Chapter 3

Elementary functions

3.1 Complex Exponential Function

Definition 3.1 ***For any complex number*** $z = x + iy$***, the complex exponential function is defined by***

$$e^z = e^x(\cos y + i \sin y).$$

It is denoted by $\exp(z)$ ***as well.***

Remark 3.1 ***The complex exponential has the same properties as the real exponential function. In particular, it is differentiable everywhere and satisfies***

$$\frac{d}{dx}e^x = e^x.$$

Theorem 3.1 ***The function*** e^z ***is entire (holomorphic on the whole complex plane), and its derivative is***

$$\frac{d}{dz}e^z = e^z.$$

Verification of the Cauchy-Riemann equations.
Let

$$e^z = e^x(\cos y + i \sin y) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y.$$

Compute the partial derivatives:

$$u_x = e^x \cos y, \qquad u_y = -e^x \sin y,$$
$$v_x = e^x \sin y, \qquad v_y = e^x \cos y.$$

We observe that

$$u_x = v_y \quad \textbf{and} \quad u_y = -v_x,$$

so the Cauchy-Riemann equations are satisfied everywhere. Therefore, e^z is analytic (entire) and

$$f'(z) = \frac{d}{dz}e^z = e^z.$$

Exercise 3.1 ***Compute the complex derivative (where it exists) of each of the following functions:***

$$(1)\ f(z) = e^{2z},$$
$$(2)\ g(z) = e^{z^2},$$
$$(3)\ h(z) = z\,e^z,$$
$$(4)\ p(z) = e^{\overline{z}},$$
$$(5)\ q(z) = \overline{z}\,e^z,$$
$$(6)\ E(z) = e^z.$$

Solution 3.1 ***(1)*** $f(z) = e^{2z}$***.***
Since $2z$ ***is holomorphic with derivative*** 2***, we apply the chain rule:***

$$\frac{d}{dz}e^{2z} = 2e^{2z}.$$

(2) $g(z) = e^{z^2}$***.***
The inner function z^2 ***is holomorphic with derivative*** $2z$***. Hence:***

$$\frac{d}{dz}e^{z^2} = 2z\,e^{z^2}.$$

(3) $h(z) = z\,e^z$***.***
Both z ***and*** e^z ***are entire, so by the product rule:***

$$h'(z) = e^z + z\,e^z = (1+z)e^z.$$

(4) $p(z) = e^{\overline{z}}$***.***
Let $z = x + iy$***. Then*** $\overline{z} = x - iy$***, and***

$$p(z) = e^{\overline{z}} = e^{x-iy} = e^x(\cos y - i\sin y).$$

Thus

$$u(x,y) = e^x\cos y, \qquad v(x,y) = -e^x\sin y.$$

We find

$$u_x = e^x\cos y, \quad u_y = -e^x\sin y, \quad v_x = -e^x\sin y, \quad v_y = -e^x\cos y.$$

The Cauchy-Riemann equations require $u_x = v_y$ ***and*** $u_y = -v_x$***, which give***

$$e^x\cos y = -e^x\cos y \quad \Rightarrow \quad \cos y = 0.$$

These are satisfied only on isolated lines $y = \frac{\pi}{2} + k\pi$, not on any open set. Therefore $p(z)$ is not holomorphic on any open domain, and does not possess a complex derivative.

***(5)** $q(z) = \overline{z}\, e^z$.*

Since $\overline{z}$ is not holomorphic, the product $\overline{z}e^z$ fails to satisfy the Cauchy?Riemann equations. Hence $q(z)$ is not holomorphic and has no complex derivative.

***(6)** $E(z) = e^z$.*

For completeness, we recall that e^z is entire and satisfies

$$\frac{d}{dz}e^z = e^z.$$

Summary of derivatives (where defined):

$$\begin{aligned}
&(1)\ \frac{d}{dz}e^{2z} = 2e^{2z},\\
&(2)\ \frac{d}{dz}e^{z^2} = 2z\, e^{z^2},\\
&(3)\ \frac{d}{dz}(ze^z) = (1+z)e^z,\\
&(4)\ e^{\overline{z}}\ \textbf{\textit{is not holomorphic}},\\
&(5)\ \overline{z}\, e^z\ \textbf{\textit{is not holomorphic}},\\
&(6)\ \frac{d}{dz}e^z = e^z.
\end{aligned}$$

3.1.1 Properties

1. Module, Argument, and Conjugate of e^z

Definition 3.2 ***Let***

$$w = e^z = e^x(\cos y + i \sin y),$$

which can also be written in polar form as

$$w = r(\cos\theta + i\sin\theta).$$

Proposition 3.1 ***By identification, we obtain:***

$$r = e^x, \qquad \theta = y + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Since $e^x > 0$ for all real x, we have:

$$|e^z| = e^x, \qquad \arg(e^z) = y + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.2 ***An important consequence is that for all $z \in \mathbb{C}$:***

$$\boxed{e^z \neq 0.}$$

Proposition 3.2 (Complex Conjugate) ***The conjugate of e^z is obtained as follows:***

$$\overline{e^z} = \overline{e^x(\cos y + i\sin y)} = e^x(\cos y - i \sin y) = e^x(\cos(-y) + i\sin(-y)) = e^{x-iy} = e^{\overline{z}}.$$

Hence,

$$\boxed{\overline{e^z} = e^{\overline{z}}.}$$

2. Algebraic Properties of e^z

Proposition 3.3 ***For any complex numbers*** z_1, z_2 ***and integer*** $n \in \mathbb{Z}$***:***

$$e^0 = 1,$$

$$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2},$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2},$$

$$(e^{z_1})^n = e^{nz_1}.$$

3. Periodicity of e^z

Proposition 3.4 ***Since*** $e^z = e^x(\cos y + i \sin y)$ ***involves the trigonometric functions*** $\cos y$ ***and*** $\sin y$***, which are both*** 2π***-periodic, it follows that:***

$$e^{z+i2\pi} = e^z.$$

Therefore, the function e^z ***is periodic with period*** $i2\pi$***.***

Hence, the function e^z is periodic with period $i2\pi$ and therefore not one-to-one (not single-valued). Different complex numbers correspond to the same image under e^z.

3.1.2 Transformation under e^z

Since $z = x + iy$, we study the image of different types of lines under the transformation

$$w = e^z.$$

Definition 3.3 ***In complex analysis, the*** w***-plane is the image plane corresponding to the complex variable***

$$w = f(z),$$

where $z = x + iy$ ***is a complex variable in the*** z***-plane (the domain).***

1. The z-plane: The z-plane represents the domain of the complex function $f(z)$. Each point in this plane corresponds to a complex number $z = x + iy$, where

$$x = \Re(z), \qquad y = \Im(z).$$

2. The w-plane: The w-plane represents the image (or codomain) of the function $f(z)$. Each point corresponds to a complex number $w = u + iv$, where

$$u = \Re(w), \qquad v = \Im(w).$$

Example 3.1 ***If*** $f(z) = e^z$***, then***

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

Hence,

$$u = e^x \cos y, \qquad v = e^x \sin y.$$

This means that each point (x, y) ***in the*** z***-plane is mapped to a point*** (u, v) ***in the*** w***-plane.***

Remark 3.3 ***Geometrically:***

- ***Vertical lines** $x =$ **constant in the** z**-plane are transformed into circles in the** w**-plane.***
- ***Horizontal lines** $y =$ **constant in the** z**-plane are transformed into rays (half-lines) in the** w**-plane.***

$$z\text{-plane} \xrightarrow{w=e^z} w\text{-plane}.$$

Case 3.1 ***Case 1:** z **represents a vertical line in the** z**-plane.***
Let

$$z = a + it, \qquad a = \textbf{constant}, \quad -\pi < t \leq \pi.$$

Then

$$w = e^z = e^a e^{it}.$$

In polar form:

$$r = e^a, \qquad \theta = t + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.4 ***This shows that every vertical line** $x = a$ **in the** z**-plane is transformed by** e^z **into a circle in the** w**-plane centered at the origin, with radius***

$$r = e^a > 0.$$

Hence, periodic regions in the** z**-plane are mapped to families of concentric circles in the** w**-plane.

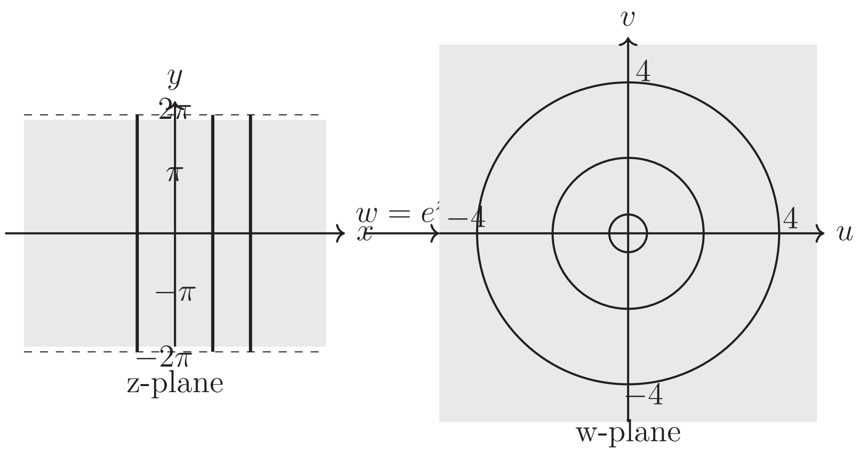


Figure 3.1: Transformation of vertical lines under $w = e^z$: the vertical strip in the z-plane maps to concentric circles in the w-plane.

