Chapter 1

Generalities on Complex Numbers

1.1 General Review of Complex Numbers

1.1.1 Properties of Complex Numbers

We define the imaginary unit i by the property

$$i^2 = -1.$$

This definition allowed mathematicians to propose **imaginary solutions** to quadratic equations with a negative discriminant, that is, equations of the form

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

Recall that the discriminant of a quadratic equation is given by

$$\Delta = b^2 - 4ac.$$

Definition 1.1 (Complex Number) *A **complex number** is any number that can be written in the form*

$$z = a + ib,$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit defined by $i^2 = -1$.

We denote

$$a = \Re(z) \quad \textit{(the real part of } z\textit{)}, \qquad b = \Im(z) \quad \textit{(the imaginary part of } z\textit{)}.$$

Remark 1.1 *Any multiple of the imaginary unit i is called a **purely imaginary number**.*

Definition 1.2 (Zero Complex Number) *The **zero complex number** is defined as*

$$0 + i0.$$

It is usually denoted simply by 0, and it is the additive identity in $\mathbb{C}$, that is,

$$z + 0 = z, \quad \forall z \in \mathbb{C}.$$

Definition 1.3 (Conjugate of a Complex Number) *Let $z = a + ib \in \mathbb{C}$, with $a, b \in \mathbb{R}$. The **conjugate** of z is defined as*

$$\overline{z} = \overline{(a + ib)} = a - ib.$$

Note that

$$\overline{\overline{z}} = z.$$

Definition 1.4 (Equality) *Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal if and only if*

$$a_1 = \Re(z_1) = a_2 = \Re(z_2) \quad \text{and} \quad b_1 = \Im(z_1) = b_2 = \Im(z_2).$$

Definition 1.5 (Arithmetic Operations) *Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. The following operations are defined:*

- ***Addition:***

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$

$$z + \overline{z} = 2\Re(z)$$

- ***Subtraction:***

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2).$$

$$z - \overline{z} = 2i\Im(z)$$

- ***Multiplication:***

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

$$z \cdot \overline{z} = |z|^2 = a^2 + b^2$$

.

Exercise 1.1 *Let $z = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Calculate the product $z \cdot \overline{z}$.*

Solution. *We have*

$$z \cdot \overline{z} = (a + ib)(a - ib) = a^2 - iab + iab - i^2b^2.$$

Since $i^2 = -1$, this simplifies to

$$z \cdot \overline{z} = a^2 + b^2.$$

Therefore,

$$z \cdot \overline{z} = |z|^2.$$

- ***Division:***

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z_2}}{z_2\,\overline{z_2}}$$

$$\frac{z_1}{z_2} = \frac{(a_1 + ib_1)\,(a_2 - ib_2)}{(a_2 + ib_2)\,(a_2 - ib_2)} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}, \quad (a_2, b_2) \neq (0, 0).$$

Remark 1.2 *These operations satisfy the usual algebraic rules: commutativity, associativity, and distributivity.*

- **Distributive Properties of Conjugation:**
  For all $z_1, z_2 \in \mathbb{C}$, the conjugation satisfies:

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2},$$

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2},$$

$$\overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2},$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \quad z_2 \neq 0,$$

Example 1.1 *Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$. Then:*

$$z_1 + z_2 = (3 + 1) + (4 - 2)i = 4 + 2i,$$

$$z_1 \cdot z_2 = (3 \cdot 1 - 4 \cdot (-2)) + i(3 \cdot (-2) + 1 \cdot 4) = 11 - 2i,$$
$$\frac{z_1}{z_2} = \frac{3 + 4i}{1 - 2i} = \frac{3 + 4i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{11 + 2i}{5} = \frac{11}{5} + \frac{2}{5}i.$$

Example 1.2 *Let*

$$z_1 = 3 + 4i, \qquad z_2 = 1 - 2i.$$

$$z_1 + z_2 = (3 + 4i) + (1 - 2i) = 4 + 2i, \qquad \overline{z_1 + z_2} = 4 - 2i,$$
$$\overline{z_1} = 3 - 4i, \qquad \overline{z_2} = 1 + 2i, \qquad \overline{z_1} + \overline{z_2} = (3 - 4i) + (1 + 2i) = 4 - 2i.$$

Hence

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}.$$

For subtraction:

$$z_1 - z_2 = (3 + 4i) - (1 - 2i) = 2 + 6i, \qquad \overline{z_1 - z_2} = 2 - 6i,$$

$$\overline{z_1} - \overline{z_2} = (3 - 4i) - (1 + 2i) = 2 - 6i.$$

Thus

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}.$$

For multiplication:

$$z_1 z_2 = (3+4i)(1-2i) = 3 - 6i + 4i - 8i^2 = 11 - 2i, \qquad \overline{z_1 z_2} = 11 + 2i,$$

$$\overline{z_1}\,\overline{z_2} = (3-4i)(1+2i) = 3 + 6i - 4i - 8i^2 = 11 + 2i.$$

Therefore

$$\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}.$$

For division (use conjugates to rationalize):

$$\frac{z_1}{z_2} = \frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1 + 2i,$$

so

$$\overline{\frac{z_1}{z_2}} = \overline{-1+2i} = -1 - 2i.$$

On the other hand

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{3-4i}{1+2i} = \frac{(3-4i)(1-2i)}{(1+2i)(1-2i)} = \frac{-5-10i}{5} = -1 - 2i.$$

Hence

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}.$$

Finally, double conjugation:

$$\overline{\overline{z_1}} = \overline{3-4i} = 3 + 4i = z_1, \qquad \overline{\overline{z_2}} = \overline{1+2i} = 1 - 2i = z_2.$$

So $\overline{\overline{z}} = z$ is verified.

