Let $f(z) = e^{iz}$. This function is entire but not bounded, since*

   $$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| = e^{-y},$$

   which grows unbounded as $y \to -\infty$. Therefore, Liouville's theorem does not apply. If f were bounded, it would have to be constant.

4.6.6 D'Alembert's Theorem (Fundamental Theorem of Algebra)

Theorem 4.15 (D'Alembert) ***Every non-constant polynomial $P(z)$ with complex coefficients has at least one complex root.***

Remarks 4.5

1. ***Any polynomial of degree $n \geq 1$ with complex coefficients has at least one root in $\mathbb{C}$.***

2. ***Polynomials grow large as $|z| \to \infty$. If a polynomial had no root, then $1/P(z)$ would be defined and entire. Boundedness of $1/P(z)$ at infinity would then force it to be constant, contradicting the fact that $P(z)$ is non-constant.***

3. ***Proof***

   - ***Suppose $P(z)$ has no roots in $\mathbb{C}$, i.e., $P(z) \neq 0$ for all $z \in \mathbb{C}$.***
   - ***Define $f(z) = 1/P(z)$. Then f is entire.***
   - ***For large $|z|$, $|P(z)| \sim |a_n||z|^n$, so***

     $$|f(z)| = \frac{1}{|P(z)|} \leq \frac{C}{|z|^n} \to 0 \quad \textit{as } |z| \to \infty,$$

     for some constant $C > 0$.
   - ***Therefore, f is bounded on $\mathbb{C}$.***
   - ***By Liouville's Theorem, f must be constant.***
   - ***Contradiction, since $P(z)$ is non-constant.***
   - ***Hence, $P(z)$ has at least one root in $\mathbb{C}$.***

4. ***Consequences: Repeating the argument allows factoring any polynomial completely into linear factors over $\mathbb{C}$:***

   $$P(z) = a_n(z - z_1)(z - z_2)\dots(z - z_n).$$

5. ***This theorem is a direct consequence of Liouville's Theorem: if a polynomial had no roots, $1/P(z)$ would be entire and bounded, leading to a contradiction.***

4.6.7 Rouché's Theorem

Theorem 4.16 ***Let f and g be holomorphic on a domain containing a simple closed contour C and its interior. If***

$$|g(z)| < |f(z)| \quad \textbf{\textit{for all}}\ z \in C,$$

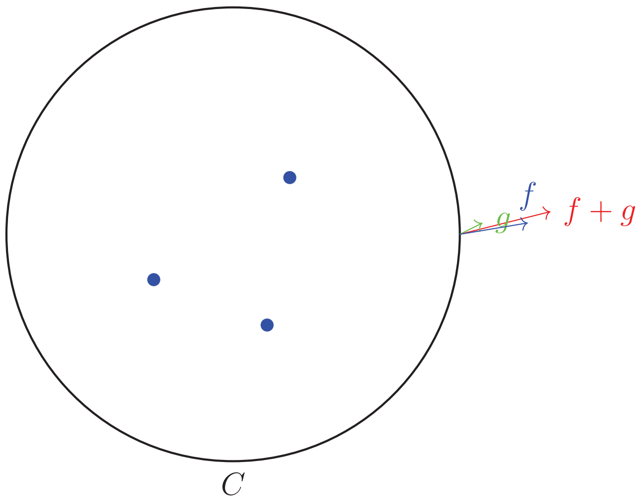
then f and $f+g$ have the same number of zeros inside C, counting multiplicities.

- f is the dominant function on the contour C.
- g is a small addition compared to f.
- The zeros of $f+g$ cannot leave the contour C, so the number of zeros inside remains the same as for f.

Example 4.24 ***Soit***

$$f(z) = z^3, \quad g(z) = 1, \quad C : |z| = 2.$$

- ***On the circle:*** $|f(z)| = |z|^3 = 8$, $|g(z)| = 1 < 8$.
- ***Therefore, f dominates g on the contour.***
- ***Conclusion:*** $f+g = z^3+1$ ***has the same number of zeros inside C as $f = z^3$, that is, 3 zeros.***



Explanation of the figure: The circle represents the contour C. The blue points are the zeros inside. The arrows show the dominant function f and the small addition g. Since f dominates g, the zeros remain inside.

Example 4.25 ***Let***

$$f(z) = z^5 + 3z^2 + 2.$$

We want to determine the number of zeros of $f(z)$ inside the unit circle

$$C : |z| = 1.$$

Step 1. Split $f(z)$ into two parts

We write

$$f(z) = g(z) + h(z),$$

where

$$g(z) = 3z^2, \qquad h(z) = z^5 + 2.$$

Step 2. Check Rouché's condition on $C : |z| = 1$

On the circle $|z| = 1$:

$$|g(z)| = |3z^2| = 3|z|^2 = 3.$$

and

$$|h(z)| = |z^5 + 2| \leq |z^5| + |2| = 1 + 2 = 3.$$

We need a strict inequality $|h(z)| < |g(z)|$. However, this is not true on all of $|z| = 1$, so let us choose another dominant part.

Step 3. Try a different splitting

Take instead:

$$g(z) = z^5, \qquad h(z) = 3z^2 + 2.$$

Then on $|z| = 1$:

$$|g(z)| = |z^5| = 1, \quad |h(z)| = |3z^2 + 2| \leq 3 + 2 = 5.$$

Here $|h(z)| > |g(z)|$, so we swap the roles: set $g(z) = 3z^2 + 2$ and $h(z) = z^5$.

Now:

$$|g(z)| \geq |3z^2| - |2| = 3 - 2 = 1, \quad |h(z)| = |z^5| = 1.$$

The inequality $|h(z)| < |g(z)|$ is true except possibly at a few points, so by Rouché's theorem, $f(z)$ and $g(z)$ have the same number of zeros inside $|z| = 1$.

Step 4. Count zeros of $g(z) = 3z^2 + 2$

We solve $3z^2 + 2 = 0$:

$$z^2 = -\frac{2}{3}, \quad z = \pm i\sqrt{\frac{2}{3}}.$$

Each of these satisfies

$$|z| = \sqrt{\frac{2}{3}} < 1.$$

Hence $g(z)$ has two zeros inside the unit circle.

Step 5. Conclusion

By Rouché's theorem, $f(z) = z^5 + 3z^2 + 2$ also has exactly two zeros inside $|z| = 1$.

$$\boxed{\textbf{Number of zeros of } f(z) \textbf{ inside } |z| = 1 \textbf{ is } 2.}$$

4.7 Solved Exercises

Exercise 4.1 ***Calculate***

$$\oint_C \frac{5z+7}{z^2-2z-3}\,dz, \quad C: |z-2| = 2,$$

using the result for $\oint 1/(z-a)\,dz$***.***

Step 1: Factor the denominator

$$z^2 - 2z - 3 = (z-3)(z+1).$$

Step 2: Partial fraction decomposition

$$\frac{5z+7}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}.$$

Solving for A ***and*** B***:***

$$A = \frac{11}{2}, \quad B = -\frac{1}{2}.$$

So

$$\frac{5z+7}{(z-3)(z+1)} = \frac{11/2}{z-3} - \frac{1/2}{z+1}.$$

Step 3: Split the integral

$$\oint_C \frac{5z+7}{(z-3)(z+1)}\,dz = \frac{11}{2}\oint_C \frac{1}{z-3}\,dz - \frac{1}{2}\oint_C \frac{1}{z+1}\,dz.$$

Step 4: Determine which poles are inside C

- $z = 3$***:*** $|3-2| = 1 < 2$. ***(inside)***

Then

$$\oint_C \frac{1}{z-3}\,dz = 2\pi i$$

- $z = -1$***:*** $|-1-2| = 3 > 2$. ***(outside)***

Then

$$\oint_C \frac{1}{z+1}\,dz = 0.$$

Step 5: Compute the integral

$$\oint_C \frac{5z+7}{z^2-2z-3}\,dz = \frac{11}{2}\cdot 2\pi i - \frac{1}{2}\cdot 0 = 11\pi i.$$

$$\boxed{\oint_C \frac{5z+7}{z^2-2z-3}\,dz = 11\pi i}.$$

Exercise 4.2 (Complex integral using Cauchy-Goursat theorem) ***Evaluate***

$$\oint_C \frac{dz}{z^2+1}, \quad C: |z| = 4$$

using the Cauchy-Goursat theorem for multiply connected domains.

