Chapter 2

Holomorphic Functions

2.1 Derivative of a Complex Function

Definition 2.1 ***Suppose that a complex function f is defined in a neighborhood of z_0. The derivative of f at z_0, denoted by $f'(z_0)$, exists and is given by:***

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

In this case, the function f is said to be differentiable at z_0.

Example 2.1 ***Apply Definition 2.1 to find the derivative of the function***

$$f(z) = z^2 - 5z$$

Solution 2.1 ***We compute the derivative using the definition:***

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - 5(z + \Delta z) - (z^2 - 5z)}{\Delta z}$$

Simplify the numerator:

$$(z+\Delta z)^2 - 5(z+\Delta z) - z^2 + 5z = z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z - z^2 + 5z = 2z\Delta z + (\Delta z)^2 - 5\Delta z = \Delta z(2z+\Delta z-$$

Divide by Δz:

$$\frac{\Delta z(2z + \Delta z - 5)}{\Delta z} = 2z + \Delta z - 5$$

Take the limit as $\Delta z \to 0$:

$$f'(z) = 2z - 5$$

2.1.1 Rules of Differentiation for Complex Functions

Definition 2.2 (Differentiable Functions) ***Let $f(z)$ and $g(z)$ be two functions defined and differentiable at z. Then the following rules of differentiation hold:***

Theorem 2.1 (Rules of Differentiation) ***$f(z)$ and $g(z)$: two functions defined and differentiable at z***

1. ***Constant Rule:***

$$\frac{d}{dz}c = 0, \qquad \frac{d}{dz}[cf(z)] = cf'(z), \quad c \in \mathbb{C}.$$

2. ***Sum Rule:***

$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z).$$

3. ***Product Rule:***

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z).$$

4. ***Quotient Rule:***

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}, \quad g(z) \neq 0.$$

5. ***Chain Rule:***

$$\frac{d}{dz}[f(g(z))] = f'(g(z))g'(z).$$

Example 2.2 ***Use the differentiation rules to compute the derivatives of the following functions:***

(a) $f(z) = 3z^4 - 5z^3 + 2z$

(b) $f(z) = \dfrac{z^2}{4z+1}$

Solution 2.2 ***(a)*** $f(z) = 3z^4 - 5z^3 + 2z$

Using the constant and power rules:

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 = 12z^3 - 15z^2 + 2$$

(b) $f(z) = \dfrac{z^2}{4z+1}$

Using the quotient rule:

$$f'(z) = \frac{(z^2)'(4z+1) - (z^2)(4z+1)'}{(4z+1)^2} = \frac{2z(4z+1) - z^2(4)}{(4z+1)^2}$$

Simplify the numerator:

$$2z(4z+1) - 4z^2 = 8z^2 + 2z - 4z^2 = 4z^2 + 2z$$

Factorize if desired:

$$f'(z) = \frac{2z(2z+1)}{(4z+1)^2}$$

2.2 Analytic Function

Definition 2.3 (Analyticity at a Point) ***A complex function $w = f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and at every point in a neighborhood of z_0.***

A complex function $f(z)$ is said to be analytic in a domain D if it is analytic at every point of this domain. In this case, the function is also called holomorphic or regular.

A function that is analytic on the entire complex plane $\mathbb{C}$ is called an entire function.

Theorem 2.2 (Polynomial and Rational Functions)

1. ***A complex polynomial function***

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \in \mathbb{N}$ and $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{C}$, is an entire function.

2. ***A rational function***

$$f(z) = \frac{p(z)}{q(z)}$$

is analytic in any domain D that does not contain any point z_0 for which $q(z_0) = 0$.

Here, $p(z)$ and $q(z)$ are two polynomial functions, and z_0 is called a singular point of $f(z)$.