Case 3.2 ***Case 2:** z **represents a horizontal line in the** z**-plane.***
Let

$$z = t + ib, \qquad b = \textbf{constant}, \quad -\infty < t \leq +\infty.$$

Then

$$w = e^z = e^t e^{ib}.$$

In polar form, we have:

$$r = e^t, \qquad \theta = b + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.5 ***This shows that each horizontal line*** $y = b$ ***in the*** z***-plane is transformed by*** e^z ***into a half-line (ray) in the*** w***-plane starting from the origin and making an angle***

$$\theta = b$$

with the positive u***-axis. The modulus*** $r = e^t$ ***varies over*** $(0, +\infty)$***.***

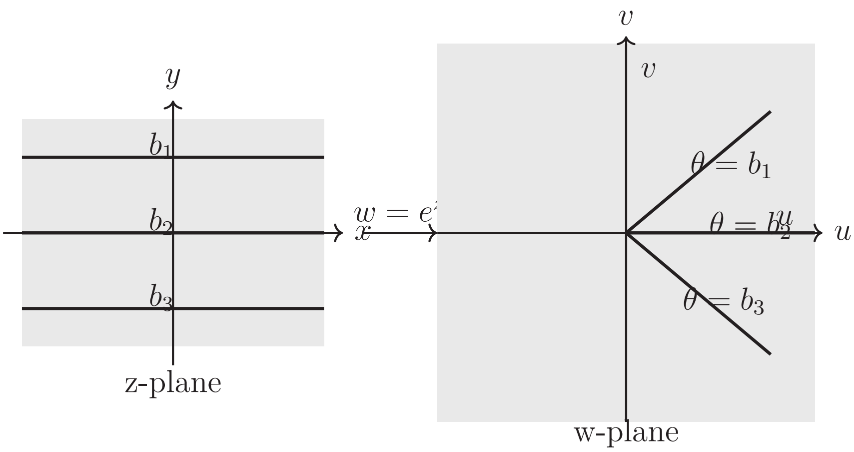


Figure 3.2: Transformation of horizontal lines under $w = e^z$: each line $y = b$ maps to a ray with argument $\theta = b$.

Proposition 3.5 ***Let*** $w = e^z$ ***with*** $z = x + iy$***. The geometric transformations of regions under the exponential mapping*** e^z ***can be summarized as follows:***

1. ***The mapping***

$$w = e^z$$

transforms the fundamental region

$$-\infty < x < \infty, \quad -\pi < y \leq \pi$$

into the set

$$|w| > 0.$$

2. ***The vertical segment***

$$x = a, \quad -\pi < y \leq \pi$$

is transformed into a circle of radius

$$|w| = e^a.$$

3. ***The horizontal line***

$$-\infty < x < \infty, \quad y = b$$

is transformed into a ray in the w***-plane making an angle***

$$\arg(w) = b.$$

3.2 Complex Logarithmic Function

As already seen in the case of the complex exponential function, the exponential mapping e^z is not single-valued as in the real case. Therefore, in the complex plane $\mathbb{C}$, for any fixed nonzero complex number $z \neq 0$, the equation

$$e^w = z$$

admits an infinite number of solutions.

In fact, because of the periodicity of the argument, the equation

$$e^w = z$$

has infinitely many solutions. This can be better understood by assuming that

$$w = u + iv$$

is a complex solution of the equation. In this case, we can write:

$$|e^w| = |z|, \qquad \arg(e^w) = \arg(z).$$

From these equalities, we obtain by identification:

$$e^u = |z|, \qquad v = \arg(z),$$

or equivalently,

$$u = \log |z|, \qquad v = \arg(z).$$

Hence, for a nonzero complex number $z \neq 0$, if $e^w = z$, then

$$w = \log |z| + i \arg(z).$$

Because the argument of z takes infinitely many values of the form

$$\arg(z) + 2n\pi, \qquad n \in \mathbb{Z},$$

the expression above defines a multi-valued function $w = G(z)$, as formally given by the following definition.

Definition 3.4 *The multi-valued function*

$$\ln z = \log |z| + i \arg(z)$$

is called the complex logarithm of z.

3.2.1 Complex Logarithm in Polar Form

If we write a nonzero complex number in polar form

$$z = re^{i\theta}, \qquad r > 0,\ \theta \in \mathbb{R},$$

then the complex logarithm of z can be written as the multi-valued expression

$$\ln z = \log r + i(\theta + 2n\pi), \qquad n \in \mathbb{Z},$$

where $\log r$ denotes the natural (Napierian) logarithm of the positive real number r.

Remark 3.6 ***The notation $\log x$ is used here for the natural logarithm (base e) of a positive real number $x > 0$.***

Example 3.2 ***Example 2. Find all complex solutions of the following equations:***

$$\textbf{\textit{(a)}}\ e^w = i; \qquad \textbf{\textit{(b)}}\ e^w = 1 + i; \qquad \textbf{\textit{(c)}}\ e^w = -2.$$

Solutions.

(a) Solve $e^w = i$.
Write $i = 1 \cdot e^{i\pi/2}$. Thus

$$w = \ln i = \log|i| + i(\arg i + 2n\pi) = 0 + i\Big(\frac{\pi}{2} + 2n\pi\Big), \qquad n \in \mathbb{Z}.$$

Hence all solutions are

$$\boxed{w = i\Big(\frac{\pi}{2} + 2n\pi\Big),\ n \in \mathbb{Z}}.$$

(b) Solve $e^w = 1 + i$.
Compute $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \pi/4$ (principal argument). Therefore

$$w = \ln(1 + i) = \log\sqrt{2} + i\Big(\frac{\pi}{4} + 2n\pi\Big) = \tfrac{1}{2}\log 2 + i\Big(\frac{\pi}{4} + 2n\pi\Big), \qquad n \in \mathbb{Z}.$$

Thus all solutions are

$$\boxed{w = \tfrac{1}{2}\log 2 + i\Big(\frac{\pi}{4} + 2n\pi\Big),\ n \in \mathbb{Z}}.$$

(c) Solve $e^w = -2$.
Write $-2 = 2e^{i\pi}$ (principal argument π). Hence

$$w = \ln(-2) = \log 2 + i(\pi + 2n\pi) = \log 2 + i\Big(\pi(1 + 2n)\Big), \qquad n \in \mathbb{Z}.$$

All solutions are

$$\boxed{w = \log 2 + i(\pi + 2n\pi),\ n \in \mathbb{Z}}.$$

3.2.2 Logarithmic Identities (multi-valued form)

The following identities are analogous to the real logarithm identities but must be understood up to integer multiples of $2\pi i$:

$$\begin{aligned}
\ln(z_1 z_2) &= \ln z_1 + \ln z_2 \quad (\textbf{mod } 2\pi i),\\
\ln\Big(z^n\Big) &= n \ln z \quad (\textbf{mod } 2\pi i), \quad n \in \mathbb{Z},\\
\ln\Big(\frac{1}{z}\Big) &= -\ln z \quad (\textbf{mod } 2\pi i),\\
\ln z &= \log|z| + i(\arg z + 2n\pi), \quad n \in \mathbb{Z}.
\end{aligned}$$

Remark 3.7 ***To obtain single-valued formulas one restricts the argument to a branch, for example the principal branch***

$$\operatorname{Ln} z = \log|z| + i\ \operatorname{Arg}(z), \qquad -\pi < \operatorname{Arg}(z) \le \pi,$$

where $\operatorname{Ln} z$ ***denotes the principal value of the complex logarithm.***

3.2.3 Principal Value of the Complex Logarithm

Definition 3.5 ***The principal value of the complex logarithm, denoted by*** $\operatorname{Ln} z$***, is defined as***

$$\operatorname{Ln} z = \log|z| + i\ \operatorname{Arg}(z),$$

where $\operatorname{Arg}(z)$ ***is the principal argument of*** z***, satisfying***

$$-\pi < \operatorname{Arg}(z) \le \pi.$$

Example 3.3 ***Example 3. Compute the principal values*** $\operatorname{Ln} z$ ***for the following equations:***

(a) $e^w = i$, ***(b)*** $e^w = 1 + i$, ***(c)*** $e^w = -2$.

Solutions.

(a) For $e^w = i$***:***

$$|i| = 1, \quad \operatorname{Arg}(i) = \frac{\pi}{2}.$$

Thus,

$$\operatorname{Ln} i = \log 1 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

(b) For $e^w = 1 + i$***:***

$$|1 + i| = \sqrt{2}, \quad \operatorname{Arg}(1 + i) = \frac{\pi}{4}.$$

Hence,

$$\operatorname{Ln}(1 + i) = \log\sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\log 2 + i\frac{\pi}{4}.$$

(c) For $e^w = -2$***:***

$$|-2| = 2, \quad \operatorname{Arg}(-2) = \pi.$$

Then,

$$\operatorname{Ln}(-2) = \log 2 + i\pi.$$

3.2.4 Inverse Relationship with the Exponential Function

If the complex exponential function

$$f(z) = e^z$$

is considered on the fundamental region

$$-\infty < x < \infty, \qquad -\pi < y \leq \pi,$$

then $f(z)$ is one-to-one on this domain. Its inverse function is the principal value of the complex logarithm:

$$f^{-1}(z) = \operatorname{Ln} z.$$

3.2.5 Analyticity of the Complex Logarithm

Since the complex logarithm is a multi-valued function, it is not continuous or differentiable over the entire complex plane. To make it analytic, we restrict its domain by removing the negative real axis.