Step 1: Identify singularities.
The function is

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

The singularities (poles) are

$$z = i \quad \textbf{\textit{and}} \quad z = -i.$$

Step 2: Define contours as sets.
- Outer contour (outer boundary):

$$\gamma_0 = \{z \in \mathbb{C} \mid |z| = 4\}$$

- Inner contour around $z = i$ (first singularity):

$$\gamma_1 = \{z \in \mathbb{C} \mid |z-i| = r\}, \quad r > 0 \ \textbf{\textit{small enough}}$$

- Inner contour around $z = -i$ (second singularity):

$$\gamma_2 = \{z \in \mathbb{C} \mid |z+i| = r\}, \quad r > 0 \ \textbf{\textit{small enough}}$$

Remark: Choosing the radius r for inner contours.
We can take any small positive r for the inner contours. The only requirements are:
1. $r > 0$ so that the contour actually surrounds the singularity.
2. r is small enough that the inner contour does not enclose any other singularities (here, each contour should enclose only one singularity).
So for our example:

$$\gamma_1 = \{z \in \mathbb{C} \mid |z-i| = r\}, \quad \gamma_2 = \{z \in \mathbb{C} \mid |z+i| = r\}, \quad r > 0 \ \textbf{\textit{small enough.}}$$

Any r satisfying $0 < r < 2$ works (since the distance between the singularities is 2). The exact value of r does not affect the integral, because of Cauchy's theorem: the integral around a simple pole depends only on whether the pole is inside, not on the radius of the contour.
Step 3: Parameterize the contours.

$$\gamma_0(t) = 4e^{it}, \quad \gamma_1(t) = i + re^{it}, \quad \gamma_2(t) = -i + re^{it}, \quad t \in [0, 2\pi]$$

$$dz = iRe^{it}dt \quad \textbf{\textit{for each contour with radius}} \ R$$

Step 4: Partial fraction decomposition.

$$\frac{1}{z^2+1} = -\frac{i}{2}\frac{1}{z-i} + \frac{i}{2}\frac{1}{z+i}$$

Step 5: Integrate around the inner contours explicitly.

Contour γ_1 around $z = i$:

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_1} -\frac{i}{2}\frac{1}{z-i}dz + \oint_{\gamma_1} \frac{i}{2}\frac{1}{z+i}dz$$

- First term:

$$\oint_{\gamma_1} -\frac{i}{2}\frac{1}{z-i}dz = -\frac{i}{2}\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = -\frac{i}{2} \cdot i \cdot 2\pi = \pi$$

- Second term (holomorphic inside γ_1):

$$\oint_{\gamma_1} \frac{i}{2}\frac{1}{z+i}dz \approx 0$$

Total:

$$\oint_{\gamma_1} f(z)dz = \pi$$

Contour γ_2 around $z = -i$:

$$\oint_{\gamma_2} f(z)dz = \oint_{\gamma_2} -\frac{i}{2}\frac{1}{z-i}dz + \oint_{\gamma_2} \frac{i}{2}\frac{1}{z+i}dz$$

- First term (holomorphic inside γ_2):

$$\oint_{\gamma_2} -\frac{i}{2}\frac{1}{z-i}dz \approx 0$$

- Second term:

$$\oint_{\gamma_2} \frac{i}{2}\frac{1}{z+i}dz = \frac{i}{2}\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = -\pi$$

Total:

$$\oint_{\gamma_2} f(z)dz = -\pi$$

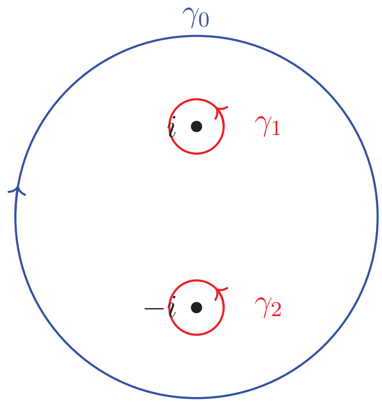
Step 6: Apply the Cauchy-Goursat theorem for multiply connected domains.

$$\oint_{\gamma_0} f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz = \pi + (-\pi) = 0$$

Conclusion:

$$\boxed{\oint_C \frac{dz}{z^2+1} = 0}$$

This confirms that the integral along the outer contour equals the sum of integrals around the inner contours, as predicted by the Cauchy-Goursat theorem for multiply connected domains.



Exercise 4.3 ***Let $f : \mathbb{C} \to \mathbb{C}$ be defined by***

$$f(z) = \begin{cases} \frac{z^2}{|z|^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

1. ***Show that f is continuous at every point $z \neq 0$.***

2. ***Consider the circle $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Compute***

$$\oint_\gamma f(z)\, dz.$$

3. ***Using Morera's theorem, determine whether f is holomorphic at $z = 0$.***

Solution 4.2 Step 1: Continuity for $z \neq 0$

For $z \neq 0$,

$$f(z) = \frac{z^2}{|z|^2} = \frac{z^2}{z\overline{z}} = \frac{z}{\overline{z}}.$$

Both z and $\overline{z}$ are continuous for $z \neq 0$, so $f(z)$ is continuous on $\mathbb{C} \setminus \{0\}$.

Step 2: Compute the integral along γ

Take $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $dz = ie^{it}dt$ and

$$f(\gamma(t)) = \frac{\gamma(t)^2}{|\gamma(t)|^2} = \frac{e^{2it}}{1} = e^{2it}.$$

Thus, the integral becomes:

$$\oint_\gamma f(z)\, dz = \int_0^{2\pi} f(\gamma(t))\, dz = \int_0^{2\pi} e^{2it} ie^{it} dt = i \int_0^{2\pi} e^{3it} dt.$$

But

$$\int_0^{2\pi} e^{3it} dt = \frac{1}{3}\left[e^{3it}\right]_0^{2\pi} = \frac{1}{3}(e^{6\pi i} - 1) = 0.$$

Hence,

$$\oint_\gamma f(z)\, dz = 0.$$

Step 3: Apply Morera's theorem

Morera's theorem states that if f ***is continuous on a domain*** D ***and*** $\oint_\gamma f(z)\,dz = 0$ ***for every closed contour*** $\gamma \subset D$***, then*** f ***is holomorphic on*** D***.***

- f ***is continuous on*** $\mathbb{C} \setminus \{0\}$***, and the integral around any small circle not enclosing other singularities is zero. - Therefore,*** f ***is holomorphic on*** $\mathbb{C} \setminus \{0\}$***.***

Step 4: Check holomorphy at $z = 0$

We examine the derivative at $z = 0$***:***

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{z/\overline{z}}{z} = \lim_{z\to 0} \frac{1}{\overline{z}}.$$

The limit depends on the path:

- Along $z = x$ ***(real axis):*** $\frac{1}{\overline{z}} = \frac{1}{x} \to \infty$***.***

- Along $z = iy$ ***(imaginary axis):*** $\frac{1}{\overline{z}} = \frac{1}{-iy} = \frac{i}{y} \to \infty$***.***

Hence, the limit does not exist, and f ***is not holomorphic at*** 0***.***

Example 4.26 *Using Cauchy's theorem, calculate*

$$\oint_C \frac{z}{z^2+9}\,dz, \quad C : |z - 2i| = 4$$

Step 1: Factor the denominator and find singularities.