Definition 1.6 (Additive Inverse) *In the system of complex numbers, every number $z \in \mathbb{C}$ has a unique additive inverse z' such that*

$$z + z' = 0.$$

This inverse is given by

$$z' = -z.$$

Definition 1.7 (Multiplicative Inverse) *In the system of complex numbers, every nonzero number $z \in \mathbb{C}$ has a unique multiplicative inverse z'' such that*

$$z \cdot z'' = 1.$$

This inverse is given by

$$z'' = \frac{1}{z} = z^{-1}.$$

The number z^{-1} is also called the reciprocal of z.

Application Compute the following powers of i:

$$i^3,\ i^4,\ i^5,\ i^6,\ i^7,\ i^8,\ i^9,\ i^{10},\ i^{11},\ i^{12}.$$

Then, deduce a general rule for i^n depending on the parity of n.

Solution 1.1 ***Recall that***

$$i^1 = i, \qquad i^2 = -1.$$

Step 1: Compute the powers successively.

$$i^3 = i^2 \cdot i = -1 \cdot i = -i,$$

$$\begin{aligned}
i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1, \\
i^5 &= i^4 \cdot i = 1 \cdot i = i, \\
i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1, \\
i^7 &= i^6 \cdot i = (-1) \cdot i = -i, \\
i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1, \\
i^9 &= i^8 \cdot i = 1 \cdot i = i, \\
i^{10} &= i^9 \cdot i = i \cdot i = i^2 = -1, \\
i^{11} &= i^{10} \cdot i = (-1) \cdot i = -i, \\
i^{12} &= i^8 \cdot i^4 = 1 \cdot 1 = 1.
\end{aligned}$$

Step 2: Observe the pattern.
The powers of i repeat every four steps:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1.$$

Step 3: General rule. For any integer $n \geq 1$, there exists an integer $k \in \mathbb{N}$ such that:

$$i^n = \begin{cases} 1, & \textbf{\textit{if }} n = 4k, \\ i, & \textbf{\textit{if }} n = 4k+1, \\ -1, & \textbf{\textit{if }} n = 4k+2, \\ -i, & \textbf{\textit{if }} n = 4k+3. \end{cases}$$

1.1.2 The Complex Plane

Definition 1.8 (Complex Plane) ***Since every complex number z can be represented by a pair of real numbers (a, b), we use a plane to represent complex numbers.***

This plane is defined by two perpendicular axes:

- ***the real axis, denoted by $\Re$ or x-axis,***
- ***the imaginary axis, denoted by $\Im$ or y-axis.***

Thus, any complex variable can be written as

$$z = x + iy, \qquad x, y \in \mathbb{R},$$

where $x = \Re(z)$ ***and*** $y = \Im(z)$***.***

We can also interpret each pair of real numbers (x, y) as the components of a 2D vector. Thus, a complex number

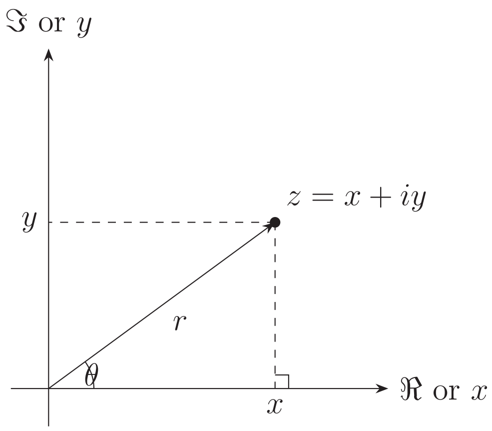
$$z = x + iy$$

can also be seen as a two-dimensional vector whose initial point is at the origin of the axes, and whose terminal point is the point (x, y).

Definition 1.9 ***The vector representation of a complex number*** $z = x + iy$ ***is the directed vector***

$$\vec{z} = (x, y),$$

where $x = \Re(z)$ ***is the projection on the real axis, and*** $y = \Im(z)$ ***is the projection on the imaginary axis.***



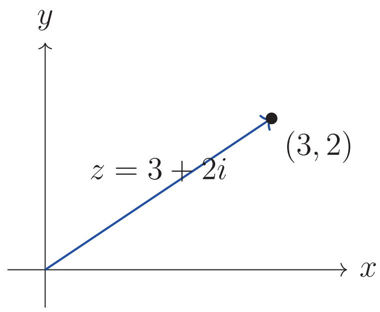
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right)$$

Figure 1.1: Complex plane (Argand plane): representation of $z = x + iy$, modulus r and argument θ.

Example 1.3 ***Let*** $z = 3 + 2i$***. Its vector representation is***

$$\vec{z} = (3, 2),$$

which is a vector starting from the origin $(0, 0)$ ***and ending at the point*** $(3, 2)$ ***in the plane.***



Definition 1.10 (Modulus of a Complex Number) ***Let*** $z = x + iy$ ***with*** $x, y \in \mathbb{R}$***. The modulus of*** z ***is the positive real number defined by***

$$|z| = \sqrt{x^2 + y^2}.$$

Proposition 1.1 (Properties of the Modulus) ***For any*** $z, z_1, z_2 \in \mathbb{C}$ ***with*** $z_2 \neq 0$***, the modulus satisfies the following properties:***

- $|z|^2 = z\,\overline{z}$,
- $|z| = \sqrt{z\,\overline{z}}$,
- $|z_1 z_2| = |z_1|\,|z_2|$,
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$,
- $|z^2| = |z|^2$.