Thus, the principal branch of the logarithmic function is defined on

$$\mathbb{C} \setminus (-\infty, 0] = \{\, z \in \mathbb{C} \mid \Re(z) > 0 \text{ or } \Im(z) \neq 0 \,\}.$$

On this domain, the principal branch is expressed as

$$f_1(z) = \log|z| + i\ \operatorname{Arg}(z) = \log r + i\theta, \qquad -\pi < \theta < \pi,$$

where $z = re^{i\theta}$ with $r > 0$. This definition excludes all points $x \in (-\infty, 0]$.

Determination of the Domain of Definition of $\ln(z)$

Let

$$z = re^{i\theta}, \quad \text{where } r > 0 \text{ and } \theta \in \mathbb{R}.$$

The complex logarithm is defined by

$$\ln(z) = \ln r + i\theta.$$

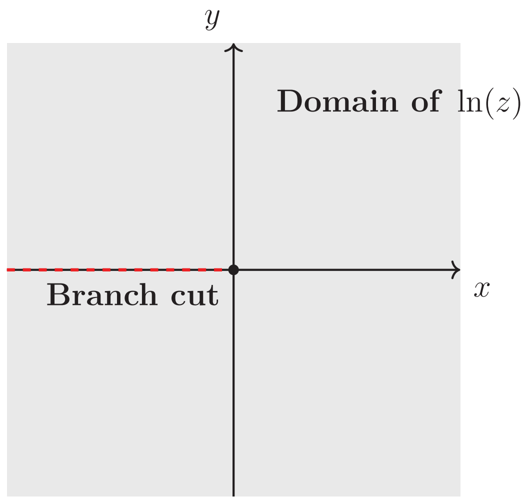
However, since θ is not unique (we can add any multiple of 2π), $\ln(z)$ is a multivalued function. To make it single-valued, we must choose a principal value for θ, usually

$$-\pi < \theta < \pi.$$

This restriction excludes the negative real axis, where $\theta = \pi$ or $\theta = -\pi$. Hence, the domain of definition of the principal branch of the logarithm is

$$D = \mathbb{C} \setminus (-\infty, 0].$$

Geometrically, this means that the branch cut (the rejected part) is the negative real axis, which we represent by a dashed line on the complex plane.



3.2.6 Analyticity of the Principal Branch of $\ln z$

Definition 3.6 ***The principal branch of the complex logarithm is the function***

$$f_1(z) = \operatorname{Ln} z = \log|z| + i \operatorname{Arg}(z),$$

defined on the domain

$$\mathbb{C} \setminus (-\infty, 0] = \{z \in \mathbb{C} \mid z \neq 0,\ -\pi < \operatorname{Arg}(z) \leq \pi\},$$

where $\operatorname{Arg}(z)$ ***denotes the principal argument satisfying*** $-\pi < \operatorname{Arg}(z) \leq \pi$***.***

Theorem 3.2 (Analyticity of the principal branch of $\ln z$) ***The principal branch*** $f_1(z) = \operatorname{Ln} z$ ***is analytic on*** $\mathbb{C} \setminus (-\infty, 0]$***. Moreover, for every*** z ***in this domain,***

$$f_1'(z) = \frac{d}{dz}\operatorname{Ln} z = \frac{1}{z}.$$

Example 3.4 ***Compute the derivatives (on the appropriate domain):***

$$\textbf{\textit{(a)}}\ F(z) = z \operatorname{Ln} z, \qquad \textbf{\textit{(b)}}\ G(z) = \operatorname{Ln}(z+1).$$

Solution 3.2 ***We work with the*** **principal branch** Ln***, defined on*** $\mathbb{C} \setminus (-\infty, 0]$ ***and satisfying***

$$\operatorname{Ln} w = \log|w| + i \operatorname{Arg}(w), \qquad -\pi < \operatorname{Arg}(w) \leq \pi.$$

(a) $F(z) = z \operatorname{Ln} z$***.*** ***Domain:*** $z \in \mathbb{C} \setminus (-\infty, 0]$ ***(so that*** $\operatorname{Ln} z$ ***is defined).***

Both z ***and*** $\operatorname{Ln} z$ ***are differentiable on this domain (indeed*** Ln ***is analytic there). Use the product rule:***

$$F'(z) = 1 \cdot \operatorname{Ln} z + z \cdot \frac{d}{dz}\big(\operatorname{Ln} z\big).$$

Since on the principal branch $\frac{d}{dz}\operatorname{Ln} z = \frac{1}{z}$***, we obtain***

$$\boxed{F'(z) = \operatorname{Ln} z + z \cdot \frac{1}{z} = \operatorname{Ln} z + 1, \qquad z \in \mathbb{C} \setminus (-\infty, 0].}$$

(b) $G(z) = \mathrm{Ln}(z+1)$***.*** ***Domain:*** $z+1 \in \mathbb{C} \setminus (-\infty, 0]$***, i.e.***

$$z \in \mathbb{C} \setminus \{\, t \in \mathbb{R} \mid t \le -1 \,\}$$

(the complex plane with the ray $(-\infty, -1]$ ***removed).***
On this domain the composition is analytic and by the chain rule

$$G'(z) = \frac{d}{dz}\mathrm{Ln}(z+1) = \frac{1}{z+1}.$$

Hence

$$\boxed{G'(z) = \frac{1}{z+1}, \qquad z \in \mathbb{C} \setminus (-\infty, -1].}$$

3.2.7 Transformations under the principal logarithm

Proposition 3.6 ***Let*** $w = \mathrm{Ln}\, z = u + iv$ ***be the principal logarithm (so*** $-\pi < v \le \pi$***). The main mapping properties are:***

1. $w = \mathrm{Ln}\, z$ ***maps the set*** $\{z \in \mathbb{C} \mid |z| > 0\}$ ***onto the horizontal strip***

$$-\infty < u < \infty, \qquad -\pi < v \le \pi.$$

2. A circle $\{z : |z| = r\}$ ***is mapped to the vertical segment***

$$u = \log r, \qquad -\pi < v \le \pi.$$

3. A ray $\{z : \arg z = \theta\}$ ***is mapped to the horizontal line***

$$-\infty < u < \infty, \qquad v = \theta \quad (-\pi < \theta \le \pi).$$

Remark 3.8 ***These correspond exactly to the inverse relations of the exponential map*** $z \mapsto e^z$***: for*** $z = re^{i\theta}$***,***

$$\mathrm{Ln}\, z = \log r + i\theta \quad (\textit{\textbf{principal value}}),$$

so modulus-level sets become vertical lines in the w***-plane and argument-level sets become horizontal lines.***

Recall the principal logarithm for a nonzero complex number $z = re^{i\theta}$ **with** $r > 0$ **and principal argument** $-\pi < \theta \le \pi$**:**

$$w = \mathrm{Ln}\, z = \log r + i\theta, \qquad w = u + iv.$$

Thus $u = \log r$ **and** $v = \theta$**. The examples below illustrate the three principal mapping properties: (1) the whole nonzero plane to the horizontal strip,**
(2) a circle to a vertical segment, and
(3) a ray to a horizontal line.

Example 3.5 ***The image of the set*** $\{z \in \mathbb{C} : |z| > 0\}$
Show that under the principal logarithm $w = \operatorname{Ln} z$ ***the set of all nonzero complex numbers***

$$\{z \in \mathbb{C} \mid |z| > 0\}$$

is mapped onto the horizontal strip

$$-\infty < u < \infty, \qquad -\pi < v \le \pi.$$

Solution 3.3 ***1. Any nonzero complex number has a polar representation***

$$z = re^{i\theta}, \qquad r > 0, \quad -\pi < \theta \le \pi,$$

where we choose the principal argument $\operatorname{Arg} z \in (-\pi, \pi]$***.***

2. Apply the principal logarithm to z***:***

$$w = \operatorname{Ln} z = \log r + i\theta.$$

3. Separate into real and imaginary parts:

$$u = \log r, \qquad v = \theta.$$

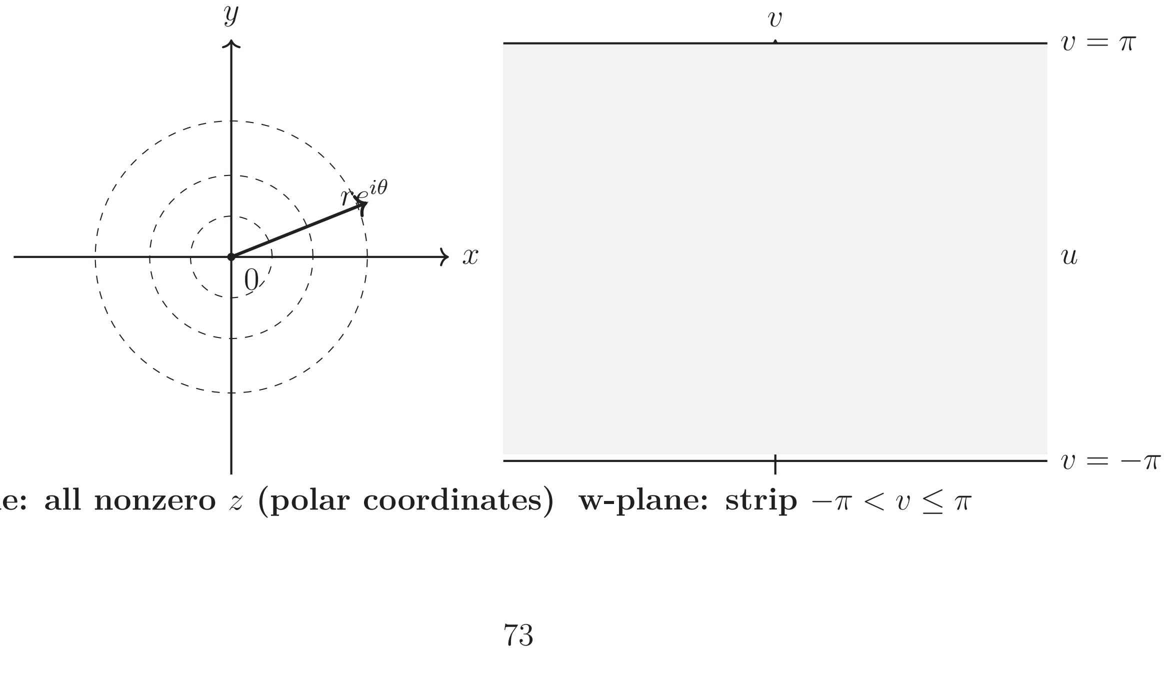
4. As r ***varies over*** $(0, \infty)$***,*** $\log r$ ***takes all real values*** $(-\infty, \infty)$***. As*** θ ***ranges over the principal interval*** $(-\pi, \pi]$***,*** v ***takes values*** $-\pi < v \le \pi$***.***