We have

$$z^2 + 9 = (z + 3i)(z - 3i).$$

Hence, the singularities (poles) are

$$z_1 = 3i, \quad z_2 = -3i.$$

Step 2: Determine which singularities are inside C***.***

The contour is centered at $2i$ ***with radius*** 4***:***

$$C = \{z \in \mathbb{C} \mid |z - 2i| = 4\}.$$

The distances are:

$$|3i - 2i| = 1 < 4 \quad \textbf{\textit{inside,}} \qquad |-3i - 2i| = 5 > 4 \quad \textbf{\textit{outside.}}$$

So only $z_1 = 3i$ ***is inside*** C***.***

Step 3: Partial fraction decomposition.

$$\frac{z}{z^2+9} = \frac{z}{(z-3i)(z+3i)} = \frac{A}{z-3i} + \frac{B}{z+3i}.$$

Multiply both sides by $(z - 3i)(z + 3i)$***:***

$$z = A(z + 3i) + B(z - 3i)$$

Set $z = 3i$***:***

$$3i = A(6i) \implies A = \frac{1}{2}$$

Set $z = -3i$***:***

$$-3i = B(-6i) \implies B = \frac{1}{2}$$

Hence:

$$\frac{z}{z^2+9} = \frac{1/2}{z-3i} + \frac{1/2}{z+3i}.$$

Step 4: Integrate along C***.***
Only the singularity inside C ***contributes:***

$$\oint_C \frac{z}{z^2+9}dz = \oint_C \frac{1/2}{z-3i}dz + \oint_C \frac{1/2}{z+3i}dz = \oint_C \frac{1/2}{z-3i}dz + 0.$$

Step 5: Apply Cauchy's integral formula.
For a simple pole z_0 ***inside*** C***:***

$$\oint_C \frac{dz}{z-z_0} = 2\pi i.$$

Hence:

$$\oint_C \frac{1/2}{z-3i}dz = \frac{1}{2} \cdot 2\pi i = \pi i.$$

Step 6: Final answer.

$$\boxed{\oint_C \frac{z}{z^2+9}dz = \pi i}$$

Second Method: Using Cauchy's Integral Formula

Consider the integral

$$\oint_C \frac{z}{z^2+9}\,dz, \quad C : |z-2i| = 4$$

where C ***is oriented counterclockwise.***

Step 1: Factor the denominator and identify singularities.

$$z^2 + 9 = (z-3i)(z+3i)$$

The singularities are

$$z_1 = 3i \quad \textbf{\textit{(inside }} C\textbf{\textit{)}}, \qquad z_2 = -3i \quad \textbf{\textit{(outside }} C\textbf{\textit{)}}.$$

Step 2: Rewrite the integrand in the form $\frac{f(z)}{z-z_0}$***.***

$$\frac{z}{z^2+9} = \frac{z}{(z-3i)(z+3i)} = \frac{\frac{z}{z+3i}}{z-3i}.$$

Let $z_0 = 3i$ ***(the singularity inside*** C***) and define***

$$f(z) = \frac{z}{z+3i}.$$

Notice that $f(z)$ is holomorphic inside and on C because its only singularity $z = -3i$ lies outside C.

Step 3: Apply Cauchy's integral formula.

Cauchy's integral formula states:

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$$

Step 4: Evaluate $f(z_0)$.

$$f(3i) = \frac{3i}{3i + 3i} = \frac{3i}{6i} = \frac{1}{2}$$

Step 5: Compute the integral.

$$\oint_C \frac{z}{z^2 + 9} dz = 2\pi i \cdot \frac{1}{2} = \pi i$$

Conclusion:

Using this second method with Cauchy's integral formula and the holomorphic function

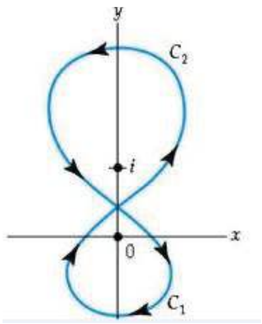
$$f(z) = \frac{z}{z + 3i},$$

we find

$$\boxed{\oint_C \frac{z}{z^2 + 9} dz = \pi i}.$$

Exercise 4.4 *Evaluate the integral*

$$\oint_C \frac{z^3 + 3}{z(z - i)^2} dz$$



where $C = C_1 + C_2$.

Part 2: Power Series and Laurent Series

4.8 Series in the Complex Plane

4.8.1 Sequence of Complex Numbers

Definition 4.16 (Sequence of Complex Numbers) ***A complex sequence*** $\{z_n\}$ ***is a correspondence that associates to each positive integer*** $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ ***a unique complex value*** $z_n \in \mathbb{C}$***. In other words, a sequence is simply a list of complex numbers ordered by their indices.***

Example. Consider the sequence defined by

$$z_n = 1 + in.$$

Its first few terms are:

$$z_1 = 1 + i, \quad z_2 = 1 + 2i, \quad z_3 = 1 + 3i, \quad z_4 = 1 + 4i, \ \ldots$$

Each term lies on a vertical line parallel to the imaginary axis in the complex plane.

We say that a sequence $\{z_n\}$ ***converges to a complex number*** L ***if the limit***

$$\lim_{n \to \infty} z_n = L$$

exists. In that case, L ***is called the limit of the sequence.***

Proposition 4.2 (Convergence of a Complex Sequence) ***A sequence of complex numbers*** $\{z_n\}$ ***converges to a complex limit***

$$L = a + ib$$

if and only if the real parts and imaginary parts converge separately, that is,

$$\Re(z_n) \to a \quad \textbf{\textit{and}} \quad \Im(z_n) \to b \quad \textbf{\textit{as}}\ n \to \infty.$$

Example 4.27 (Convergence of a Complex Sequence) ***Determine whether the sequence***

$$z_n = \frac{3 + ni}{n + 2ni}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$. If it does, find its limit.

Solution: We multiply the numerator and the denominator by the conjugate of the denominator:

$$z_n = \frac{(3+ni)(n-2ni)}{(n+2ni)(n-2ni)} = \frac{(3+ni)(n-2ni)}{n^2+4n^2}.$$

Expanding the numerator gives

$$(3+ni)(n-2ni) = 3n - 6ni + n^2 i - 2n^2 i^2 = (2n^2+3n) + i(n^2-6n).$$

Hence,

$$z_n = \frac{2n^2+3n}{5n^2} + i\,\frac{n^2-6n}{5n^2}.$$

Now, we identify the real and imaginary parts:

$$\Re(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \longrightarrow \frac{2}{5},$$

$$\Im(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \longrightarrow \frac{1}{5},$$

as $n \to \infty$.

Conclusion: From Theorem 6.1, since both the real and imaginary parts converge, the sequence (z_n) converges in $\mathbb{C}$ to

$$a + ib = \frac{2}{5} + \frac{i}{5}.$$

$$\boxed{\lim_{n\to\infty} z_n = \frac{2}{5} + \frac{i}{5}.}$$

4.8.2 Series of Complex Numbers

Definition 4.17 ***Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers. The expression***

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is called an infinite series of complex terms. We define its partial sums by

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + z_3 + \cdots + z_n.$$

The series is said to converge if the sequence (S_n) approaches a finite limit L as $n \to \infty$; in that case, we write

$$\sum_{k=1}^{\infty} z_k = L,$$

and we say that the series converges to L, or equivalently, that L is the sum of the series.

Geometric Series

Definition 4.18 ***A geometric series is an infinite series of the form***

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{4.1}$$

where a ***and*** z ***are complex numbers.***

The n***th partial sum of this series is given by***

$$S_n = a + az + az^2 + \cdots + az^{n-1} = a\,\frac{1-z^n}{1-z}.$$

If $|z| < 1$***, then*** $z^n \to 0$ ***as*** $n \to \infty$***, and consequently***

$$S_n \longrightarrow \frac{a}{1-z}.$$

Hence, for all complex numbers z ***satisfying*** $|z| < 1$***, the geometric series*** (4.32) ***converges and its sum is***

$$\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots . \tag{4.2}$$

On the other hand, when $|z| \geq 1$***, the sequence of partial sums*** (S_n) ***does not converge, and therefore the geometric series*** (4.32) ***diverges.***

Remark 4.20 ***Two important cases of geometric series arise for specific values of the parameters*** a ***and*** z***.***

1. ***When*** $a = 1$ ***and*** z ***is any complex number such that*** $|z| < 1$***, the geometric series becomes***
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots . \tag{4.3}$$
2. ***Similarly, by replacing*** z ***with*** $-z$ ***in*** (4.3)***, we obtain***
$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots , \tag{4.4}$$
which is valid whenever $|z| < 1$***.***

These two identities are special forms of the geometric series (4.32) ***and are often used in analytic computations and power series expansions.***

Remark 4.21 ***For any complex number*** $z \neq 1$***, we have the finite geometric identity***

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}. \tag{4.5}$$

If we decompose the left-hand side of (4.5) ***as***

$$\frac{1-z^n}{1-z} = \frac{1}{1-z} - \frac{z^n}{1-z},$$

we can isolate $\frac{1}{1-z}$ ***and obtain the alternative expression***

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1-z}. \tag{4.6}$$

Equation (4.6) ***provides an explicit formula for the remainder term*** $\frac{z^n}{1-z}$***, which tends to zero as*** $n \to \infty$ ***whenever*** $|z| < 1$***.***

Example 4.28 ***Consider the infinite series***

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots \tag{4.7}$$

which is a geometric series.