Definition 1.11 (Distance Between Two Complex Numbers) ***Let*** $z_1 = x_1 + iy_1$ ***and*** $z_2 = x_2 + iy_2$ ***be two complex numbers. The distance between*** z_1 ***and*** z_2 ***in the complex plane is defined as***

$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

1.1.3 Polar and Exponential Form of Complex Numbers

Polar Form

From a simple geometric viewpoint, the components x and y of a complex number

$$z = x + iy$$

can be expressed in terms of the polar parameters r and θ as

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad \Longrightarrow \qquad z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

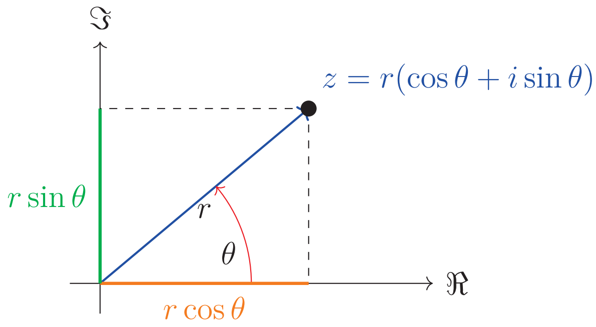
Here:

$$r = |z| = \sqrt{x^2 + y^2} \in \mathbb{R}^+, \qquad -\pi \leq \theta < \pi.$$

The angle θ of the vector z is measured in radians, positive in the counterclockwise direction. This angle θ is called the argument of z and is denoted by

$$\theta = \arg(z).$$

Remark 1.3 ***The argument*** $\arg(z)$ ***is not unique because of the*** 2π***-periodicity of the trigonometric functions*** $\cos\theta$ ***and*** $\sin\theta$***.***



Exercise 1.2 ***Write the complex number***

$$z = -\sqrt{3} - i$$

in polar form.

Solution 1.2 ***We are given the complex number***

$$z = -\sqrt{3} - i.$$

Step 1: Compute the modulus r***.***

$$r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2.$$

Step 2: Determine the argument θ***. We know that***

$$\tan\theta = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Thus, the reference angle is

$$\theta_0 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Since $x < 0$ ***and*** $y < 0$***, the point lies in the third quadrant. Therefore,***

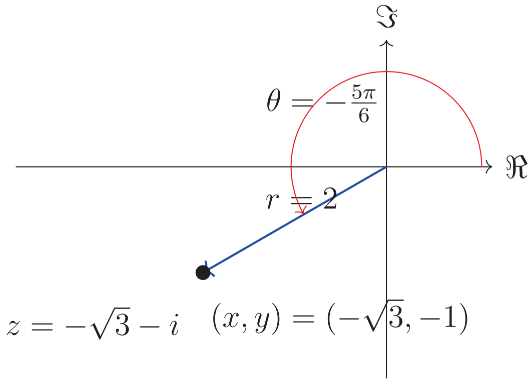
$$\theta = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}.$$

Step 3: Write z ***in polar form.***

$$z = r(\cos\theta + i\sin\theta) = 2\Big(\cos\Big(-\tfrac{5\pi}{6}\Big) + i\sin\Big(-\tfrac{5\pi}{6}\Big)\Big).$$

Final Answer:

$$z = 2\left(\cos\left(-\tfrac{5\pi}{6}\right) + i\sin\left(-\tfrac{5\pi}{6}\right)\right).$$



1. Principal Argument

Definition 1.12 (Argument and Principal Argument) ***Let*** $z \in \mathbb{C} \setminus \{0\}$ ***and write*** $z = r(\cos\theta + i\sin\theta)$ ***with*** $r > 0$ ***and*** $\theta \in \mathbb{R}$***. The set of all possible values of*** θ ***is called the*** **argument** ***of*** z ***and is denoted by*** $\arg(z)$***.***

The ***principal argument*** $\mathrm{Arg}(z)$ ***is the unique argument chosen in the interval***

$$-\pi \leq \mathrm{Arg}(z) < \pi.$$

Remark 1.4 ***Because*** $\cos(\theta)$ ***and*** $\sin(\theta)$ ***are*** 2π***-periodic, the argument is not unique: if*** θ ***is an argument of*** z ***then*** $\theta + 2k\pi$ ***is also an argument for every*** $k \in \mathbb{Z}$***. The principal argument selects the representative in*** $[-\pi, \pi)$***.***

2.Multiplication and Division in Polar Form

Let

$$z_1 = r_1\Big(\cos\theta_1 + i\sin\theta_1\Big), \qquad z_2 = r_2\Big(\cos\theta_2 + i\sin\theta_2\Big),$$

with $r_1, r_2 > 0$**.**

Proposition 1.2 (Product and Quotient in Polar Form) ***The product and the quotient are given by***

$$z_1 z_2 = r_1 r_2\Big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big),$$

and, for $z_2 \neq 0$***,***

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big).$$

Equivalently,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2),$$

up to addition of integer multiples of 2π ***(and taking principal arguments when needed).***