5. Therefore the image is exactly the horizontal strip

$$-\infty < u < \infty, \qquad -\pi < v \le \pi.$$

Remark 3.9

- ***The negative real axis corresponds to*** $\theta = \pi$ ***and maps to the top boundary*** $v = \pi$***.***
- ***The origin*** $z = 0$ ***is not in the domain of*** Ln***, so it does not map to any*** w***.***
- ***The principal logarithm is single-valued only after fixing*** $\theta \in (-\pi, \pi]$***. If we allowed other branches,*** v ***would differ by multiples of*** 2π***.***



z-plane: all nonzero z **(polar coordinates) w-plane: strip** $-\pi < v \le \pi$

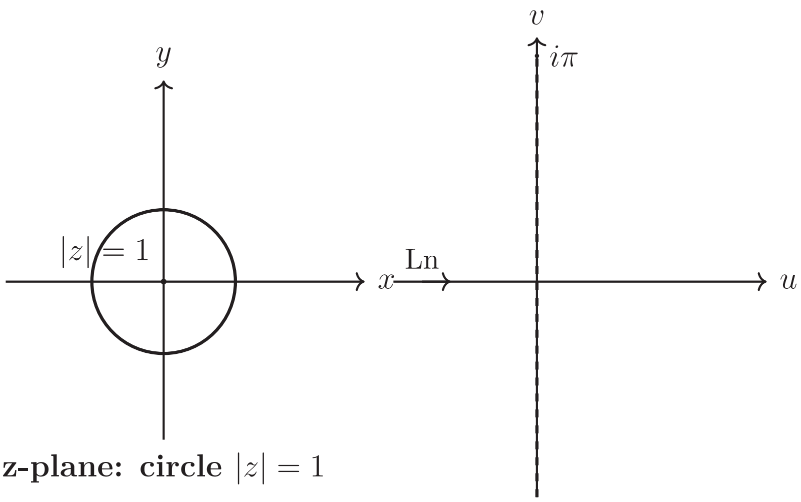
Example 3.6 ***A circle*** $|z| = r$ ***maps to the vertical segment*** $u = \log r$
Let $|z| = 1$***. Find its image under the principal logarithm and explain why the image is a vertical segment.***

Solution 3.4

1. ***Points on the circle*** $|z| = 1$ ***can be written*** $z = e^{i\theta}$ ***with*** $-\pi < \theta \leq \pi$***.***
2. ***Compute the principal logarithm:***
$$w = \operatorname{Ln} z = \log 1 + i\theta = i\theta.$$
3. ***Thus***
$$u = \Re(w) = 0, \qquad v = \Im(w) = \theta, \quad -\pi < \theta \leq \pi.$$
4. ***Therefore the image is the vertical segment*** $\{w : u = 0,\ -\pi < v \leq \pi\}$***.***
5. ***For a general circle*** $|z| = r$ ***the same reasoning gives*** $u = \log r$ ***and the vertical segment*** $\{w : u = \log r,\ -\pi < v \leq \pi\}$***.***

Remark 3.10

- ***The endpoint corresponding to*** $\theta = \pi$ ***(i.e.*** $z = -1$***) maps to*** $w = i\pi$***. The value*** $\theta = -\pi$ ***is not an independent principal argument, so the parameterization interval is*** $-\pi < \theta \leq \pi$***.***
- ***The mapping is one-to-one when restricted appropriately (e.g. restrict domain to the circle with specified principal arguments).***



z-plane: circle $|z| = 1$

w-plane: vertical segment $u = 0$, $-\pi < v \leq \pi$

Example 3.7 ***A ray*** $\arg z = \theta$ ***maps to the horizontal line*** $v = \theta$ ***Let the ray*** $\arg z = \pi/4$ ***(all*** $z = re^{i\pi/4}$ ***with*** $r > 0$***). Find its image under the principal logarithm.***

Solution 3.5

1. ***Any point on the ray has the form*** $z = re^{i\pi/4}$ ***where*** $r > 0$ ***and the principal argument*** $\pi/4$ ***belongs to*** $(-\pi, \pi]$***.***

2. Apply the principal logarithm:

$$w = \operatorname{Ln} z = \log r + i\frac{\pi}{4}.$$

3. Separate parts:

$$u = \log r, \qquad v = \frac{\pi}{4}.$$

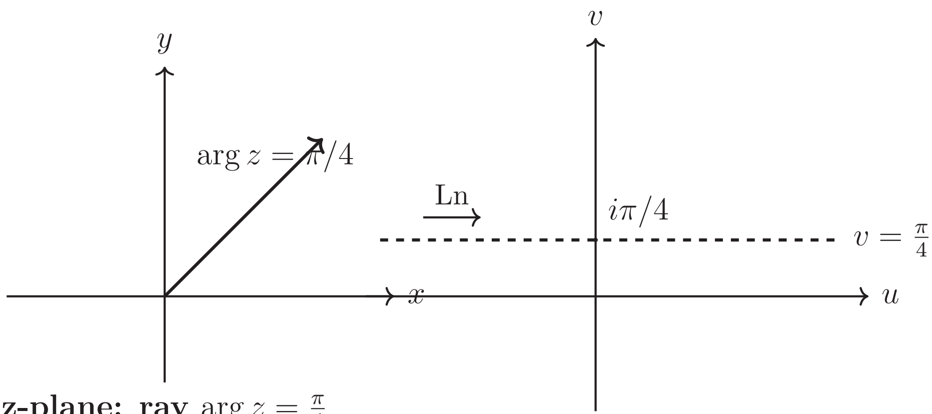
4. As r ranges over $(0, \infty)$, $u = \log r$ ranges over $(-\infty, \infty)$. Therefore the image is the horizontal line

$$\{w : -\infty < u < \infty,\ v = \tfrac{\pi}{4}\}.$$

5. Orientation note: increasing r corresponds to increasing u. For example $z = 1 \cdot e^{i\pi/4}$ maps to $w = i\pi/4$, and $z = e \cdot e^{i\pi/4}$ maps to $w = 1 + i\pi/4$.

Remark 3.11
- *The ray does not include the origin;* Ln *is not defined at* 0.
- *The horizontal line is full in u (unbounded both ways).*

TikZ drawing.



z-plane: ray $\arg z = \frac{\pi}{4}$

w-plane: horizontal line $v = \frac{\pi}{4}$

1. **Compute the image under** Ln **of the circle** $|z| = e^{-1}$ **and draw the result.**
2. **Compute the image under** Ln **of the ray** $\arg z = -\frac{3\pi}{4}$**.**
3. **Show that the set** $\{z : 1 < |z| < e$ **and** $0 < \arg z < \frac{\pi}{2}\}$ **is mapped to the rectangle**

$$0 < u < 1, \qquad 0 < v < \tfrac{\pi}{2}$$

in the w**-plane.**

3.3 Complex Powers

We already know how to manipulate an integer power of a complex number such as z^n. However, when the exponent is itself a complex number, the computation of an expression like $(1+i)^i$ is less straightforward. In such cases, we use the complex exponential and logarithmic functions.