It can be written in the standard form (4.32) ***with***

$$a = \frac{1+2i}{5} \quad \textbf{\textit{and}} \quad z = \frac{1+2i}{5}.$$

To determine whether the series converges, we compute the modulus of z***:***

$$|z| = \frac{|1+2i|}{5} = \frac{\sqrt{1^2+2^2}}{5} = \frac{\sqrt{5}}{5} < 1.$$

Since $|z| < 1$***, the series*** (4.7) ***converges.***

Using the general formula for the sum of a convergent geometric series,

$$\sum_{k=1}^{\infty} az^{k-1} = \frac{a}{1-z},$$

we obtain

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1-\frac{1+2i}{5}} = \frac{1+2i}{5-(1+2i)} = \frac{1+2i}{4-2i}.$$

To simplify this expression, multiply numerator and denominator by the conjugate $4+2i$***:***

$$\frac{1+2i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{(1+2i)(4+2i)}{(4-2i)(4+2i)} = \frac{4+2i+8i+4i^2}{16+4} = \frac{4-4+10i}{20} = \frac{i}{2}.$$

Therefore, the geometric series (4.7) ***is convergent and its sum is***

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{i}{2}. \tag{4.8}$$

4.8.3 Convergence Criteria

Theorem 4.17 (Necessary Condition for Convergence) ***Let*** $\sum_{k=1}^{\infty} z_k$ ***be an infinite series of complex numbers. If this series converges, then the sequence of its terms tends to zero, that is,***

$$\lim_{k\to\infty} z_k = 0.$$

Remark 4.22 ***This condition is*** **necessary** ***but not*** **sufficient**. ***In other words, even if*** $\lim_{k\to\infty} z_k = 0$***, the series*** $\sum_{k=1}^{\infty} z_k$ ***may still diverge. For example, the harmonic series***

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges, although its general term tends to zero.

Theorem 4.18 (The n**th-Term Test for Divergence)** ***If***

$$\lim_{n\to\infty} z_n \neq 0,$$

then the series

$$\sum_{k=1}^{\infty} z_k$$

diverges.

Remark 4.23 ***This test provides a simple way to detect divergence: if the individual terms of a series do not approach zero, the sequence of partial sums cannot converge. However, the converse is not true- the condition*** $\lim_{n\to\infty} z_n = 0$ ***does*** **not** ***guarantee convergence.***

Theorem 4.19 (Ratio Test for Complex Series) ***Let***

$$\sum_{k=1}^{\infty} z_k$$

be a series whose terms z_k ***are nonzero complex numbers, and suppose that***

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Then:

(i) If $L < 1$***, the series converges absolutely.***

(ii) If $L > 1$ ***or*** $L = \infty$***, the series diverges.***

(iii) If $L = 1$***, the test gives no information (the result is inconclusive).***

Theorem 4.20 (Root Test for Complex Series) ***Let***

$$\sum_{k=1}^{\infty} z_k$$

be a series of complex terms. Assume that the following limit exists:

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L.$$

Then:

(i) If $L < 1$***, the series converges absolutely.***

(ii) If $L > 1$ ***or*** $L = \infty$***, the series diverges.***

(iii) If $L = 1$***, the test is inconclusive.***

4.8.4 Complex Polynomials

Definition 4.19 ***A complex polynomial function is a function of the form***

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ ***and*** n ***is a non-negative integer.***
The degree of the polynomial is n ***if*** $a_n \neq 0$***.***

Example 4.29

$$P(z) = 3z^2 + 2z + 1 \quad \textbf{and} \quad Q(z) = z^3 - iz + 2$$

are complex polynomials of degree 2 ***and*** 3***, respectively.***

Remark 4.24 ***Polynomials are the simplest examples of entire functions, since they are differentiable (and hence holomorphic) everywhere in*** $\mathbb{C}$***.***

4.8.5 From Polynomials to Power Series

Polynomials involve a finite number of terms. It is natural to ask what happens if we allow an infinite number of terms of the form $a_n(z - z_0)^n$**.**

This leads us to the notion of a power series, which can be viewed as an infinite-degree polynomial.

Definition 4.20 (Complex Power Series) ***A complex power series centered at*** $z_0 \in \mathbb{C}$ ***is an infinite series of the form***

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where (a_n) ***is a sequence of complex numbers.***

For each fixed z**, the series may converge or diverge. The set of all** $z \in \mathbb{C}$ **for which the series converges is called its domain of convergence.**

4.8.6 Radius and Disk of Convergence

Definition 4.21 (Radius of Convergence) ***There exists a real number*** $R \geq 0$***, possibly*** $R = +\infty$***, such that:***

$$\begin{cases} \textbf{\textit{The series converges absolutely for all }} z \textbf{\textit{ such that }} |z - z_0| < R, \\ \textbf{\textit{The series diverges for all }} z \textbf{\textit{ such that }} |z - z_0| > R. \end{cases}$$

The number R ***is called the radius of convergence, and the disk***

$$D(z_0, R) = \{ z \in \mathbb{C} : |z - z_0| < R \}$$

is called the disk of convergence.

Theorem 4.21 (Radius of Convergence) ***For a general power series***

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

the limit involved in the ratio test depends only on the coefficients a_k***. Specifically, if***

1. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$***, then the radius of convergence is***

$$R = \frac{1}{L}.$$

2. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$***, then the radius of convergence is*** $R = \infty$***.***

3. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$***, then the radius of convergence is*** $R = 0$***.***

A similar results follows from the root test using

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Theorem 4.22 (Root Test and Radius of Convergence) ***If a power series***

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

satisfies

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0,$$

then the radius of convergence is given by

$$R = \frac{1}{L}.$$

If the limit is $L = 0$***, then*** $R = \infty$***; if*** $L = \infty$***, then*** $R = 0$***.***

Example 4.30 *For the series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we can check that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|z|}{n+1} \to 0 \quad \text{when } n \to \infty.$$

This means that the series converges for every value of z in the complex plane.

Therefore, the radius of convergence is infinite:

$$R = +\infty.$$

Because the series converges everywhere, the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

is well defined for all complex numbers z.

Each term $\frac{z^n}{n!}$ is a polynomial, so it is easy to see that $f(z)$ is smooth and differentiable everywhere. That is why we say that e^z is an entire function, which means "holomorphic on the whole complex plane."

Example 4.31 *Consider the power series*

$$\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}.$$

By applying the ratio test, we examine the limit

$$\lim_{n\to\infty} \left| \frac{\dfrac{z^{n+2}}{n+1}}{\dfrac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |z| = |z|. \tag{4.9}$$

Hence, the series converges absolutely for $|z| < 1$.

The circle of convergence is therefore defined by $|z| = 1$, and the corresponding radius of convergence is $R = 1$.

On the circle $|z| = 1$, the series does not converge absolutely, since

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is the classical divergent harmonic series. However, the series may converge at some points on the circle. For instance, at $z = -1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

is the convergent alternating harmonic series. In fact, this particular series converges for every point on the unit circle except $z = 1$.

4.8.7 Taylor Series

Power Series and Their Associated Functions

There exists a one-to-one correspondence between any complex number z located within the circle of convergence and the value to which the series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges. In this sense, a power series *defines* or *represents* a function f. For each specific z inside the circle of convergence, the number L to which the power series converges is defined as the value of f at z; that is,

$$f(z) = L.$$

In this section, we present some important properties concerning the nature of this function f.

In the previous section, we established that every power series possesses a radius of convergence R. Throughout this discussion, we will assume that the power series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

has either a positive or an infinite radius of convergence R.

Proposition 4.3 ***Suppose that f has a power series expansion***

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad z \in D(z_0, R),$$

and let $0 < r < R$. Then we have

$$\sum_{n=0}^{\infty} |a_n| r^n = \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| d\theta.$$

Differentiation and Integration of Power Series

The following three theorems show that a function f, defined by a power series, is continuous, differentiable, and integrable within its circle of convergence.

Theorem 4.23 (Continuity) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

represents a continuous function f within its circle of convergence $|z-z_0| = R$.