2.2.1 Differentiability and Cauchy-Riemann Equations

Theorem 2.3 (Differentiability Implies Continuity) ***If f is differentiable at a point z_0 in a domain D, then f is necessarily continuous at z_0.***

Theorem 2.4 (L'Hôpital's Rule for Complex Functions) ***Suppose f and g are two functions analytic at z_0 with $f(z_0) = 0$ and $g(z_0) = 0$ but $g'(z_0) \neq 0$. Then:***

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Example 2.3 ***Use L'Hôpital's Rule to compute the following limit:***

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}.$$

Solution 2.3 ***Let $f(z) = z^2 - 4z + 5$ and $g(z) = z^3 - z - 10i$.***

$$f'(z) = 2z - 4, \quad g'(z) = 3z^2 - 1$$

Evaluate at $z_0 = 2 + i$:

$$f'(2+i) = 2(2+i) - 4 = 4 + 2i - 4 = 2i$$

$$g'(2+i) = 3(2+i)^2 - 1 = 3(4+4i-1) - 1 = 3(3+4i) - 1 = 9 + 12i - 1 = 8 + 12i$$

Thus, using L'Hôpital's rule:

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i} = \frac{f'(2+i)}{g'(2+i)} = \frac{2i}{8+12i} = \frac{2i(8-12i)}{(8+12i)(8-12i)} = \frac{24+16i}{208} = \frac{3}{26} + \frac{2}{26}i = \frac{3}{26} + \frac{1}{13}i$$

Theorem 2.5 (Cauchy-Riemann Equations) ***Let*** $f(z) = u(x,y) + iv(x,y)$ ***be differentiable at a point*** $z = x + iy$***. Then the first-order partial derivatives of*** u ***and*** v ***exist and satisfy the Cauchy-Riemann equations at*** z***:***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Analyticity Criteria

Example 2.4 ***Verify that the function***

$$f(z) = z^2 + z$$

satisfies the Cauchy-Riemann equations.

Solution 2.4 ***Write*** $f(z) = u(x,y) + iv(x,y)$ ***with*** $z = x + iy$***:***

$$f(z) = (x+iy)^2 + (x+iy) = x^2 - y^2 + x + i(2xy + y)$$

So, $u(x,y) = x^2 - y^2 + x$ ***and*** $v(x,y) = 2xy + y$***.***

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x + 1 = 2x + 1 \quad ?$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2y = -2y \quad ?$$

Thus, the Cauchy-Riemann equations are satisfied everywhere, so $f(z)$ ***is analytic in*** $\mathbb{C}$***.***

Theorem 2.6 (Non-Analyticity Criterion) ***If the Cauchy-Riemann equations are not satisfied at every point*** z ***in a domain*** D***, then the function***

$$f(z) = u(x,y) + iv(x,y)$$

cannot be analytic on D***.***

Example 2.5 ***Show that the function***

$$f(z) = 2x^2 + i(y^2 - x)$$

is not analytic at any point.

Solution 2.5 ***Identify*** $u(x, y) = 2x^2$ ***and*** $v(x, y) = y^2 - x$***.***

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 4x, \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = -1, \quad \frac{\partial v}{\partial y} = 2y$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 4x = 2y \quad \textit{(not true for all } x, y\textit{)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 0 = 1 \quad \textit{(false)}$$

Since the Cauchy-Riemann equations fail everywhere, $f(z)$ ***is not analytic at any point.***

Theorem 2.7 (Analyticity Criterion) ***Suppose the real functions*** $u(x, y)$ ***and*** $v(x, y)$ ***are continuous and have first-order partial derivatives in a domain*** D***.***

If u ***and*** v ***satisfy the Cauchy-Riemann equations at every point of*** D***, then the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is analytic in D***.***

Example 2.6 ***Let***

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}, \quad z = x + iy$$

Determine the domain $D = \mathbf{Dom}(f)$ ***and show that*** f ***is analytic on*** D***.***

Solution 2.6 ***We can write*** $f(z) = u(x, y) + iv(x, y)$ ***with***

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

Domain: $x^2 + y^2 \neq 0 \Rightarrow D = \mathbb{C} \setminus \{0\}$***.***

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, $f(z)$ ***is analytic on*** $D = \mathbb{C} \setminus \{0\}$***.***

2.3 Theorems on Differentiability and Harmonic Functions

2.3.1 Theorems on Differentiability

Theorem 2.8 ***If the real functions $u(x, y)$ and $v(x, y)$ are continuous and have first-order partial derivatives in a neighborhood of a point z, and if they satisfy the Cauchy-Riemann equations at z, then the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is differentiable at z, and its derivative is given by:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Theorem 2.9 ***Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on a domain D. If $f'(z) = 0$ on D, then***

$$f(z) = c \quad \textbf{on } D,$$

where c is a complex constant.