3. Integer Powers and De Moivre's Formula

Proposition 1.3 (Integer Powers) ***Let*** $z = r(\cos\theta + i\sin\theta)$ ***and*** $n \in \mathbb{N}$***. Then***

$$z^n = r^n\Big(\cos(n\theta) + i\sin(n\theta)\Big).$$

In particular, if $r = 1$ ***then the classical De Moivre formula holds:***

$$\Big(\cos\theta + i\sin\theta\Big)^n = \cos(n\theta) + i\sin(n\theta).$$

Remark 1.5 ***The formula extends to*** $n \in \mathbb{Z}$ ***(negative integers) by using*** $z^{-n} = (z^{-1})^n$ ***and the polar expression of*** z^{-1}***.***

4. Nth Roots of a Complex Number

Let $z = r(\cos\theta + i\sin\theta)$ **with** $r > 0$**. We seek all** w **such that** $w^n = z$**.**

Proposition 1.4 (Formula for the n**th Roots)** ***All solutions*** w ***of*** $w^n = z$ ***are given by***

$$w_k = \rho\Big(\cos\varphi_k + i\sin\varphi_k\Big), \qquad k = 0, 1, \ldots, n-1,$$

where

$$\rho = r^{1/n}, \qquad \varphi_k = \frac{\theta + 2k\pi}{n}.$$

Thus there are exactly n ***distinct*** n***th roots:***

$$w_k = r^{1/n}\left(\cos\Big(\frac{\theta + 2k\pi}{n}\Big) + i\sin\Big(\frac{\theta + 2k\pi}{n}\Big)\right), \quad k = 0, \ldots, n-1.$$

The root corresponding to $k = 0$ ***(with argument taken in the principal branch) is called the*** principal nth root ***of*** z***.***

Exercise 1.3 ***Find all fourth roots of***

$$1 + i.$$

(That is, compute w ***such that*** $w^4 = 1 + i$***.)***

Solution 1.3 ***We seek all complex numbers*** w ***such that***

$$w^4 = 1 + i.$$

Step 1: Put the right-hand side in polar form. ***Write*** $1 + i$ ***in polar (exponential) form. Its modulus is***

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

and a principal argument is

$$\mathrm{Arg}(1 + i) = \frac{\pi}{4}.$$

Hence

$$1 + i = \sqrt{2}\, e^{i\pi/4}.$$

Step 2: Apply the formula for n-th roots. If $w^4 = 1+i$ and we set $w = \rho e^{i\varphi}$, then

$$\rho^4 = \sqrt{2}, \qquad 4\varphi = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}.$$

Thus

$$\rho = (\sqrt{2})^{1/4} = 2^{1/8}, \qquad \varphi = \frac{\pi/4 + 2k\pi}{4} = \frac{\pi}{16} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

Step 3: List the four distinct roots. Taking $k = 0, 1, 2, 3$ gives the four distinct fourth roots:

$$w_k = 2^{1/8}\, e^{i\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)} = 2^{1/8}\left(\cos\left(\frac{\pi}{16}+\frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)\right), \quad k = 0, 1, 2, 3.$$

Remark (principal root). The principal fourth root corresponds to $k = 0$:

$$w_0 = 2^{1/8}\, e^{i\pi/16}.$$

Check. For any k,

$$w_k^4 = \left(2^{1/8}\right)^4 e^{i\left(4(\pi/16+k\pi/2)\right)} = 2^{1/2} e^{i(\pi/4+2k\pi)} = \sqrt{2}\, e^{i\pi/4} = 1+i,$$

as required.

Exponential Form and Euler's Formula

Definition 1.13 (Exponential Form) ***Using Euler's formula (see below), any complex number can be written in exponential form as***

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg(z)$ (or $\mathrm{Arg}(z)$ for the principal value).

Proposition 1.5 (Euler's Formula) ***For every real θ,***

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Remark 1.6 ***From Euler's formula the polar and exponential forms are equivalent:***

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Multiplication then becomes particularly simple:

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)},$$

and similarly for powers and roots. Endeed, $w = \rho(\cos\varphi + i\sin\varphi) = \rho e^{i\varphi}$ and $w^n = \rho^n e^{in\varphi}$. Hence, when $w^n = z = re^{i\theta}$ identify moduli $\rho^n = r$ and arguments $n\varphi = \theta + 2k\pi$, then solve for ρ and φ.

1.1.4 Application: Quadratic Formula in Complex Numbers

Consider a quadratic equation with complex coefficients:

$$az^2 + bz + c = 0, \quad a \neq 0.$$

We can solve it in a manner similar to real numbers. First, compute the discriminant:

$$\Delta = b^2 - 4ac.$$

In general, the roots are given by:

$$z_i = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad i = 1, 2.$$

Exercise 1.4 ***Solve the following quadratic equations in the set of complex numbers:***

1. $z^2 + 4z + 5 = 0$
2. $z^2 + (1 - i)z - 3i = 0$

Solution 1.4 ***1. Solve*** $z^2 + 4z + 5 = 0$***:***
The discriminant is

$$\Delta = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4.$$

Since $\Delta < 0$***, the roots are complex:***

$$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

So the solutions are:

$$\boxed{z_1 = -2 + i, \quad z_2 = -2 - i}.$$

2. Solve $z^2 + (1 - i)z - 3i = 0$***:***
The discriminant is

$$\Delta = b^2 - 4ac = (1 - i)^2 - 4 \cdot 1 \cdot (-3i).$$

Compute $(1 - i)^2$***:***

$$(1 - i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i.$$

Then:

$$\Delta = -2i - 4(-3i) = -2i + 12i = 10i.$$

Now, compute the square root of $10i$***:***

$$10i = 10\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \implies \sqrt{10i} = \sqrt{10}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{5}(1 + i).$$