Theorem 4.24 (Term-by-Term Differentiation) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

can be differentiated term by term within its circle of convergence $|z-z_0|=R$.
Differentiating the series term by term yields:

$$\frac{d}{dz}\left(\sum_{k=0}^{\infty} a_k(z-z_0)^k\right) = \sum_{k=0}^{\infty} a_k \frac{d}{dz}\left((z-z_0)^k\right) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}.$$

Theorem 4.25 (Term-by-Term Integration) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

can be integrated term by term within its circle of convergence $|z-z_0|=R$, ***for every contour*** C ***lying entirely inside the circle of convergence.***

The theorem states that

$$\int_C \left(\sum_{k=0}^{\infty} a_k(z-z_0)^k\right) dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k\, dz = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1} + \textbf{constant},$$

whenever the contour C lies in the interior of the circle of convergence $|z-z_0|=R$. Suppose a power series represents a function f within the circle of convergence $|z-z_0|=R$; that is,

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots . \tag{4.10}$$

Taylor Series

It follows from Theorem 4.24 that the derivatives of f are given by the following series:

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots . \tag{4.11}$$

$$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k (z-z_0)^{k-2} = 2\cdot 1\, a_2 + 3\cdot 2\, a_3(z-z_0) + \cdots . \tag{4.12}$$

$$f^{(3)}(z) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k (z-z_0)^{k-3} = 3\cdot 2\cdot 1\, a_3 + \cdots . \tag{4.13}$$

There is a relationship between the coefficients a_k in (4.10) and the derivatives of f. Evaluating (4.10), (4.11), (4.12), and (4.13) at $z=z_0$ gives:

$$f(z_0) = a_0, \quad f'(z_0) = 1!\, a_1, \quad f''(z_0) = 2!\, a_2, \quad f^{(3)}(z_0) = 3!\, a_3.$$

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respectively.
In general,

$$f^{(n)}(z_0) = n!\, a_n, \quad \textbf{or equivalently,} \quad a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0. \tag{4.14}$$

When $n = 0$ in (4.14), we interpret the zero-order derivative as $f(z_0)$ and note that $0! = 1$, so that the formula gives $a_0 = f(z_0)$.
Substituting (4.14) into (4.10), we obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \tag{4.15}$$

The series given in (4.15) is called the Taylor series for f centered at z_0. A Taylor series with center $z_0 = 0$ is referred to as a Maclaurin series, and is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, z^k. \tag{4.16}$$

If we are given a function f that is analytic in some domain D, can we represent it by a power series of the form (4.15) or (4.16)?

Theorem 4.26 (Taylor's Theorem) ***Let f be analytic within a domain D, and let z_0 be a point in D. Then f has the series representation***

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \tag{4.17}$$

which is valid for the largest circle C with center at z_0 and radius R that lies entirely within D.

Example 4.32 (Some Important Maclaurin Series) ***The following are some of the most common Maclaurin series expansions:***

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{4.18}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}. \tag{4.19}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}. \tag{4.20}$$

Example 4.33 ***We aim to find the Maclaurin expansion of*** $f(z) = \frac{1}{(1-z)^2}$.

1. Find the Maclaurin expansion of $g(z) = \frac{1}{1-z}$,

2. Find the Maclaurin expansion of $\frac{d}{dz}\frac{1}{1-z}$ ***3- Deduce the Maclaurin expansion of*** $f(z) = \frac{1}{(1-z)^2}$.

Solution 4.3 ***For*** $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

we can differentiate both sides of this expression with respect to z. ***Thus,***

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}(1) + \frac{d}{dz}(z) + \frac{d}{dz}(z^2) + \frac{d}{dz}(z^3) + \cdots ,$$

which simplifies to

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}.$$

Motivating Question

We know that if a complex function $f(z)$ is analytic in a neighborhood of a point z_0, it can be expanded into a Taylor series about z_0.

But what happens if $f(z)$ is not analytic at z_0, even though it is analytic in some region around this point?

Can we still represent $f(z)$ by a series expansion similar to the Taylor series?

The answer is yes. In this situation, we use the ***Laurent series***, which generalizes the Taylor expansion by allowing both positive and negative powers of $(z - z_0)$.

4.9 Laurent Series

4.9.1 Singular Point

Definition 4.22 ***If a complex function*** f ***fails to be analytic at a point*** $z = z_0$, ***then this point is called a singularity or singular point of the function. For instance, the complex numbers*** $z = 2i$ ***and*** $z = -2i$ ***are singularities of the function***

$$f(z) = \frac{z}{z^2+4},$$

because f ***is discontinuous at each of these points.***

In this section, we introduce a new type of "power series" expansion of f about an ***isolated singularity*** z_0. This new expansion, called the ***Laurent series***, involves both negative and nonnegative integer powers of $(z - z_0)$.

4.9.2 Laurent Series Expansion

Theorem 4.27 *Let f be a complex function that is analytic in an annular region*

$$R_1 < |z - z_0| < R_2,$$

where R_1 and R_2 are real numbers such that $0 \leq R_1 < R_2 \leq \infty$. Then f can be represented by a Laurent series about the point z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta,$$

and C is a positively oriented simple closed contour contained in the annulus.

Remark 4.25

- *The terms with $n \geq 0$ form the analytic part of the series.*
- *The terms with $n < 0$ form the principal part.*
- *If the principal part contains only finitely many terms, the singularity is called a pole.*
- *If infinitely many negative powers appear, the singularity is called an essential singularity.*

Example 4.34 *Consider the complex function*

$$f(z) = \frac{8z + 1}{z(z - 1)}$$

and the region $0 < |z| < 1$.

1. *Find the Laurent series of $f(z)$ centered at $z_0 = 0$.*
2. *Identify the principal part and the analytic part.*
3. *Illustrate the region of convergence and indicate the principal and analytic parts on a diagram.*

Solution 4.4 *Step 1: Partial fraction decomposition.*

$$f(z) = \frac{8z + 1}{z(z - 1)} = \frac{A}{z} + \frac{B}{z - 1}.$$

Multiplying both sides by $z(z - 1)$:

$$8z + 1 = A(z - 1) + Bz.$$

Compare coefficients:

$$A = -1, \quad B = 9$$

Hence:

$$f(z) = -\frac{1}{z} + \frac{9}{z-1}.$$

Step 2: Expand $\frac{9}{z-1}$ ***for*** $|z| < 1$ ***using a geometric series.***

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n \quad \Rightarrow \quad \frac{9}{z-1} = -9\sum_{n=0}^{\infty} z^n.$$

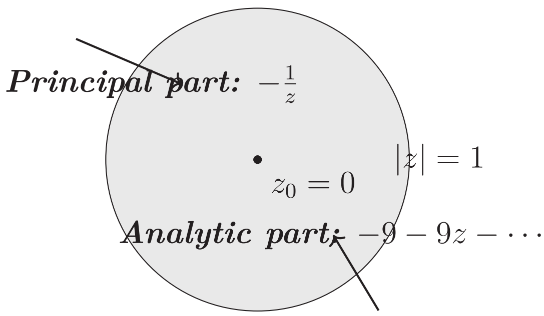
Step 3: Combine terms to obtain the Laurent series.

$$f(z) = -\frac{1}{z} - 9 - 9z - 9z^2 - \cdots = -\frac{1}{z} - \sum_{n=0}^{\infty} 9z^n$$

Step 4: Identify principal and analytic parts.

- ***Principal part (negative powers):*** $-\frac{1}{z}$
- ***Analytic part (non-negative powers):*** $-9 - 9z - 9z^2 - \cdots = -\sum_{n=0}^{\infty} 9z^n$

Step 5: Diagram of the region and series parts.



Conclusion:
The Laurent series of $f(z)$ ***around*** $z_0 = 0$***, valid for*** $0 < |z| < 1$***, is***

$$\boxed{f(z) = -\frac{1}{z} - 9 - 9z - 9z^2 - \cdots = -\frac{1}{z} - \sum_{n=0}^{\infty} 9z^n}$$

with the principal part consisting of $-\frac{1}{z}$ ***and the analytic part*** $-9-9z-9z^2-\cdots$***.***

4.9.3 Isolated Singular Points

Definition 4.23 ***Let $f(z)$ be a complex function. A point $z_0 \in \mathbb{C}$ is called a*** **singular point** ***or*** **singularity** ***of f if f is not analytic at z_0, but is analytic at some point arbitrarily close to z_0.***

A singularity z_0 is said to be an **isolated singularity** ***if there exists a small radius $r > 0$ such that $f(z)$ is analytic for all z in the punctured disk***

$$0 < |z - z_0| < r.$$

Example 4.35 ***1. The function***

$$f(z) = \frac{1}{z}$$

has a singularity at $z_0 = 0$. For any small disk around 0, $f(z)$ is analytic for all $z \neq 0$. Hence $z_0 = 0$ is an isolated singularity.