Definition 3.7 (Complex Power) ***If*** α ***is a complex number and*** $z \neq 0$***, the complex power is defined as***

$$z^\alpha = e^{\alpha \ln z}.$$

For integer exponents n***, the power function*** z^n ***is single-valued, while for complex exponents*** α***, the function*** z^α ***is multivalued because of the multivalued nature of*** $\ln z$***.***

Definition 3.8 (Principal Value) ***If*** α ***is a complex number and*** $z \neq 0$***, the function***

$$z^\alpha = e^{\alpha \operatorname{Ln} z}$$

is called the principal value of the complex power z^α***, where*** $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$ ***with*** $\operatorname{Arg} z \in (-\pi, \pi]$***.***

Remark 3.12 ***The function*** z^α ***inherits the multivalued character of the logarithm function, and different branches correspond to different possible values of the argument*** $\operatorname{Arg} z = \theta + 2k\pi$***.***

—

Example 3.8 ***Compute the complex power*** i^{2i}***.***

Solution 3.6 ***We know that***

$$i = e^{i(\pi/2 + 2k\pi)}, \qquad k \in \mathbb{Z}.$$

Hence,

$$\ln i = i\left(\frac{\pi}{2} + 2k\pi\right), \quad \textit{and therefore} \quad i^{2i} = e^{2i \ln i} = e^{-2(\frac{\pi}{2} + 2k\pi)} = e^{-\pi(1+4k)}.$$

Thus, the general values are

$$\boxed{i^{2i} = \{e^{-\pi(1+4k)} : k \in \mathbb{Z}\}.}$$

For the principal value ($k = 0$***),***

$$\boxed{i^{2i}\Big|_{\text{principal}} = e^{-\pi} \approx 0.0432139183.}$$

—

Example 3.9 ***Compute the complex power*** $(1+i)^i$***.***

Solution 3.7 ***We write***

$$1+i=\sqrt{2}\,e^{i(\pi/4+2k\pi)}.$$

Thus,

$$\ln(1+i)=\tfrac{1}{2}\ln 2+i\Big(\tfrac{\pi}{4}+2k\pi\Big).$$

Then,

$$(1+i)^i=e^{i\ln(1+i)}=e^{i(\frac{1}{2}\ln 2+i(\frac{\pi}{4}+2k\pi))}=e^{-(\frac{\pi}{4}+2k\pi)}\,e^{i(\frac{1}{2}\ln 2)}.$$

Hence,

$$\boxed{(1+i)^i=e^{-(\frac{\pi}{4}+2k\pi)}\,e^{i(\frac{1}{2}\ln 2)}.}$$

For the principal value ($k=0$ ***),***

$$(1+i)^i\Big|_{\text{principal}}=e^{-\pi/4}\,e^{i(\frac{1}{2}\ln 2)}\approx 0.4288290063+0.1548717525\,i.$$

3.3.1 Analyticity of the Complex Power

We recall that, on the principal branch domain $D=\mathbb{C}\setminus(-\infty,0]$, the principal branch

$$f_1(z)=z^\alpha=e^{\alpha\,\mathrm{Ln}\,z},\qquad -\pi<\mathrm{Arg}\,z<\pi,$$

is analytic and satisfies

$$f_1'(z)=\frac{d}{dz}\Big(e^{\alpha\mathrm{Ln}\,z}\Big)=\alpha\,z^{\alpha-1}.$$

Example 3.10 ***Find the derivative of the principal value of z^i at the point $z=1+i$.***

Solution 3.8 ***Step 1 - general derivative formula. For $\alpha=i$ we have, on the principal branch,***

$$\frac{d}{dz}z^i=i\,z^{i-1}.$$

Step 2 - principal logarithm of the point. For $z_0=1+i$ the principal polar data are

$$|1+i|=\sqrt{2},\qquad \mathrm{Arg}(1+i)=\frac{\pi}{4}.$$

Thus the principal logarithm is

$$\mathrm{Ln}(1+i)=\ln|1+i|+i\,\mathrm{Arg}(1+i)=\tfrac{1}{2}\ln 2+i\frac{\pi}{4}.$$

Set

$$a:=\tfrac{1}{2}\ln 2,\qquad b:=\frac{\pi}{4}.$$

Step 3 - compute z_0^{i-1}. Using the principal logarithm,

$$z_0^{i-1}=e^{(i-1)\mathrm{Ln}(1+i)}=e^{(i-1)(a+ib)}.$$

Expand the product in the exponent:

$$(i-1)(a+ib)=(ia-a)+(i^2b-ib)=(-a-b)+i(a-b).$$

Therefore

$$z_0^{i-1} = e^{-a-b}\, e^{i(a-b)}.$$

Step 4 - multiply by i to get the derivative.

$$f'(1+i) = i\, z_0^{i-1} = i\, e^{-a-b}\, e^{i(a-b)}.$$

We may write the final result in rectangular form by multiplying i with the complex exponential:

$$i\, e^{i\phi} = i(\cos\phi + i\sin\phi) = -\sin\phi + i\cos\phi, \qquad \phi := a - b.$$

Hence

$$\boxed{f'(1+i) = e^{-a-b}\Big(-\sin(a-b) + i\cos(a-b)\Big)}$$

with $a = \frac{1}{2}\ln 2$ and $b = \frac{\pi}{4}$.

Numerical approximation. Using $a \approx 0.3465735903$ and $b \approx 0.7853981634$,

$$a - b \approx -0.4388245731, \qquad -a - b \approx -1.1319717537.$$

Thus

$$z_0^{i-1} \approx 0.2918503794 - 0.1369786269\, i,$$

and

$$\boxed{f'(1+i) \approx 0.1369786269 + 0.2918503794\, i.}$$

(These digits are rounded; keep more if required.)

Remark 3.13 ***All steps used the principal branch Ln (i.e. $\mathrm{Arg} \in (-\pi, \pi]$). If a different branch were used the values of z^{i-1} and hence of $f'(1+i)$ would differ by the corresponding $2k\pi$ shifts in the argument.***

3.4 Complex Trigonometric Functions

Definition 3.9 (Complex Sine and Cosine) ***The complex sine and cosine functions are defined for every complex number z by:***

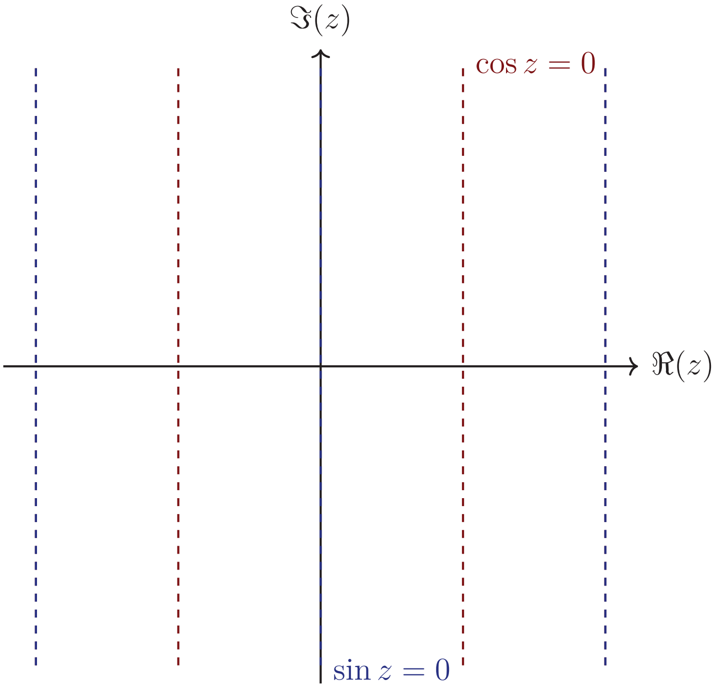
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

These definitions are direct extensions of the real trigonometric functions to the complex plane. They are entire (analytic on all of $\mathbb{C}$) since they are built from the exponential function, which is entire.

Definition 3.10 (Derived Complex Trigonometric Functions) ***Using the complex sine and cosine, we define the other trigonometric functions in the same way as in the real case:***

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Remark 3.14 ***Unlike in the real case, these functions are*** meromorphic ***on*** $\mathbb{C}$***: they are analytic everywhere except at the zeros of their denominators. For instance,*** $\tan z$ ***and*** $\sec z$ ***are undefined when*** $\cos z = 0$***, and*** $\cot z$ ***and*** $\csc z$ ***are undefined when*** $\sin z = 0$***.***



Complex z-plane: singular lines of $\tan z$ and $\cot z$

Example 3.11 ***Express each of the following complex trigonometric values in the form*** $a + ib$***:***

(a) $\cos i$***,***

(b) $\sin(2 + i)$***,***

(c) $\tan(\pi - 2i)$***.***

Solution 3.9 ***We use the standard identities (valid for complex arguments):***

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \qquad \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

and for tangent the convenient formula

$$\tan(x + iy) = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

(a) $\cos i$***. Put*** $x = 0,\ y = 1$ ***(since*** $i = 0 + 1 \cdot i$***). Using*** $\cos(0) = 1,\ \sin(0) = 0$***,***

$$\cos i = \cos 0 \cosh 1 - i \sin 0 \sinh 1 = \cosh 1.$$

Therefore

$$\boxed{\cos i = \cosh 1\ +\ 0 \cdot i.}$$

Numerically $\cosh 1 = \frac{e+e^{-1}}{2} \approx 1.5430806348$***.***

(b) $\sin(2+i)$***. Here*** $x = 2,\ y = 1$***. Apply the identity:***

$$\sin(2+i) = \sin 2 \cosh 1 + i \cos 2 \sinh 1.$$

Thus in $a + ib$ ***form:***

$$\boxed{\sin(2+i) = \Big(\sin 2 \cosh 1\Big) \ + \ i\Big(\cos 2 \sinh 1\Big).}$$

Numerically (rounded):

$$\sin 2 \approx 0.9092974268, \qquad \cos 2 \approx -0.4161468365,$$

$$\cosh 1 \approx 1.5430806348, \qquad \sinh 1 \approx 1.1752011936.$$

So

$$\sin 2 \cosh 1 \approx 1.4031192506, \qquad \cos 2 \sinh 1 \approx -0.4890561256,$$

and

$$\boxed{\sin(2+i) \approx 1.4031192506 - 0.4890561256\, i.}$$

(c) $\tan(\pi - 2i)$***. Write*** $z = \pi - 2i$ ***so*** $x = \pi,\ y = -2$***. Use the tangent formula above:***

$$\tan(\pi - 2i) = \frac{\sin(2\pi) + i \sinh(-4)}{\cos(2\pi) + \cosh(-4)}.$$

Compute the elementary values:

$$\sin(2\pi) = 0, \qquad \cos(2\pi) = 1, \qquad \sinh(-4) = -\sinh 4, \qquad \cosh(-4) = \cosh 4.$$