The roots are:

$$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(1-i) \pm \sqrt{5}(1+i)}{2} = \frac{-1+i \pm \sqrt{5}(1+i)}{2}.$$

Separating the two roots:

$$z_1 = \frac{-1+\sqrt{5}}{2} + i\frac{1+\sqrt{5}}{2}, \qquad z_2 = \frac{-1-\sqrt{5}}{2} + i\frac{1-\sqrt{5}}{2}.$$

So the solutions are:

$$\boxed{z_1 = \frac{-1+\sqrt{5}}{2} + i\frac{1+\sqrt{5}}{2}, \qquad z_2 = \frac{-1-\sqrt{5}}{2} + i\frac{1-\sqrt{5}}{2}}.$$

1.1.5 Sets of Points in the Complex Plane

In this section, we introduce the essential definitions and terminology for sets of points in the complex plane.

1. The Circle

Let

$$z_0 = x_0 + iy_0 \in \mathbb{C}.$$

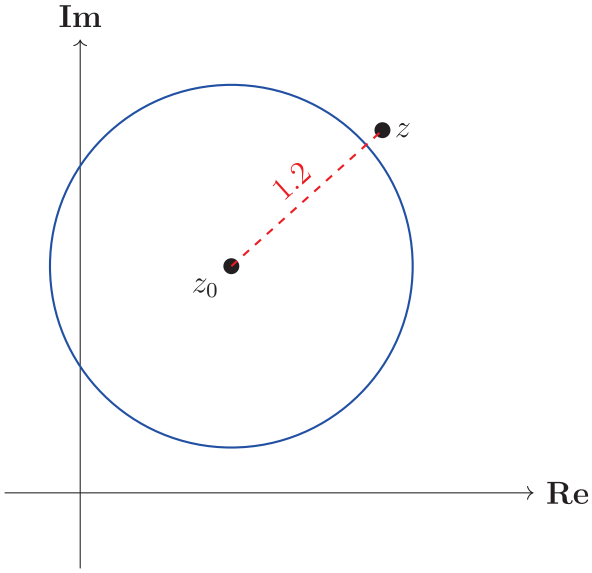
Recall that the distance between any point $z = x + iy$ **and** z_0 **is**

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Definition 1.14 (Circle in the Complex Plane) ***The set of all points*** z ***that satisfy***

$$|z - z_0| = \rho, \qquad \rho > 0,$$

is called a circle with center z_0 ***and radius*** ρ***. In other words, it is the set of all points at a fixed distance*** ρ ***from*** z_0***.***



Center: $z_0 = x_0 + iy_0$, **Radius:** $|z - z_0| = 1.2$

2. Disk and Neighborhood

Definition 1.15 (Disk) ***Let*** $z_0 \in \mathbb{C}$ ***and*** $\rho > 0$***. The disk of radius*** ρ ***centered at*** z_0 ***is the set of all points*** z ***satisfying***

$$|z - z_0| \leq \rho.$$

This includes all points on the circle $|z - z_0| = \rho$ ***as well as all points inside the circle.***

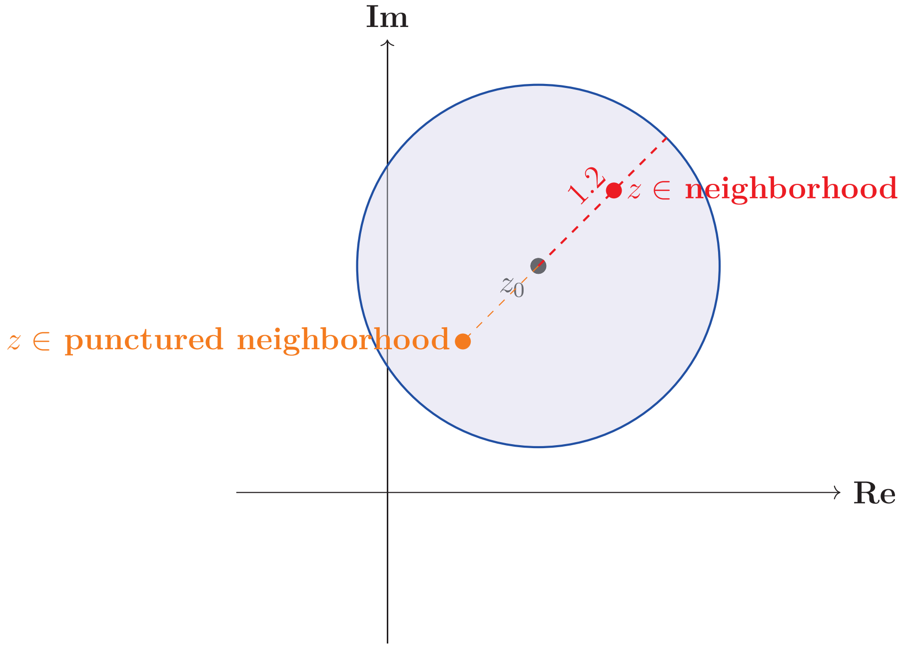
Definition 1.16 (Neighborhood of a Point) ***Let*** $z_0 \in \mathbb{C}$ ***and*** $\rho > 0$***. The neighborhood of*** z_0 ***or open disk with radius*** ρ ***and center*** z_0 ***is the set of points***

$$|z - z_0| < \rho.$$

These points lie strictly inside the disk, excluding the boundary circle.