2. The function

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)}$$

has singularities at $z = i$ and $z = -i$. Each of these points is isolated because the function is analytic in a neighborhood around each singularity, excluding the singular point itself.

3. The logarithm function

$$f(z) = \ln z$$

has a branch point at $z_0 = 0$, which is also considered a singularity. It is isolated if we consider a domain that avoids the branch cut along the negative real axis.

Classification of Isolated Singular Points

- An isolated singularity allows the function to be expanded in a Laurent series in a punctured disk around the singular point.
- If singular points accumulate (i.e., there is no punctured disk where the function is analytic), the singularity is not isolated.
- An isolated singular point $z = z_0$ of a complex function f is classified according to the principal part of its Laurent expansion. Let the Laurent expansion of f around z_0 be

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

and let the principal part be

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}.$$

. Isolated singularities are classified into three types:
1. *Removable singularities*
2. *Poles*

3. ***Essential singularities***
This classification is based on the behavior of the function near the singular point and the terms of the Laurent series.

1. **Removable singularity: If the principal part is zero, that is, all coefficients $a_{-k} = 0$, then $z = z_0$ is called a *removable singularity.* In this case, f can be redefined at z_0 to become analytic.**
2. **Pole: If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a *pole.* If the last nonzero term in the principal part is $a_{-n}/(z - z_0)^n$ with $n \geq 1$, then $z = z_0$ is a *pole of order* n. In particular, if $n = 1$, the principal part contains exactly one term with coefficient a_{-1}, and the pole is called a *simple pole.***
3. **Essential singularity: If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an *essential singularity.***

All these cases can be summarized in the following table.

$z = z_0$	Laurent Series for $0 < \lvert z - z_0 \rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\dfrac{a_{-n}}{(z - z_0)^n} + \dfrac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z - z_0)^2} + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

Example 4.36 (Removable Singularity) ***Dividing the Maclaurin series for*** $\sin z$ ***by*** z***, we obtain***

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (4.21)$$

From equation (4.40)***, we see that all the coefficients in the principal part of the Laurent series are zero. Hence,***

$$z = 0 \text{ is a removable singularity of the function } f(z) = \frac{\sin z}{z}. \qquad (4.22)$$

Example 4.37 (Poles and Essential Singularity) ***Dividing the terms of the Maclaurin series***

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad (4.23)$$

by z^2 ***gives***

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots, \quad 0 < |z| < \infty \qquad (4.24)$$

From equation (4.24)***, we see that the coefficient*** $a_{-1} \neq 0$***, and hence***

$$z = 0 \text{ is a simple pole of the function } f(z) = \frac{\sin z}{z^2}. \qquad (4.25)$$

4.9.4 Zeros of a Function

Recall, a number z_0 is a *zero* of a function f if

$$f(z_0) = 0.$$

Definition 4.24 ***We say that an analytic function f has a* zero of order n *at $z = z_0$ if z_0 is a zero of f and of its first $n-1$ derivatives, that is,***

$$f(z_0) = 0, \quad f'(z_0) = 0, \quad f''(z_0) = 0, \ \ldots, \quad f^{(n-1)}(z_0) = 0, \quad \textbf{\textit{but}} \quad f^{(n)}(z_0) \neq 0. \quad (4.26)$$

***A zero of order n is also referred to as a zero of* multiplicity n.**

Example:
For

$$f(z) = (z-5)^3,$$

we have $f(5) = 0, f'(5) = 0, f''(5) = 0, f^{(3)}(5) = 6 \neq 0$.
Thus f has a zero of order (or multiplicity) 3 at $z_0 = 5$.
A zero of order 1 is called a *simple zero*.

Theorem 4.28 (Zero of Order n) ***A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if it can be written as***

$$f(z) = (z - z_0)^n \, \phi(z), \tag{4.27}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Theorem 4.29 (Pole of Order n) ***A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if it can be written as***

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}, \tag{4.28}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Definition 4.25 (Zeros Again) ***A zero $z = z_0$ of an analytic function f is* isolated *in the sense that there exists some neighborhood of z_0 in which***

$$f(z) = 0$$

at every point z except at $z = z_0$ itself.

As a consequence, if z_0 is a zero of a nontrivial analytic function f, then the function

$$\frac{1}{f(z)}$$

has an isolated singularity at the point $z = z_0$.

Theorem 4.30 (Pole of Order n) *Let g and h be functions analytic at $z = z_0$. If h has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function*

$$f(z) = \frac{g(z)}{h(z)} \tag{4.29}$$

has a pole of order n at $z = z_0$.

Example 4.38 (Poles of a Rational Function) *Consider the rational function*

$$f(z) = \frac{2z + 5}{(z-1)(z+5)(z-2)^4}. \tag{4.30}$$

Let us denote the numerator by

$$N(z) = 2z + 5. \tag{4.31}$$

We observe that

$$N(1) = 7 \neq 0, \quad N(-5) = -5 \neq 0, \quad N(2) = 9 \neq 0.$$

Hence, the zeros of the denominator determine the poles of $f(z)$:

- *$z = 1$ and $z = -5$ are* simple poles*, because the corresponding factors in the denominator are of order 1.*
- *$z = 2$ is a* pole of order 4*, because the factor $(z-2)^4$ appears in the denominator.*

Definition 4.26 (Holomorphic function with isolated poles) *A function f defined on a domain $D \subseteq \mathbb{C}$ is said to be holomorphic with isolated poles if*

- *f is holomorphic on D except at a finite or countable set of points $\{z_1, z_2, \ldots\} \subset D$,*
- *and at each such point z_k, f can be written as*

$$f(z) = \frac{g(z)}{(z - z_k)^{n_k}},$$

where $g(z)$ is holomorphic at z_k and $g(z_k) \neq 0$.

In other words, f is holomorphic everywhere in D except at isolated singularities, and each singularity is a pole of finite order.

Example 4.39 *The function*

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)}$$

is holomorphic on $\mathbb{C} \setminus \{-1, 1\}$, and has two isolated poles at $z = 1$ and $z = -1$, with $g(z) = 1$.

Definition 4.27 (Meromorphic Function) ***Let $D \subseteq \mathbb{C}$ be an open domain. A function $f : D \to \mathbb{C}$ is said to be meromorphic on D if f is holomorphic on D except at a discrete set of points $\{z_1, z_2, \ldots\} \subset D$, where each z_k is an isolated pole of f.***

Equivalently, around each such point z_k, there exists an integer $m \geq 1$ and a function g holomorphic at z_k with $g(z_k) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_k)^m}.$$

4.9.5 Argument Principle

Theorem 4.31 (Argument Principle) ***Let f be a meromorphic function in a simply connected domain Ω. Let C be a closed contour contained in Ω, enclosing all zeros and poles of f within Ω. Then we have:***

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N(f) - P(f),$$

where $N(f)$ and $P(f)$ denote, respectively, the number of zeros and poles of f inside C (each counted with their multiplicities).

Example 4.40 ***Let***

$$f(z) = z^3 - 1.$$

We will use the Argument Principle to determine the number of zeros of $f(z)$ inside the unit circle

$$C : |z| = 1.$$

Step 1. Compute $\frac{f'(z)}{f(z)}$

$$f'(z) = 3z^2, \qquad \frac{f'(z)}{f(z)} = \frac{3z^2}{z^3 - 1}.$$

Step 2. Apply the Argument Principle
According to the theorem,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N - P,$$

where:

- **N is the number of zeros of $f(z)$ inside C,**
- **P is the number of poles of $f(z)$ inside C.**

Since $f(z)$ is a polynomial, it has no poles ($P = 0$).
Step 3. Evaluate the change of argument
For $|z| = 1$, set $z = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. Then,

$$f(z) = e^{3i\theta} - 1.$$

As θ goes from 0 to 2π, the term $e^{3i\theta}$ describes three full turns around the origin. Subtracting 1 shifts the circle but does not change the number of turns around the origin. Hence, the image $f(C)$ winds three times around the origin.