Thus

$$\tan(\pi - 2i) = \frac{i(-\sinh 4)}{1 + \cosh 4} = -i\,\frac{\sinh 4}{1 + \cosh 4}.$$

Use the identities $\sinh 4 = 2 \sinh 2 \cosh 2$ ***and*** $1 + \cosh 4 = 2\cosh^2 2$ ***to simplify:***

$$\frac{\sinh 4}{1 + \cosh 4} = \frac{2 \sinh 2 \cosh 2}{2 \cosh^2 2} = \tanh 2.$$

Hence

$$\boxed{\tan(\pi - 2i) = -i\ \tanh 2 = 0\ +\ \Big(-\tanh 2\Big)\, i.}$$

Numerically $\tanh 2 \approx 0.9640275801$***, so***

$$\boxed{\tan(\pi - 2i) \approx -0.9640275801\, i.}$$

Remark 3.15 ***Note the useful special cases:*** $\cos(iy) = \cosh y$ ***and*** $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$***. When an identity reduces an expression to purely real or purely imaginary form, it is often possible to simplify further using hyperbolic identities (as in part (c)).***

3.4.1 Trigonometric Identities

As for real trigonometric functions, the complex trigonometric functions satisfy the following identities:

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$$

$$\sin(2z) = 2 \sin z \cos z, \qquad \cos(2z) = \cos^2 z - \sin^2 z$$

3.4.2 Periodicity

It follows directly that the complex trigonometric functions sine and cosine are periodic, and their period is 2π:

$$\sin(z + 2\pi) = \sin z, \qquad \cos(z + 2\pi) = \cos z$$

3.4.3 Zeros of Complex Sine and Cosine Functions

The complex sine and cosine functions have zeros, analogously to the real case, for the following values:

$$\sin z = 0 \quad \textbf{if and only if} \quad z = n\pi$$

$$\cos z = 0 \quad \textbf{if and only if} \quad z = \frac{(2n+1)\pi}{2}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

3.4.4 Analyticity

The derivatives of the complex sine and cosine functions are obtained from their definitions in terms of the complex exponential function. We have:

$$\frac{d}{dz}(\sin z) = \cos z, \qquad \frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\tan z) = \sec^2 z, \qquad \frac{d}{dz}(\cot z) = -\csc^2 z$$

$$\frac{d}{dz}(\sec z) = \sec z\,\tan z, \qquad \frac{d}{dz}(\csc z) = -\csc z\,\cot z$$

3.5 Complex Hyperbolic Functions

Definition 3.11 ***The complex hyperbolic functions are defined by:***

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

From these, the other hyperbolic functions are deduced as follows:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

It should also be noted that the hyperbolic functions $\sinh z$ and $\cosh z$ are *entire* functions, since the exponential function e^z itself is entire.

3.5.1 Derivatives of Complex Hyperbolic Functions

The derivatives of the complex hyperbolic functions can be obtained directly from their definitions in terms of the complex exponential function. We have:

$$\frac{d}{dz}(\sinh z) = \cosh z, \qquad \frac{d}{dz}(\cosh z) = \sinh z,$$

$$\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \qquad \frac{d}{dz}(\coth z) = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}(\operatorname{sech} z) = -\operatorname{sech} z\,\tanh z, \qquad \frac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z\,\coth z.$$

3.5.2 Relations with Complex Trigonometric Functions

It is quite interesting to observe the relationship between hyperbolic and trigonometric complex functions, since their expressions have the same form up to a constant factor.

Let us replace z by iz in the definitions above. We obtain:

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = i\sin z.$$

Equivalently,

$$\sin z = -i\sinh(iz).$$

Similarly, we can establish the correspondence between the different hyperbolic and trigonometric functions as follows:

$$\sin z = -i\sinh(iz), \qquad \cos z = \cosh(iz),$$

$$\sinh z = -i\sin(iz), \qquad \cosh z = \cos(iz),$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i\sinh z}{\cosh z} = i\tanh z.$$

3.5.3 Identities of Complex Hyperbolic Functions

The equalities above can be used to establish the following identities for the complex hyperbolic functions:

$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sinh(2z) = 2 \sinh z \cosh z, \qquad \cosh(2z) = \cosh^2 z + \sinh^2 z$$

3.5.4 Inverse Trigonometric and Hyperbolic Functions

Inverse Trigonometric Functions

When seeking solutions to the equation $\sin w = z$**, it is necessary to define the inverse function such that** $z = \sin^{-1} w$**.**

Starting from the definition of the complex sine function:

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = z,$$

we obtain:

$$e^{2iw} - 2iz\, e^{iw} - 1 = 0.$$

This is a quadratic equation in e^{iw}**, whose solution is:**

$$e^{iw} = iz + \sqrt{1 - z^2}.$$

Taking the logarithm, we find:

$$iw = \ln\Big[iz + \sqrt{1 - z^2} \Big], \qquad \textbf{hence} \qquad w = -i \ln\Big[iz + \sqrt{1 - z^2} \Big].$$

This leads to the definition of the inverse sine function:

$$\boxed{\sin^{-1} z = -i \ln\Big[iz + \sqrt{1 - z^2} \Big]} \quad \textbf{(Inverse sine)}.$$

This function is *multivalued*, since it involves the complex logarithm, which is itself multivalued.

By applying the same procedure, we define the other inverse trigonometric functions:

$$\boxed{\cos^{-1} z = -i \ln\Big[z + i\sqrt{1 - z^2} \Big]} \quad \textbf{(Inverse cosine)}$$

$$\boxed{\tan^{-1} z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)} \quad \textbf{(Inverse tangent)}.$$

The derivatives of these functions are defined only on domains where the functions are continuous and differentiable (that is, after choosing suitable branches of these multivalued functions):

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\cos^{-1} z = -\frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}.$$

3.6 Inverse Hyperbolic Functions

In the same way as for trigonometric functions, we can define the inverse hyperbolic functions as the solutions of equations of the form $\sinh w = z$**. We then obtain:**

$$\sinh^{-1} z = \ln\left[z + \sqrt{z^2+1}\right] \quad \textbf{(Inverse hyperbolic sine)}$$

$$\cosh^{-1} z = \ln\left[z + i\sqrt{z^2-1}\right] \quad \textbf{(Inverse hyperbolic cosine)}$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \quad \textbf{(Inverse hyperbolic tangent)}$$

For suitable branches defined on a domain D **where these functions are continuous and differentiable, we have:**

$$\frac{d}{dz}\left(\sinh^{-1} z\right) = \frac{1}{\sqrt{z^2+1}}, \qquad \frac{d}{dz}\left(\cosh^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}, \qquad \frac{d}{dz}\left(\tanh^{-1} z\right) = \frac{1}{1-z^2}.$$

Exercise 3.2 ***(1) Find all possible values of*** $\sin^{-1}(\sqrt{5})$***.***

(2) Find the derivative of $\sin^{-1} z$ ***at*** $z = i$***, and verify that this point belongs to the domain of definition.***

(3) Assume that $\cosh^{-1} z$ ***represents a branch of the inverse hyperbolic cosine defined using the complex square root and logarithm branches***

$$f_2(z) = \sqrt{r}\, e^{i\theta/2}, \quad 0 < \theta < 2\pi.$$

(4) Compute:

(a) $\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right)$***;***

(b) $\left.\frac{d}{dz}\cosh^{-1} z\right|_{z=\frac{\sqrt{2}}{2}}$***.***

Solution 3.10 ***(1) Using the formula***

$$\sin^{-1} z = -i \ln\left[iz + \sqrt{1-z^2}\right],$$

we get for $z = \sqrt{5}$***:***

$$\sin^{-1}(\sqrt{5}) = -i \ln\left[i\sqrt{5} + \sqrt{1-5}\right] = -i \ln\left[i\sqrt{5} + 2i\right] = -i \ln\left[i(\sqrt{5}+2)\right].$$

Hence,

$$\boxed{\sin^{-1}(\sqrt{5}) = \frac{\pi}{2} + i \ln(\sqrt{5}+2) + 2k\pi, \quad k \in \mathbb{Z}.}$$

(2) The derivative is

$$\frac{d}{dz}\left(\sin^{-1} z\right) = \frac{1}{\sqrt{1-z^2}}.$$

At $z = i$***:***

$$\frac{d}{dz}\left(\sin^{-1} z\right)\Big|_{z=i} = \frac{1}{\sqrt{1-i^2}} = \frac{1}{\sqrt{2}}.$$

The point $z = i$ ***belongs to the domain since*** $1 - z^2 = 2 \neq 0$***.***

$$\boxed{\frac{d}{dz}\left(\sin^{-1} z\right)\Big|_{z=i} = \frac{1}{\sqrt{2}}.}$$

(a) Using

$$\cosh^{-1} z = \ln\left[z + i\sqrt{z^2-1}\right],$$

for $z = \frac{\sqrt{2}}{2}$***:***

$$\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right) = \ln\left[\frac{\sqrt{2}}{2} + i\sqrt{\frac{1}{2}-1}\right] = \ln\left[\frac{\sqrt{2}}{2} + i\frac{i}{\sqrt{2}}\right] = \ln(1) = 0.$$

$$\boxed{\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right) = 0.}$$

(b) The derivative is

$$\frac{d}{dz}\left(\cosh^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}.$$