When and Why We Use the Theorem $f'(z) = u_x + iv_x$?

Remark 2.1 ***There are two different ways to compute or study the derivative of a complex function $f(z)$:***

(a) The direct (formal) method, which uses the usual differentiation rules (sum, product, chain rule, etc.);

(b) The analytic (fundamental) method, which starts from the definition

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy,$$

and uses the theorem

$$f'(z) = u_x + iv_x,$$

under the condition that the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

are satisfied.

Example 2.7 ***(a) Direct Method (formal differentiation).***

When $f(z)$ is written only in terms of z (for example e^z, z^2, z^3, etc.), we can apply the usual differentiation rules, because such functions are known to be **holomorphic everywhere*****.***

For instance,

$$\frac{d}{dz}(z^2) = 2z.$$

In this case, there is no need to express $f(z)$ in terms of $u(x, y)$ and $v(x, y)$.

Example 2.8 ***(b) Analytic Method (definition via partial derivatives).***

The theorem

$$f'(z) = u_x + iv_x$$

is fundamental, because it allows us to determine whether a function is actually complex differentiable. ***That is, it provides a test for holomorphy using the Cauchy-Riemann equations.***

If u ***and*** v ***satisfy***

$$u_x = v_y, \qquad u_y = -v_x,$$

and u, v ***are continuous, then*** $f(z)$ ***is differentiable (holomorphic) at that point, and***

$$f'(z) = u_x + iv_x.$$

Remark 2.2 ***The theorem*** $f'(z) = u_x + iv_x$ ***is particularly useful in the following situations:***

(1) When we do not know whether $f(z)$ ***is holomorphic or not.***

Example: $f(z) = \overline{z}$ ***or*** $f(z) = \overline{z}\,e^z$***. These look simple, but they are not holomorphic. To check this, we must write*** $f = u+iv$ ***and verify the Cauchy-Riemann equations.***

(2) When we need to prove that a function is entire (holomorphic on all of $\mathbb{C}$***).***

In such cases, we show that u, v ***satisfy the Cauchy-Riemann equations everywhere.***

(3) When we study the local geometric behavior of a complex function.

The quantities u_x, v_x, u_y, v_y ***describe how*** f ***transforms small neighborhoods (rotations, dilations, conformality, etc.). This relies directly on*** $f'(z) = u_x + iv_x$***.***

Method	Purpose	Advantage	When to Use It
Usual rules (direct)	Compute $f'(z)$ quickly for known holomorphic functions	Simple and fast	When $f(z)$ is expressed only in terms of z, e.g. $e^z, z^n, \sin z, \cos z$
Analytic definition $f'(z) = u_x + iv_x$	Check if $f(z)$ is holomorphic (Cauchy-Riemann test)	Fundamental for analytic verification	When f depends on $\overline{z}$, or when one must prove holomorphy

Remark 2.3 ***In short:***

The formula $f'(z) = u_x + iv_x$ ***is essential for proving holomorphy,***

while direct rules are used to compute derivatives of already known entire functions.

2.3.2 Harmonic Functions

Definition 2.4 ***A real-valued function $\varphi(x, y)$ of two real variables is said to be harmonic in a domain D if it possesses continuous first and second partial derivatives in D and satisfies the Laplace equation:***

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

2.3.3 Harmonic Functions and Their Conjugates

Theorem 2.10 ***Suppose the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is analytic on a domain D. Then, the real functions $u(x, y)$ and $v(x, y)$ are harmonic on D.

Example 2.9 ***For the entire function***

$$f(z) = z^2,$$

find $u(x, y)$ and $v(x, y)$, and show that they are harmonic on $\mathbb{C}$.