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = 3.$$

Step 4. Conclusion
Therefore,

$$N - P = 3 \quad \Rightarrow \quad N = 3.$$

$$\boxed{f(z) = z^3 - 1 \text{ has three zeros inside the unit circle } |z| = 1.}$$

Corollary 4.3 ***If f is analytic inside and on a positively oriented simple closed contour C, and if f has no zeros on C, then***

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N(f),$$

where $N(f)$ denotes the number of zeros of f inside C (counted with multiplicities).

Remark 4.26 ***The definition of the Argument Principle can also be expressed in terms of the total variation of the argument of $f(z)$ along the contour C:***

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N(f) - P(f),$$

where:

- $\arg f(z)$ ***denotes the argument of $f(z)$, i.e.***

$$f(z) = |f(z)|e^{i\arg f(z)}.$$

- $\Delta_C \arg f(z)$ ***represents the total change in the argument of $f(z)$ as z traverses the contour C.***

Example 4.41 ***Let the function***

$$f(z) = \frac{2z+1}{z}.$$

Determine the number of times the variable $w = f(z)$ winds around the origin when z traverses the unit circle $|z| = 1$ once in the positive (counterclockwise) direction.

Solution 4.5 ***Step 1: Identify zeros and poles.***
- Zeros: $f(z) = 0 \implies 2z + 1 = 0 \implies z = -\frac{1}{2}$. Multiplicity: 1 Inside the unit circle: $|-1/2| = 1/2 < 1$.
- Poles: $f(z)$ has a pole at $z = 0$. Multiplicity: 1 Inside the unit circle: $|0| = 0 < 1$.

Step 2: Apply the argument principle.
The argument principle states:

$$\Delta_\Gamma \arg f(z) = 2\pi(N - P),$$

where N ***is the number of zeros inside*** Γ ***and*** P ***is the number of poles inside*** Γ***.***
Here: $N = 1$***,*** $P = 1$***, so***

$$\Delta_\Gamma \arg f(z) = 2\pi(1 - 1) = 0.$$

Step 3: Conclusion.
The number of times $w = f(z)$ ***winds around the origin is***

$$\boxed{0}.$$

Example 4.42 ***Let***

$$f(z) = \frac{2z + 1}{z(3z + 1)}.$$

Solution 4.6 ***Step 1: Identify zeros and poles.***
- Zeros: $2z + 1 = 0 \implies z = -\frac{1}{2}$ ***(multiplicity 1) - Poles:*** $z = 0$ ***and*** $3z + 1 = 0 \implies z = -\frac{1}{3}$ ***(multiplicity 1 each)***
Step 2: Check which are inside $|z| < 1$***.***
- Zero: $z = -1/2$***: inside - Poles:*** $z = 0$ ***and*** $z = -1/3$ ***: both inside***
So $N = 1$***,*** $P = 2$***.***
Step 3: Apply the argument principle.

$$\Delta_\Gamma \arg f(z) = 2\pi(N - P) = 2\pi(1 - 2) = -2\pi.$$

Step 4: Conclusion.
The negative sign indicates a clockwise rotation around the origin. Hence, the number of windings of $w = f(z)$ ***around the origin is***

$$\boxed{\frac{\Delta_\Gamma \arg f(z)}{2\pi} = -1}.$$

4.10 Solved Exercises

Exercise 4.5 ***Consider the power series***

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z - 1 - i)^k.$$

1. ***Determine the radius of convergence*** R ***of this series using the ratio test.***

2. ***Find the domain of absolute convergence in the complex plane.***

3. ***Identify the analytic function $f(z)$ represented by this power series.***

Solution 4.7 ***Solution. Set $w = z - 1 - i$. The series can be written as***

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} w^k.$$

(1) **Radius of convergence via the ratio test.**
Denote the general coefficient by $a_k = \dfrac{(-1)^{k+1}}{k!}$. Apply the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+2}/(k+1)!}{(-1)^{k+1}/k!}\right| = \frac{1}{k+1}.$$

Taking the limit as $k \to \infty$ gives

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0.$$

By the ratio test (or the standard consequence for power series), if this limit is 0 then the radius of convergence is $R = \infty$.

(2) **Domain of absolute convergence.**
Since $R = \infty$, the series converges absolutely for every complex z. In other words, the domain of absolute convergence is the whole complex plane $\mathbb{C}$.

(3) **Identification of the analytic function $f(z)$.**
Recall the Taylor expansion of the exponential:

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \qquad (\textbf{\textit{valid for all }} t \in \mathbb{C}).$$

Replace t by $-w = -(z-1-i)$:

$$e^{-w} = \sum_{k=0}^{\infty} \frac{(-w)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} w^k.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} w^k = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} w^k = -(e^{-w} - 1) = 1 - e^{-w}.$$

Returning to z (with $w = z - 1 - i$), we obtain the closed form

$$f(z) = 1 - \exp\Big(-(z-1-i)\Big).$$

Because the exponential is entire, f is entire as well (consistent with the radius $R = \infty$). The equality above holds for all $z \in \mathbb{C}$ by uniqueness of power-series expansion about the center $z_0 = 1 + i$.

Conclusion.** **The radius of convergence is** $R = \infty$**; the series converges absolutely for every** $z \in \mathbb{C}$**; and it represents the entire function

$$\boxed{f(z) = 1 - \exp\Big(-(z-1-i)\Big)}.$$

Exercise 4.6 ***Find the Taylor (Maclaurin) series expansion of the function***

$$f(z) = \frac{1}{1-z}$$

about $z_0 = 0$***. Determine the radius of convergence.***

Solution 4.8 ***We compute the Maclaurin expansion by observing that for*** $|z| < 1$ ***the geometric series converges:***

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k. \tag{4.32}$$

One may also obtain the same result by differentiating the partial sums or by using the Taylor coefficient formula

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Indeed, $f^{(k)}(z) = k!(1-z)^{-k-1}$***, hence*** $f^{(k)}(0) = k!$ ***and***

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{k!}{k!} = 1, \qquad k \geq 0, \tag{4.33}$$

which gives the series in (4.32)***.***

Finally, since the geometric series converges for $|z| < 1$***, the radius of convergence is***

$$R = 1. \tag{4.34}$$

Thus the Maclaurin series of $f(z) = \dfrac{1}{1-z}$ ***is*** $\sum_{k=0}^{\infty} z^k$ ***with radius of convergence*** $R = 1$***.***

Exercise 4.7 ***Find the Taylor series of***

$$f(z) = \frac{1}{1+z^2}$$

about a general center z_0 ***(with*** $z_0 \neq \pm i$***). Determine the coefficients of the expansion and the radius of convergence.***

Solution 4.9 ***Write*** f ***by partial fractions using the factorization*** $1 + z^2 = (z - i)(z + i)$***:***

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Fix a center z_0 ***with*** $z_0 \neq \pm i$. ***For each simple pole*** $a \in \{i, -i\}$ ***we use the geometric-type expansion***

$$\frac{1}{z-a} = \frac{1}{z_0 - a} \cdot \frac{1}{1 - \dfrac{z - z_0}{z_0 - a}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(z_0 - a)^{k+1}},$$

which is valid when $\left|\dfrac{z - z_0}{z_0 - a}\right| < 1$, ***i.e.*** $|z - z_0| < |z_0 - a|$.

Applying this to $a = i$ ***and*** $a = -i$ ***and combining gives the Taylor expansion about*** z_0***:***

$$\boxed{\frac{1}{1+z^2} = \sum_{k=0}^{\infty} a_k (z - z_0)^k} \tag{4.35}$$

with the coefficients

$$a_k \;=\; \frac{1}{2i}\left(\frac{1}{(z_0 - i)^{k+1}} - \frac{1}{(z_0 + i)^{k+1}}\right), \qquad k \geq 0. \tag{4.36}$$

The region of validity is determined by the nearest singularity to z_0. ***The series*** (4.35) ***converges for***

$$|z - z_0| < R, \qquad R = \min\{|z_0 - i|,\ |z_0 + i|\}. \tag{4.37}$$

Special case. ***For the Maclaurin expansion (***$z_0 = 0$***) the formulas simplify. From*** (4.36) ***with*** $z_0 = 0$ ***we obtain***

$$a_k = \frac{1}{2i}\left(\frac{1}{(-i)^{k+1}} - \frac{1}{(i)^{k+1}}\right).$$

After simplification one gets the well-known Maclaurin series (valid for $|z| < 1$***):***

$$\frac{1}{1+z^2} = \sum_{m=0}^{\infty} (-1)^m z^{2m}. \tag{4.38}$$

(Only even powers appear; coefficients of odd powers are zero.)