A punctured neighborhood (or deleted neighborhood) excludes the center itself and is defined by the strict inequalities

$$0 < |z - z_0| < \rho.$$



Disk: $|z - z_0| \leq 1.2$, **Neighborhood:** $|z - z_0| < 1.2$, **Punctured Neighborhood:** $0 < |z - z_0| < 1.2$

Remark 1.7 *"Punctured neighborhood" mean neighborhood with the center punctured/excluded.*

3. Open Sets, Interior, Boundary, and Exterior Points

Definition 1.17 (Interior Point) *A point $z \in \mathbb{C}$ is called an interior point of a set S in the complex plane if there exists a neighborhood of z that is entirely contained in S.*

Definition 1.18 (Open Set) *A set S is called open if every point of S is an interior point.*

Definition 1.19 (Limit Point and Boundary) *A point z_0 is called a limit point of a set S if every neighborhood of z_0 contains at least one point in S and at least one point outside S.*

The set of all limit points of S forms the boundary of S, denoted by ∂S.

Definition 1.20 (Exterior Point) *A point $z \in \mathbb{C}$ is called an exterior point of S if it is neither an interior point nor a boundary point of S.*

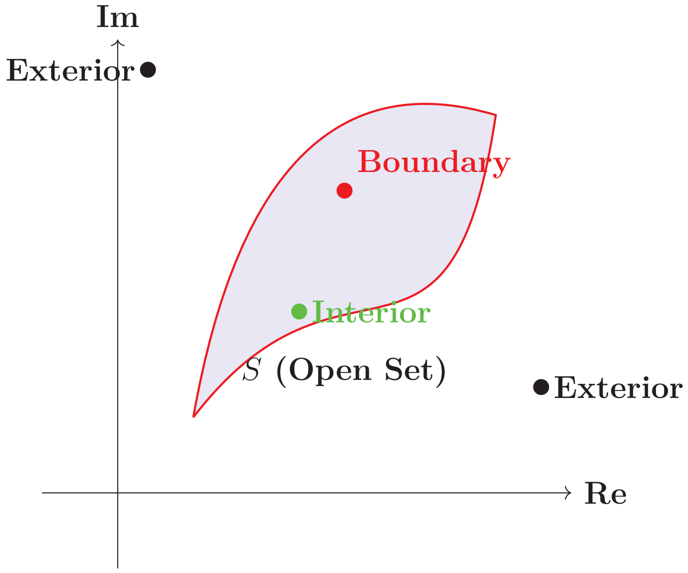


Figure: Open set S, interior points (green), boundary points (red), exterior points (black).

4. Annulus (Circular Ring)

Definition 1.21 (Annulus) *Consider two sets of points around a center $z_0 \in \mathbb{C}$:*

- $S_1 = \{z \in \mathbb{C} \mid |z - z_0| > \rho_1\}$, *representing points outside the circle of radius ρ_1 centered at z_0.*
- $S_2 = \{z \in \mathbb{C} \mid |z - z_0| < \rho_2\}$, *representing points inside the circle of radius ρ_2 centered at z_0.*

If $0 < \rho_1 < \rho_2$, then the set of points satisfying the strict inequalities simultaneously,

$$\rho_1 < |z - z_0| < \rho_2,$$

is the intersection of S_1 and S_2, and it is called a circular annulus.

Remark 1.8 *If we take $\rho_1 = 0$, we recover the definition of an open disk with the center excluded, i.e., a punctured neighborhood.*

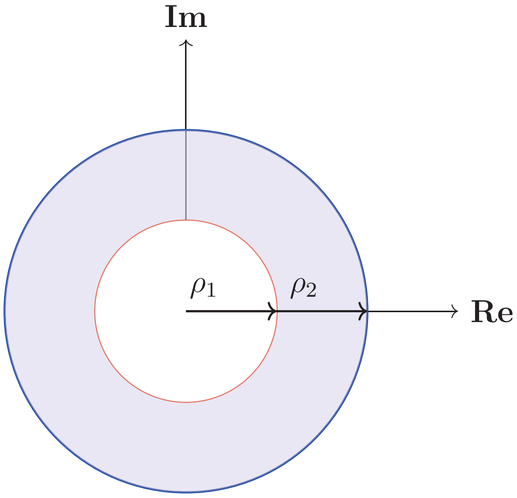
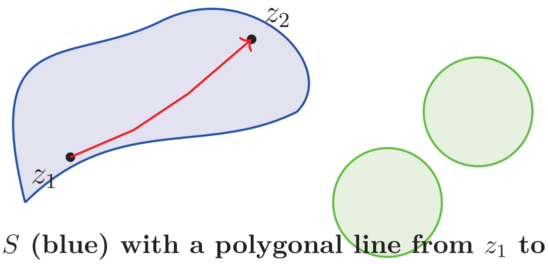


Figure: Circular annulus (blue area) with inner radius ρ_1 (red) and outer radius ρ_2 (blue), centered at z_0.