Solution 2.7 ***Let $z = x + iy$. Then:***

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Hence,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Compute the second partial derivatives:

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2,$$

$$\frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0.$$

Then:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0.$$

Thus, both u and v satisfy Laplace?s equation and are harmonic on $\mathbb{C}$.

Definition 2.5 (Conjugate Harmonic Function) ***Suppose the real function $u(x, y)$ is harmonic on a domain D. Then, it is possible to find another real harmonic function $v(x, y)$ such that u and v satisfy the Cauchy-Riemann equations.***

In this case, the complex function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic on D.

The function $v(x, y)$ is called the harmonic conjugate of $u(x, y)$.

Example 2.10 ***1. Verify that*** $u(x,y) = x^3 - 3xy^2 - 5y$ ***is harmonic.***

2. Find the harmonic conjugate $v(x,y)$ ***of*** $u(x,y)$***.***

Solution 2.8 ***1. Compute the second partial derivatives:***

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy - 5.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Then:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

so $u(x,y)$ ***is harmonic. ?***

2. To find $v(x,y)$***, use the Cauchy-Riemann equations:***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Substitute:

$$3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad -6xy - 5 = -\frac{\partial v}{\partial x}.$$

Hence,

$$\frac{\partial v}{\partial x} = 6xy + 5, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

Integrate the first equation with respect to x***:***

$$v(x,y) = 3x^2y + 5x + h(y),$$

where $h(y)$ ***is a function of*** y ***only.***

Differentiate with respect to y***:***

$$\frac{\partial v}{\partial y} = 3x^2 + h'(y).$$

Compare with $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$***:***

$$h'(y) = -3y^2 \quad \Rightarrow \quad h(y) = -y^3 + C.$$

Therefore:

$$v(x,y) = 3x^2y - y^3 + 5x + C.$$

Conclusion:

$$u(x,y) = x^3 - 3xy^2 - 5y, \quad v(x,y) = 3x^2y - y^3 + 5x + C,$$

and the analytic function is

$$f(z) = u + iv = (x^3 - 3xy^2 - 5y) + i(3x^2y - y^3 + 5x + C).$$

2.4 Solved Exercises

Exercise 2.1 ***Continuity of a Complex Function at a Point***

Let the complex function

$$f(z) = \frac{z^2 - 1}{z - 1}.$$

1. ***Determine the domain of definition of*** $f(z)$.
2. ***Study the continuity of*** f ***at the point*** $z_0 = 1$.
3. ***Determine whether*** f ***can be made continuous at*** $z_0 = 1$ ***by defining a suitable value*** $f(1)$.

Solution 2.9 ***1. Domain of definition.***

The function is given by:

$$f(z) = \frac{z^2 - 1}{z - 1}.$$

The denominator vanishes at $z = 1$, ***hence the function is not defined at this point.***

Therefore, the domain of definition is:

$$D = \mathbb{C} \setminus \{1\}.$$

2. Continuity at $z_0 = 1$.

To check continuity, we compute the limit of $f(z)$ ***as*** $z \to 1$.

We have:

$$f(z) = \frac{z^2 - 1}{z - 1} = \frac{(z-1)(z+1)}{z-1} = z + 1, \quad \textbf{\textit{for }} z \neq 1.$$

Then:

$$\lim_{z \to 1} f(z) = \lim_{z \to 1}(z + 1) = 2.$$

However, $f(1)$ ***is not defined.***

Thus, f ***is discontinuous at*** $z_0 = 1$ ***because it is not defined there.***

3. Possible extension of continuity.

If we define $f(1) = 2$, ***then for all*** $z \in \mathbb{C}$:

$$f(z) = \begin{cases} \dfrac{z^2 - 1}{z - 1}, & z \neq 1, \\ 2, & z = 1, \end{cases}$$

we obtain a continuous function on $\mathbb{C}$.

Indeed, the limit at $z_0 = 1$ ***equals the defined value:***

$$\lim_{z \to 1} f(z) = 2 = f(1).$$

Hence, the extended function is continuous on the whole complex plane.