Partant de la formule générale obtenue précédemment,

$$a_k = \frac{1}{2i}\left(\frac{1}{(z_0 - i)^{k+1}} - \frac{1}{(z_0 + i)^{k+1}}\right),$$

prenons $z_0 = 1$. **Alors** $1 - i = \sqrt{2}\, e^{-i\pi/4}$ **et** $1 + i = \sqrt{2}\, e^{i\pi/4}$. **D'où**

$$\frac{1}{(1-i)^{k+1}} - \frac{1}{(1+i)^{k+1}} = (\sqrt{2})^{-(k+1)}\Big(e^{i(k+1)\pi/4} - e^{-i(k+1)\pi/4}\Big) = 2i\,(\sqrt{2})^{-(k+1)} \sin\Big((k+1)\tfrac{\pi}{4}\Big).$$

En multipliant par $1/(2i)$ **on obtient la forme réelle et simple des coefficients :**

$$\boxed{a_k \;=\; 2^{-\frac{k+1}{2}} \sin\Big((k+1)\tfrac{\pi}{4}\Big), \qquad k \geq 0.}$$

Ainsi la série de Taylor de $\dfrac{1}{1+z^2}$ autour de $z_0 = 1$ s'écrit

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} a_k (z-1)^k, \qquad a_k = 2^{-\frac{k+1}{2}} \sin\Big((k+1)\tfrac{\pi}{4}\Big).$$

Les premiers coefficients (et premiers termes) sont :

$$\begin{aligned}
a_0 &= 2^{-1/2} \sin\tfrac{\pi}{4} = \tfrac{1}{2},\\
a_1 &= 2^{-1} \sin\tfrac{\pi}{2} = \tfrac{1}{2},\\
a_2 &= 2^{-3/2} \sin\tfrac{3\pi}{4} = \tfrac{1}{4},\\
a_3 &= 2^{-2} \sin\pi = 0,\\
a_4 &= 2^{-5/2} \sin\tfrac{5\pi}{4} = -\tfrac{1}{8},
\end{aligned}$$

donc

$$\frac{1}{1+z^2} = \tfrac{1}{2} + \tfrac{1}{2}(z-1) + \tfrac{1}{4}(z-1)^2 + 0 \cdot (z-1)^3 - \tfrac{1}{8}(z-1)^4 + \cdots .$$

Le rayon de convergence est la distance du centre à la singularité la plus proche,

$$R = \min\{|1-i|,\ |1+i|\} = \sqrt{2}.$$

Remarque : la formule compacte $a_k = 2^{-(k+1)/2} \sin\big((k+1)\pi/4\big)$ permet de générer tous les coefficients (on voit la périodicité des signes/coefs via la valeur du sinus).

Exercise 4.8 (Laurent Series Expansion) ***Consider the complex function***

$$f(z) = \frac{1}{1+z}.$$

1. ***Find the Laurent series of $f(z)$ centered at $z_0 = 0$ for the region $|z| > 1$.***
2. ***Identify the principal part and the analytic part of the series.***

Solution 4.10 ***Step 1: Rewrite $f(z)$ in a suitable form for expansion.***
For $|z| > 1$, we can factor out $1/z$:

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z} = \frac{1}{z} \cdot \frac{1}{1-(-1/z)}.$$

Step 2: Expand using the geometric series.
For $|1/z| < 1$ (i.e., $|z| > 1$), we can write

$$\frac{1}{1-(-1/z)} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n}.$$

Step 3: Multiply by $1/z$ to obtain the Laurent series.

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} (-1)^n z^{-n-1}.$$

Step 4: Identify the principal part and the analytic part.
- The series contains only negative powers of z, so the principal part is

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$$

- There is no part with non-negative powers of z, so the analytic part is 0.

Conclusion:
The Laurent series of $f(z)$ around $z_0 = 0$ valid for $|z| > 1$ is

$$\boxed{f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots}$$

with the principal part consisting of all terms and the analytic part equal to zero.

Exercise 4.9 (Classification of Isolated Singularities) ***Classify the singularities of the following complex functions and determine their type (removable, pole, or essential):***

1. $f_1(z) = \frac{\sin z}{z}$
2. $f_2(z) = \frac{1}{z^3}$
3. $f_3(z) = e^{1/z}$

Solution 4.11 ***1. Function*** $f_1(z) = \frac{\sin z}{z}$
The Maclaurin series for $\sin z$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{4.39}$$

Dividing by z gives the Laurent series around $z = 0$:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{4.40}$$

***All coefficients in the principal part are zero. Therefore, $z = 0$ is a* removable singularity.**

2. Function $f_2(z) = \frac{1}{z^3}$
The function has a singularity at $z = 0$, and its Laurent series is

$$f_2(z) = \frac{1}{z^3}. \tag{4.41}$$

***The principal part contains a finite number of terms (here, only $1/z^3$), hence $z = 0$ is a* pole of order 3.**

3. Function $f_3(z) = e^{1/z}$
Expanding around $z = 0$ gives the Laurent series

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \tag{4.42}$$

***The principal part contains infinitely many terms, so $z = 0$ is an* essential singularity.**

Exercise 4.10 (Poles and Zeros of Order n) ***Consider the following complex functions:***

1. $f_1(z) = \frac{(z-1)^3}{(z+2)^2}$
2. $f_2(z) = z^4(z-3)$
3. $f_3(z) = \frac{z^2+1}{(z-2)^5}$

For each function, identify all zeros and poles, and determine their order.

Solution 4.12 ***1. Function*** $f_1(z) = \frac{(z-1)^3}{(z+2)^2}$

1. ***Zeros:***

   - The zeros of a function occur where the numerator vanishes. - The numerator is $(z-1)^3$***. Solve*** $(z-1)^3 = 0$***:***

$$z - 1 = 0 \implies z = 1.$$

   - Since the factor is raised to the power 3***,*** $z = 1$ ***is a zero of*** **order 3.**

2. ***Poles:***

   - Poles occur where the denominator vanishes. - The denominator is $(z+2)^2$***. Solve*** $(z+2)^2 = 0$***:***

$$z + 2 = 0 \implies z = -2.$$

   - The exponent 2 ***indicates a*** **pole of order 2** ***at*** $z = -2$***.***

3. ***Remark:***

   - Near $z = -2$***, the function behaves as*** $f_1(z) \sim \frac{\textit{constant}}{(z+2)^2}$***, which confirms the pole of order*** 2***.***

2. Function $f_2(z) = z^4(z-3)$

1. ***Zeros:***

   - Solve $z^4(z-3) = 0$***. Factorization shows two contributions:***

$$z^4 = 0 \implies z = 0 \textit{ with multiplicity 4},$$
$$z - 3 = 0 \implies z = 3 \textit{ with multiplicity 1}.$$

   - Therefore, $z = 0$ ***is a zero of order*** 4***, and*** $z = 3$ ***is a zero of order*** 1***.***

2. ***Poles:***

   - The function is a polynomial; it has no denominator. - Hence, there are **no poles.**

3. Function $f_3(z) = \frac{z^2+1}{(z-2)^5}$

1. ***Zeros:***

   - *Zeros occur where* $z^2 + 1 = 0$***:***

   $$z^2 = -1 \implies z = i,\ z = -i.$$

   - *Each factor* $(z - i)$ ***and*** $(z + i)$ ***appears only once, so both zeros are of*** **order 1*.***

2. ***Poles:***

   - *Poles occur where* $(z - 2)^5 = 0$***:***

   $$z - 2 = 0 \implies z = 2.$$

   - *The factor is raised to the 5th power, so* $z = 2$ ***is a*** **pole of order 5*.***

3. ***Remark:***

   - *Near* $z = 2$***, the function behaves as***

   $$f_3(z) \sim \frac{\textit{constant}}{(z - 2)^5},$$

   confirming the pole of order 5.