At $z = \frac{\sqrt{2}}{2}$***:***

$$\frac{d}{dz}\left(\cosh^{-1} z\right)\Big|_{z=\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{\frac{1}{2}-1}} = \frac{1}{\sqrt{-\frac{1}{2}}} = \frac{1}{i/\sqrt{2}} = -i\sqrt{2}.$$

$$\boxed{\frac{d}{dz}\left(\cosh^{-1} z\right)\Big|_{z=\frac{\sqrt{2}}{2}} = -i\sqrt{2}.}$$

3.7 Solved Exercises

Exercise 3.3 ***Compute the following complex derivatives:***

$$\textbf{(a)}\ f(z) = iz^4(z^2 - e^z), \qquad \textbf{(b)}\ g(z) = \exp\{z^2 - (1+i)z + 3\}.$$

Solution 3.11 ***(a) Let***

$$f(z) = iz^4(z^2 - e^z).$$

We use the product rule:

$$f'(z) = i\left[(z^4)'(z^2 - e^z) + z^4(z^2 - e^z)'\right].$$

Compute each derivative:

$$(z^4)' = 4z^3, \qquad (z^2 - e^z)' = 2z - e^z.$$

Hence

$$\begin{aligned} f'(z) &= i\left[4z^3(z^2 - e^z) + z^4(2z - e^z)\right] \\ &= i\left(4z^5 - 4z^3e^z + 2z^5 - z^4e^z\right) \\ &= i\left(6z^5 - e^z(4z^3 + z^4)\right). \end{aligned}$$

Therefore,

$$\boxed{f'(z) = i\left(6z^5 - e^z(4z^3 + z^4)\right)}.$$

(b) Let

$$g(z) = \exp\{z^2 - (1+i)z + 3\}.$$

We apply the chain rule:

$$g'(z) = \exp\{z^2 - (1+i)z + 3\} \cdot (2z - (1+i)).$$

Thus,

$$\boxed{g'(z) = (2z - (1+i))\, e^{z^2 - (1+i)z + 3}}.$$

Second Method

Solution 3.12 ***(a) Function*** $f(z) = iz^4(z^2 - e^z)$***.***

Let $z = x + iy$***. Then***

$$f(z) = i(x+iy)^4\left[(x+iy)^2 - e^{x+iy}\right].$$

We expand each term:

$$(x+iy)^2 = (x^2 - y^2) + 2ixy, \qquad e^{x+iy} = e^x(\cos y + i\sin y).$$

Hence

$$\begin{aligned} z^2 - e^z &= (x^2 - y^2 - e^x\cos y) + i(2xy - e^x\sin y), \\ z^4 &= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3). \end{aligned}$$

Now, let

$$z^4(z^2 - e^z) = (A + iB), \quad \textbf{\textit{so that}}\ f(z) = i(A + iB) = iA - B.$$

Thus

$$u(x, y) = -B, \qquad v(x, y) = A.$$

Since u ***and*** v ***are differentiable functions of*** x ***and*** y***, the function*** f ***is holomorphic everywhere (product and sum of entire functions).***

Therefore, by definition of the complex derivative:

$$f'(z) = u_x + iv_x.$$

We compute symbolically (using standard differentiation rules for holomorphic functions):

$$\begin{aligned} f'(z) &= i\left[4z^3(z^2 - e^z) + z^4(2z - e^z)\right] \\ &= i\left(6z^5 - e^z(4z^3 + z^4)\right). \end{aligned}$$

Hence

$$\boxed{f'(z) = u_x + iv_x = i\Big(6z^5 - e^z(4z^3 + z^4)\Big)}.$$

(b) Function $g(z) = e^{\,z^2-(1+i)z+3}$***.***

Let

$$z = x + iy \quad \textbf{\textit{and}} \quad \Phi(z) = z^2 - (1 + i)z + 3 = (x^2 - y^2 - x + y + 3) + i(2xy - y - x).$$

Then

$$e^{\Phi(z)} = e^{x^2-y^2-x+y+3}\Big[\cos(2xy - y - x) + i\sin(2xy - y - x)\Big].$$

Hence

$$\begin{aligned} u(x, y) &= e^{x^2-y^2-x+y+3}\cos(2xy - y - x), \\ v(x, y) &= e^{x^2-y^2-x+y+3}\sin(2xy - y - x). \end{aligned}$$

We compute

$$u_x = e^{x^2-y^2-x+y+3}\Big[(2x - 1)\cos(2xy - y - x) - (2y - 1)\sin(2xy - y - x)\Big],$$

$$v_x = e^{x^2-y^2-x+y+3}\Big[(2x - 1)\sin(2xy - y - x) + (2y - 1)\cos(2xy - y - x)\Big].$$

Thus, by definition,

$$f'(z) = u_x + iv_x = e^{z^2-(1+i)z+3}\,(2z - (1 + i)).$$

$$\boxed{g'(z) = u_x + iv_x = (2z - (1 + i))\, e^{\,z^2-(1+i)z+3}}.$$

Exercise 3.4 ***Compute the complex power*** $(-3)^{\,i/\pi}$***.***

Solution 3.13 ***We write***

$$-3 = 3e^{i(\pi+2k\pi)}, \qquad k \in \mathbb{Z}.$$

Thus,

$$\ln(-3) = \ln 3 + i(\pi + 2k\pi),$$

and hence

$$(-3)^{\,i/\pi} = e^{(i/\pi)\ln(-3)} = e^{(i/\pi)(\ln 3 + i(\pi+2k\pi))} = e^{-(1+2k)}\, e^{\,i(\ln 3)/\pi}.$$

Therefore,

$$\boxed{(-3)^{\,i/\pi} = e^{-(1+2k)}\, e^{\,i(\ln 3)/\pi}.}$$

For the principal value $(k = 0)$***,***

$$(-3)^{\,i/\pi}\Big|_{\text{principal}} = e^{-1}\, e^{\,i(\ln 3)/\pi} \approx 0.3456138434 + 0.1260410825\, i.$$

Exercise 3.5 ***Compute the complex power*** $(2i)^{\,1-i}$***.***

Solution 3.14 ***We write***

$$2i = 2e^{i(\pi/2+2k\pi)}.$$

Hence,

$$\ln(2i) = \ln 2 + i(\pi/2 + 2k\pi).$$

Then,

$$(2i)^{1-i} = e^{(1-i)\ln(2i)} = e^{(1-i)(\ln 2 + i(\pi/2+2k\pi))}.$$

Expanding the exponent:

$$(1-i)(\ln 2 + iB) = (\ln 2 + B) + i(B - \ln 2), \quad \textbf{\textit{with }} B = \frac{\pi}{2} + 2k\pi.$$

Hence,

$$(2i)^{1-i} = e^{\ln 2 + B}\, e^{\,i(B-\ln 2)} = 2e^{B}\, e^{\,i(B-\ln 2)}.$$

Thus,

$$\boxed{(2i)^{1-i} = 2e^{\,(\frac{\pi}{2}+2k\pi)}\, e^{\,i(\frac{\pi}{2}+2k\pi-\ln 2)}.}$$

For the principal value $(k = 0)$***,***

$$(2i)^{1-i}\Big|_{\text{principal}} = 2e^{\pi/2}\, e^{\,i(\pi/2-\ln 2)} \approx 6.1474175340 + 7.4008126711\, i.$$

Remark 3.16 ***The above examples illustrate how the multivalued nature of the complex logarithm leads to multiple values of complex powers. The*** **principal value** ***corresponds to taking*** $\operatorname{Arg} z$ ***within*** $(-\pi, \pi]$***.***

Expression	Principal Value (approx.)
i^{2i}	$e^{-\pi} \approx 0.0432139183$
$(1+i)^i$	$0.4288290063 + 0.1548717525\, i$
$(-3)^{\,i/\pi}$	$0.3456138434 + 0.1260410825\, i$
$(2i)^{\,1-i}$	$6.1474175340 + 7.4008126711\, i$

Exercise 3.6 (Analyticity and derivatives of complex trigonometric functions) ***Show that the complex sine and cosine functions are entire (analytic on $\mathbb{C}$) and compute the derivatives***

$$\frac{d}{dz}\sin z, \quad \frac{d}{dz}\cos z, \quad \frac{d}{dz}\tan z, \quad \frac{d}{dz}\cot z, \quad \frac{d}{dz}\sec z, \quad \frac{d}{dz}\csc z.$$

State clearly the domains where the derivatives of $\tan z$, $\cot z$, $\sec z$, $\csc z$ ***are defined.***

Solution 3.15 ***1. Entirety of*** $\sin z$ ***and*** $\cos z$***.***

Recall the exponential definitions for all $z \in \mathbb{C}$***:***

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The complex exponential e^w ***is entire (holomorphic on*** $\mathbb{C}$***). Since*** $\sin z$ ***and*** $\cos z$ ***are linear combinations and compositions of entire functions, they are entire as well. In particular they are differentiable everywhere in*** $\mathbb{C}$***.***

2. Derivative of $\sin z$ ***and*** $\cos z$***.***

Differentiate the exponential expressions term-by-term (valid for entire functions):

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

Thus

$$\frac{d}{dz}\sin z = \cos z, \qquad \frac{d}{dz}\cos z = -\sin z.$$

3. Derivative of $\tan z$***.***

By definition $\tan z = \dfrac{\sin z}{\cos z}$***. Use the quotient rule on the domain where*** $\cos z \neq 0$***:***

$$\frac{d}{dz}\tan z = \frac{\cos z \cdot (\cos z) - \sin z \cdot (-\sin z)}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z.$$