Exercise 2.2 ***Study of Differentiability at a Point***
Let the complex function

$$f(z) = x^2 + iy^2, \quad \textbf{\textit{where}}\ z = x + iy.$$

1. ***Write*** $f(z)$ ***in the form*** $f(z) = u(x, y) + iv(x, y)$ ***and determine the functions*** $u(x, y)$ ***and*** $v(x, y)$***.***

2. ***Study whether*** f ***is differentiable at the point*** $z_0 = 0$ ***by using the definition:***

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

3. ***Verify differentiability at*** $z_0 = 0$ ***by applying the Cauchy-Riemann equations.***

4. ***Conclude whether*** f ***is analytic on any domain of*** $\mathbb{C}$***.***

Solution 2.10 ***1. We have:***

$$f(z) = x^2 + iy^2,$$

hence

$$u(x, y) = x^2, \qquad v(x, y) = y^2.$$

2. Differentiability by definition.
Let $z_0 = 0$***, and consider the limit***

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}.$$

Since $\Delta z = \Delta x + i\Delta y$***,***

$$f(\Delta z) = (\Delta x)^2 + i(\Delta y)^2.$$

Thus,

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2 + i(\Delta y)^2}{\Delta x + i\Delta y}.$$

We test the limit along two different paths:

- ***If we approach along the real axis:*** $\Delta y = 0 \Rightarrow \Delta z = \Delta x$***,***

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2}{\Delta x} = \Delta x \to 0.$$

- ***If we approach along the imaginary axis:*** $\Delta x = 0 \Rightarrow \Delta z = i\Delta y$***,***

$$\frac{f(\Delta z)}{\Delta z} = \frac{i(\Delta y)^2}{i\Delta y} = \Delta y \to 0.$$

Both give 0***. However, to be sure the limit exists, let us take the general path*** $\Delta y = k\Delta x$***:***

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2 + i(k\Delta x)^2}{\Delta x(1+ik)} = \Delta x\frac{1+ik^2}{1+ik}.$$

As $\Delta x \to 0$***, the expression tends to*** 0 ***independently of*** k***. Hence, the limit exists and***

$$f'(0) = 0.$$

Therefore, f ***is differentiable at*** $z_0 = 0$***.***

3. ***Using the Cauchy-Riemann equations.***

We have

$$u(x,y) = x^2, \quad v(x,y) = y^2.$$

Compute the partial derivatives:

$$u_x = 2x, \quad u_y = 0, \quad v_x = 0, \quad v_y = 2y.$$

At $z_0 = 0$ ***(i.e.*** $x = 0, y = 0$***):***

$$u_x = 0, \quad v_y = 0, \quad u_y = 0, \quad v_x = 0.$$

Then, at $z_0 = 0$***, the Cauchy-Riemann equations***

$$u_x = v_y \quad \textbf{and} \quad u_y = -v_x$$

are satisfied.

Thus, f ***is differentiable at*** $z_0 = 0$***, and its derivative is***

$$f'(0) = u_x + iv_x = 0 + i(0) = 0.$$

4. ***Analyticity on a domain.***

For f ***to be analytic, the Cauchy-Riemann equations must hold at every point.***

In general,

$$u_x = 2x, \quad v_y = 2y,$$

so $u_x = v_y$ ***only when*** $x = y$***. Therefore, the C-R equations are not satisfied everywhere in*** $\mathbb{C}$***.***

Hence f ***is not analytic on any domain of*** $\mathbb{C}$***, but it is differentiable only at the single point*** $z_0 = 0$***.***

Exercise 2.3 ***Let*** $u(x, y) = x^2 - y^2$***.***

1. ***Verify that*** $u(x, y)$ ***is harmonic.***
2. ***Find its harmonic conjugate*** $v(x, y)$***.***
3. ***Construct the analytic function*** $f(z) = u(x, y) + iv(x, y)$***.***