5. Domain

Definition 1.22 (Domain) ***Let S be a set of points in the complex plane. If any two points $z_1, z_2 \in S$ can be connected by a polygonal line composed of a finite sequence of line segments joined end-to-end, and this line lies entirely within S, then the set S is said to be connected.***

An open connected set is called a domain.



Connected domain S (blue) with a polygonal line from z_1 to z_2

Non-connected set (green) example

Region

Definition 1.23 (Region) ***A region in the complex plane is any set of points that may include all, some, or none of its boundary points.***

According to this definition:

- ***An open set, which contains none of its boundary points, is a region.***
- ***A closed region contains all its boundary points. For example, the closed disk***

  $$|z - z_0| \leq \rho$$

  is a closed region.

- ***An open region such as the disk***

$$|z - z_0| < \rho$$

is called an open disk.

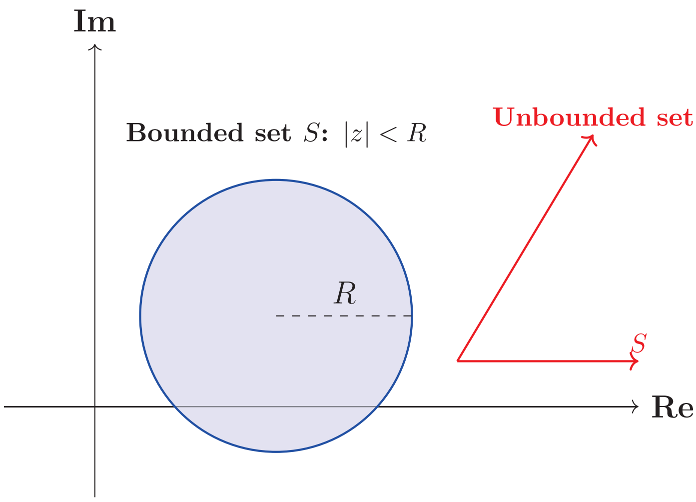
- ***If the center z_0 is excluded, the set is called a punctured region or punctured disk.***

Bounded Set

Definition 1.24 (Bounded Set) ***A set S in the complex plane is said to be bounded if there exists a real number $R > 0$ such that***

$$|z| < R \quad \textit{for all } z \in S.$$

Otherwise, the set is called unbounded.



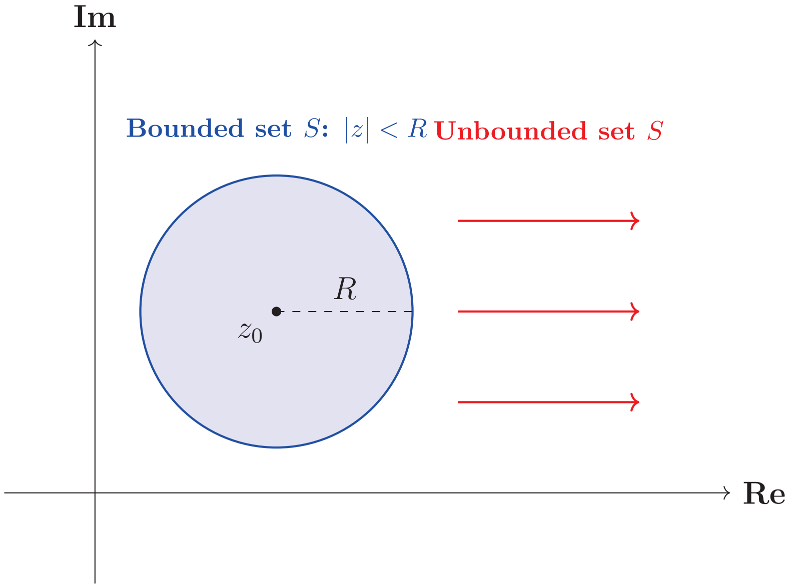
In the figure above:

- The blue disk represents a bounded set S. Its center is $z_0 = 2 + i$ and all points inside satisfy

$$|z - z_0| < R.$$

The dashed line shows the radius R from the center z_0 to a point on the boundary. All points of this set are within a finite distance from z_0.

- The arrows indicate an unbounded set S. They illustrate that the set extends beyond the visible portion of the plane. Since the points can continue indefinitely in some directions, there is no finite R that contains all points, making it unbounded.



In the figure above:

- The blue disk represents a bounded set S. Its center is z_0 and all points inside satisfy $|z - z_0| < R$, so there exists a finite radius R containing all points.
- The red arrows represent an unbounded set S. The arrows indicate that the points in this set can extend indefinitely in the complex plane. Therefore, no finite R can contain all points, making the set unbounded.
- The dashed line in the blue disk shows the radius R from the center z_0 to a point on the boundary.

1.2 Complex Functions, Limits, Continuity, and the Complex Infinity

1.2.1 Definition of a Complex Function

Definition 1.25 ***A complex function f is a mapping from a subset D of the complex plane $\mathbb{C}$ to $\mathbb{C}$:***

$$f : D \subset \mathbb{C} \to \mathbb{C}, \quad z \mapsto f(z),$$

where $z = x + iy \in D$ and

$$f(z) = u(x, y) + iv(x, y), \quad \textit{with } u(x, y), v(x, y) \in \mathbb{R}.$$

-The set D is called the domain of definition of f.
- $u(x, y)$ is the real part of $f(z)$
- $v(x, y)$ is the imaginary part of $f(z)$
- $f(z) = u(x, y) + iv(x, y)$ is the Algebric form of $f(z)$.