So $\dfrac{d}{dz}\tan z = \sec^2 z$ ***on the domain*** $\{z \in \mathbb{C} : \cos z \neq 0\}$ ***(i.e. excluding the poles of*** $\tan z$***, which occur at*** $z = (2n+1)\pi/2$***,*** $n \in \mathbb{Z}$***).***

4. Derivative of $\cot z$***.***

$\cot z = \dfrac{\cos z}{\sin z}$***. Use the quotient rule where*** $\sin z \neq 0$***:***

$$\frac{d}{dz}\cot z = \frac{\sin z \cdot (-\sin z) - \cos z \cdot \cos z}{\sin^2 z} = -\frac{\sin^2 z + \cos^2 z}{\sin^2 z} = -\frac{1}{\sin^2 z} = -\csc^2 z.$$

So $\dfrac{d}{dz}\cot z = -\csc^2 z$ ***on*** $\{z \in \mathbb{C} : \sin z \neq 0\}$ ***(excluding poles*** $z = n\pi$***,*** $n \in \mathbb{Z}$***).***

5. Derivative of $\sec z$***.***
$\sec z = 1/\cos z$***. Differentiate on the domain*** $\cos z \neq 0$***:***

$$\frac{d}{dz}\sec z = \frac{d}{dz}\left(\frac{1}{\cos z}\right) = -\frac{-\sin z}{\cos^2 z} = \frac{\sin z}{\cos^2 z} = \frac{1}{\cos z}\cdot\frac{\sin z}{\cos z} = \sec z\ \tan z.$$

Thus $\dfrac{d}{dz}\sec z = \sec z\ \tan z$ ***for*** $\cos z \neq 0$***.***

6. Derivative of $\csc z$***.***
$\csc z = 1/\sin z$***. Differentiate on*** $\sin z \neq 0$***:***

$$\frac{d}{dz}\csc z = \frac{d}{dz}\left(\frac{1}{\sin z}\right) = -\frac{\cos z}{\sin^2 z} = -\csc z\ \cot z.$$

So $\dfrac{d}{dz}\csc z = -\csc z\ \cot z$ ***for*** $\sin z \neq 0$***.***

Summary. The derivatives (with their domains) are:

$$\frac{d}{dz}\sin z = \cos z \quad (\forall z \in \mathbb{C}), \qquad \frac{d}{dz}\cos z = -\sin z \quad (\forall z \in \mathbb{C}),$$
$$\frac{d}{dz}\tan z = \sec^2 z \quad (\cos z \neq 0), \qquad \frac{d}{dz}\cot z = -\csc^2 z \quad (\sin z \neq 0),$$
$$\frac{d}{dz}\sec z = \sec z\ \tan z \quad (\cos z \neq 0), \qquad \frac{d}{dz}\csc z = -\csc z\ \cot z \quad (\sin z \neq 0).$$

This completes the exercise.

Exercise 3.7 ***(1) Find all possible values of*** $\cos^{-1}(2i)$***.***

(2) Compute the derivative of $\tanh^{-1} z$ ***at*** $z = \frac{i}{2}$***, and verify that this point belongs to the domain of definition.***

(3) Assume that $\sinh^{-1} z$ ***represents a branch of the inverse hyperbolic sine defined using the complex square root and logarithm branches***

$$f_2(z) = \sqrt{r}\, e^{i\theta/2}, \quad 0 < \theta < 2\pi.$$

Compute:

(a) $\sinh^{-1}\left(\frac{3}{2}\right)$***;***

(b) $\left.\dfrac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}}$***.***

Solution 3.16 ***(1) Values of*** $\cos^{-1}(2i)$***.***

We use the standard complex formula

$$\cos^{-1} z = -i\ln\left(z + \sqrt{z^2 - 1}\right),$$

where the square root and the logarithm are multivalued (hence the inverse cosine is multivalued).

Take $z = 2i$***. Compute***

$$z^2 - 1 = (2i)^2 - 1 = -4 - 1 = -5.$$

Thus

$$\sqrt{z^2 - 1} = \sqrt{-5} = \pm i\sqrt{5}.$$

Two possible choices give

$$z + \sqrt{z^2 - 1} = 2i \pm i\sqrt{5} = i(2 \pm \sqrt{5}).$$

Now

$$\cos^{-1}(2i) = -i \ln\Big(i(2 \pm \sqrt{5})\Big).$$

Write $i(2 \pm \sqrt{5}) = r\, e^{i\theta}$ ***with***

$$r = |2 \pm \sqrt{5}|, \qquad \theta = \arg\Big(i(2 \pm \sqrt{5})\Big).$$

Case $2 + \sqrt{5}$ ***(positive):***

$$i(2 + \sqrt{5}) = (2 + \sqrt{5})e^{i\pi/2}, \qquad \ln\Big(i(2 + \sqrt{5})\Big) = \ln(2 + \sqrt{5}) + i\Big(\frac{\pi}{2} + 2k\pi\Big).$$

Hence

$$\cos^{-1}(2i) = -i\bigg(\ln(2 + \sqrt{5}) + i\Big(\frac{\pi}{2} + 2k\pi\Big)\bigg) = \frac{\pi}{2} + 2k\pi \;-\; i\ln(2 + \sqrt{5}), \quad k \in \mathbb{Z}.$$

Case $2 - \sqrt{5}$ ***(negative; note*** $2 - \sqrt{5} < 0$***): let*** $a = \sqrt{5} - 2 > 0$***. Then*** $2 - \sqrt{5} = -a$ ***and***

$$i(2 - \sqrt{5}) = i(-a) = ae^{-i\pi/2}.$$

So

$$\ln\Big(i(2 - \sqrt{5})\Big) = \ln(a) + i\Big(-\frac{\pi}{2} + 2k\pi\Big),$$

and

$$\cos^{-1}(2i) = -i\bigg(\ln(a) + i\Big(-\frac{\pi}{2} + 2k\pi\Big)\bigg) = -\frac{\pi}{2} + 2k\pi \;-\; i\ln(\sqrt{5} - 2), \quad k \in \mathbb{Z}.$$

Compact form of the multivalued results:

$$\boxed{\cos^{-1}(2i) = \pm\frac{\pi}{2} + 2k\pi \;-\; i\ln\left|2 \pm \sqrt{5}\right|, \qquad k \in \mathbb{Z},}$$

where the two signs correspond to the two choices of the square root. (The explicit forms found above are usually written as)

$$\boxed{\cos^{-1}(2i) = \frac{\pi}{2} + 2k\pi - i\ln(2 + \sqrt{5}) \quad \textbf{\textit{or}} \quad \cos^{-1}(2i) = -\frac{\pi}{2} + 2k\pi - i\ln(\sqrt{5} - 2).}$$

(2) Derivative of $\tanh^{-1} z$ ***at*** $z = \frac{i}{2}$***.***

Recall

$$\tanh^{-1} z = \frac{1}{2}\ln\!\left(\frac{1+z}{1-z}\right)$$

and its derivative (on domains where $1 - z^2 \neq 0$***) is***

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}.$$

Evaluate at $z = \dfrac{i}{2}$***:***

$$1 - z^2 = 1 - \left(\frac{i}{2}\right)^2 = 1 - \left(-\frac{1}{4}\right) = 1 + \frac{1}{4} = \frac{5}{4}.$$

Thus

$$\left.\frac{d}{dz}\tanh^{-1} z\right|_{z=\frac{i}{2}} = \frac{1}{5/4} = \frac{4}{5}.$$

Since $1-z^2 \neq 0$ ***at*** $z = i/2$***, the derivative is defined there. Also the principal formula for*** $\tanh^{-1}$ ***is analytic except at points where*** $1 \pm z = 0$ ***or equivalently where*** $z = \pm 1$ ***(and more generally where the logarithm branch cut is chosen). The point*** $z = i/2$ ***is not a singularity, so it belongs to the domain for the standard branches.***

$$\boxed{\left.\frac{d}{dz}\tanh^{-1} z\right|_{z=\frac{i}{2}} = \frac{4}{5}.}$$

(3) Inverse hyperbolic sine at $\frac{3}{2}$ ***and its derivative.***

We use

$$\sinh^{-1} z = \ln\!\left(z + \sqrt{z^2+1}\right),$$

with the principal (positive) square root for real positive arguments.

(a) For $z = \dfrac{3}{2}$ ***(real, positive),***

$$z^2 + 1 = \frac{9}{4} + 1 = \frac{13}{4}, \qquad \sqrt{z^2+1} = \frac{\sqrt{13}}{2}.$$

Therefore

$$\sinh^{-1}\!\left(\tfrac{3}{2}\right) = \ln\!\left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right) = \ln\!\left(\frac{3+\sqrt{13}}{2}\right).$$

$$\boxed{\sinh^{-1}\!\left(\tfrac{3}{2}\right) = \ln\!\left(\frac{3+\sqrt{13}}{2}\right).}$$

(b) The derivative formula (on domains where the chosen square root is nonzero) is

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{z^2+1}}.$$

Hence at $z = \dfrac{3}{2}$***,***

$$\left.\frac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}} = \frac{1}{\sqrt{\frac{13}{4}}} = \frac{1}{\frac{\sqrt{13}}{2}} = \frac{2}{\sqrt{13}}.$$

$$\boxed{\left.\frac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}} = \frac{2}{\sqrt{13}}.}$$