Harmonic and Analytic Functions

Exercise 2.4 ***Let*** $f(z) = z^3$***.***

1. ***Write*** f ***in terms of*** x, y***.***

$$z^3 = (x + iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

So

$$u(x, y) = x^3 - 3xy^2, \qquad v(x, y) = 3x^2y - y^3.$$

2. ***Show*** u ***and*** v ***are harmonic on*** $\mathbb{C}$***.***

Compute second partials for u***:***

$$u_{xx} = \frac{\partial^2}{\partial x^2}(x^3 - 3xy^2) = 6x, \qquad u_{yy} = \frac{\partial^2}{\partial y^2}(x^3 - 3xy^2) = -6x.$$

Hence

$$u_{xx} + u_{yy} = 6x - 6x = 0.$$

For v***:***

$$v_{xx} = \frac{\partial^2}{\partial x^2}(3x^2y - y^3) = 6xy, \qquad v_{yy} = \frac{\partial^2}{\partial y^2}(3x^2y - y^3) = -6xy,$$

so

$$v_{xx} + v_{yy} = 6xy - 6xy = 0.$$

Thus u ***and*** v ***satisfy Laplace's equation everywhere, so they are harmonic on*** $\mathbb{C}$***.***

Exercise 2.5 ***Let*** $u(x, y) = x^2 - y^2$***.***

1. ***Verify*** u ***is harmonic.***

$$u_{xx} = 2, \qquad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 2 - 2 = 0.$$

So u ***is harmonic.***

2. ***Find a harmonic conjugate*** $v(x, y)$.

We seek v ***such that the Cauchy-Riemann (C-R) equations hold:***

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x \Rightarrow v_x = 2y.$$

Integrate $v_y = 2x$ ***with respect to*** y***:***

$$v(x, y) = 2xy + h(x),$$

where $h(x)$ ***is an arbitrary function of*** x.

Differentiate this with respect to x***:***

$$v_x = 2y + h'(x),$$

but from C-R we need $v_x = 2y$. ***Hence*** $h'(x) = 0$ ***so*** h ***is constant. We may take constant*** 0 ***(it only changes*** f ***by an additive constant).***

Thus one harmonic conjugate is

$$v(x, y) = 2xy + C.$$

3. ***Construct the analytic function*** $f(z) = u + iv$.

Choosing $C = 0$,

$$f(z) = x^2 - y^2 + i(2xy) = (x + iy)^2 = z^2,$$

which is analytic (indeed entire).

Exercise 2.6 ***Let*** $u(x, y) = e^x \cos y$.

1. ***Prove that*** $u(x, y)$ ***is harmonic.***
2. ***Find its harmonic conjugate*** $v(x, y)$.
3. ***Write the corresponding analytic function*** $f(z)$.

Solution 2.11 ***Let*** $u(x, y) = e^x \cos y$.

1. ***Prove*** u ***is harmonic.***

Compute second partials:

$$u_x = e^x \cos y, \quad u_{xx} = e^x \cos y,$$

$$u_y = -e^x \sin y, \quad u_{yy} = -e^x \cos y.$$

Hence

$$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$$

So u ***is harmonic.***

2. ***Find a harmonic conjugate*** $v(x, y)$***.***

   We need v ***satisfying C-R:***

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin y = -v_x \Rightarrow v_x = e^x \sin y.$$

   Integrate $v_y = e^x \cos y$ ***with respect to*** y***:***

$$v(x, y) = e^x \sin y + h(x).$$

   Differentiate this with respect to x***:***

$$v_x = e^x \sin y + h'(x).$$

   Equate to required $v_x = e^x \sin y$ ***so*** $h'(x) = 0$***. Thus*** h ***is constant. Take*** $h \equiv 0$***. Therefore a harmonic conjugate is***

$$v(x, y) = e^x \sin y + C.$$

3. ***Write the analytic function*** $f(z)$***.***

   Choosing $C = 0$***,***

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy} = e^z,$$

   so $f(z) = e^z$ ***(analytic on*** $\mathbb{C}$***).***

Exercise 2.7 ***Let*** $f(z) = \dfrac{1}{z} = \frac{x-iy}{x^2+y^2}$***.***

1. ***Express*** $f(z) = u(x, y) + iv(x, y)$***.***
2. ***Verify that*** u ***and*** v ***satisfy the Cauchy-Riemann equations.***
3. ***Show that*** u ***and*** v ***are harmonic on*** $\mathbb{C} \setminus \{0\}$***.***

Exercise 2.8 ***Let*** $u(x, y) = x^3 - 3xy^2 - 5y$***.***

1. ***Verify that*** $u(x, y)$ ***is harmonic.***
2. ***Determine the harmonic conjugate*** $v(x, y)$***.***
3. ***Deduce the corresponding analytic function*** $f(z)$***.***

Solution 2.12 ***Let*** $f(z) = \dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2}$***.***

1. ***Express*** f ***as*** $u + iv$.

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

So

$$u(x,y) = \frac{x}{x^2+y^2}, \qquad v(x,y) = -\frac{y}{x^2+y^2},$$

defined for $(x,y) \neq (0,0)$, ***i.e. on*** $D = \mathbb{C} \setminus \{0\}$.

2. ***Verify Cauchy-Riemann equations.***

Compute partial derivatives (for $(x,y) \neq (0,0)$***):***

$$u_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2},$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}, \qquad v_y = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus

$$u_x = v_y, \qquad u_y = -v_x,$$

so the C-R equations hold on D.

3. ***Show*** u ***and*** v ***are harmonic on*** D.

Compute Laplacians (one gives the idea; the other is analogous):

For u***:***

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right), \quad u_{yy} = \frac{\partial}{\partial y}\left(\frac{-2xy}{(x^2+y^2)^2}\right).$$

After simplification (algebraic but straightforward), one finds

$$u_{xx} + u_{yy} = 0 \quad \textit{for } (x,y) \neq (0,0).$$

Likewise for v, $v_{xx} + v_{yy} = 0$. ***Hence*** u ***and*** v ***are harmonic on*** $D = \mathbb{C} \setminus \{0\}$.

(Alternatively: $1/z$ ***is analytic on*** $\mathbb{C} \setminus \{0\}$***; therefore its real and imaginary parts are harmonic there.)***

Exercise 2.9 ***Let*** $u(x,y) = x^3 - 3xy^2 - 5y$.

1. ***Verify*** u ***is harmonic.***

Compute second partials:

$$u_x = 3x^2 - 3y^2, \quad u_{xx} = 6x,$$

$$u_y = -6xy - 5, \quad u_{yy} = -6x.$$

Thus

$$u_{xx} + u_{yy} = 6x - 6x = 0,$$

so u ***is harmonic.***

2. ***Determine the harmonic conjugate*** $v(x, y)$***.***

Use C-R:

$$u_x = 3x^2 - 3y^2 = v_y, \qquad u_y = -6xy - 5 = -v_x \Rightarrow v_x = 6xy + 5.$$

Integrate v_x ***w.r.t.*** x***:***

$$v(x, y) = 3x^2y + 5x + h(y),$$

where $h(y)$ ***is a function of*** y ***only.***

Differentiate w.r.t. y***:***

$$v_y = 3x^2 + h'(y).$$

Compare with required $v_y = 3x^2 - 3y^2$ ***to get***

$$h'(y) = -3y^2 \quad \Rightarrow \quad h(y) = -y^3 + C.$$

Therefore a harmonic conjugate is

$$v(x, y) = 3x^2y - y^3 + 5x + C,$$

where C ***is an arbitrary real constant.***

3. ***Deduce the analytic function*** $f(z) = u + iv$***.***

Taking $C = 0$ ***for simplicity,***

$$f(z) = \Big(x^3 - 3xy^2 - 5y\Big) + i\Big(3x^2y - y^3 + 5x\Big).$$

One may check this f ***is analytic on the domain where*** u ***is defined (all*** $\mathbb{C}$***).***

Remark 2.4 ***In each exercise the harmonic conjugate is determined up to an additive real constant. When constructing the analytic function*** $f(z) = u + iv$***, that constant adds an imaginary constant*** iC ***to*** f***, which does not affect analyticity.***