Example 1.4 *Consider the function*

$$f(z) = z^2, \quad z \in \mathbb{C}.$$

Its domain of definition is the entire complex plane $\mathbb{C}$. If $z = x + iy$, then

$$f(z) = (x + iy)^2 = x^2 - y^2 + i(2xy),$$

where

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Example 1.5 *Consider the function*

$$f(z) = \frac{1}{z(z^2 + 1)}.$$

This function is not defined for the values of z that make the denominator zero. Solving

$$z(z^2 + 1) = 0 \quad \Longrightarrow \quad z = 0,\ z = i,\ z = -i.$$

Hence, the domain of definition of f is

$$D = \mathbb{C} \setminus \{0,\ i,\ -i\}.$$

1.2.2 Uniform and Multiform Complex Functions

Definition 1.26 ***In the strict mathematical sense, a function f defined on a domain $D \subset \mathbb{C}$ is a mapping that assigns to each point $z \in D$ one and only one value $w = f(z) \in \mathbb{C}$.***

A complex function is said to be uniform (or single-valued) if it satisfies this property throughout its domain.

However, in complex analysis, we sometimes encounter expressions such as $\sqrt{z}$, $\log z$, or $z^{1/n}$, which can take several possible values for the same z. These are not functions in the strict sense, but are called multiform functions (or multi-valued functions). Each distinct determination (or branch) of such an expression defines a true single-valued function on a suitable subset of $\mathbb{C}$.

Example 1.6 (Uniform function) *The function*

$$f(z) = z^2$$

is uniform, because for each $z \in \mathbb{C}$ there exists exactly one value of $f(z)$.

Example 1.7 (Multiform expression) *The expression*

$$f(z) = \sqrt{z}$$

is multiform because for each $z \neq 0$, there are two possible values:

$$\sqrt{z} = \pm\sqrt{r}\, e^{i\theta/2}, \quad \textit{where } z = re^{i\theta}.$$

To obtain a true function, one must choose a branch - for example, by restricting the argument to $-\pi < \theta \leq \pi$, defining the principal branch of $\sqrt{z}$.

1.2.3 Limit of a Complex Function

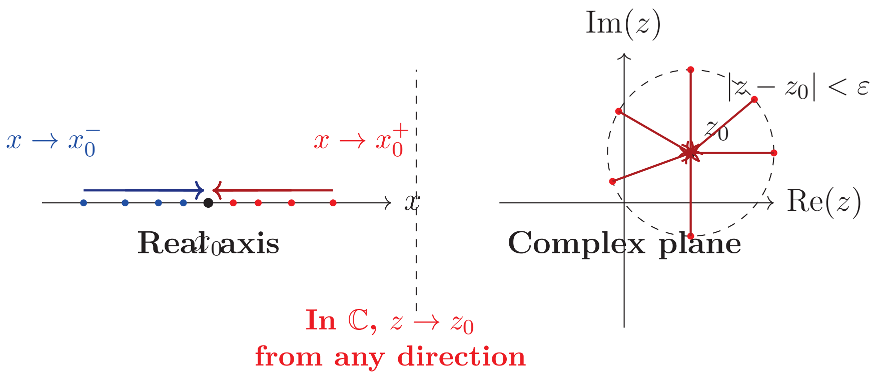
Definition 1.27 *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a complex function, and let z_0 be a limit point of D. We say that $f(z)$ tends to a limit $w_0 \in \mathbb{C}$ when $z \to z_0$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon.$$

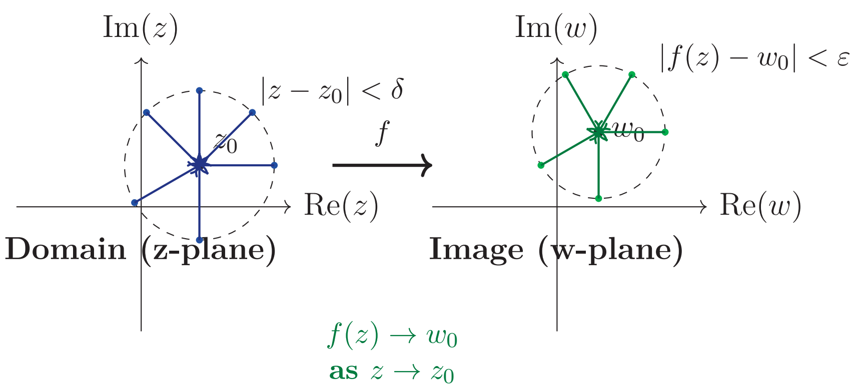
Symbolically, we write

$$\lim_{z \to z_0} f(z) = w_0.$$

Remark 1.9 *Geometrically, this means that when the point z approaches z_0 from any direction in the complex plane, the image point $f(z)$ approaches w_0 in the image plane. This definition extends the concept of limit from real functions (where z and $f(z)$ lie on $\mathbb{R}$) to complex functions (where both belong to $\mathbb{C}$).*



Approach of the variable: real vs. complex



Meaning of $\lim_{z \to z_0} f(z) = w_0$

Example 1.8

$$f(z) = z^2, \quad z_0 = 1 + i$$

$$\lim_{z \to 1+i} f(z) = (1 + i)^2 = 2i.$$

Criterion for the Non-Existence of a Limit

If a function f approaches two distinct complex numbers $L_1 \neq L_2$ along two different paths passing through z_0, then

$$\lim_{z \to z_0} f(z)$$

does not exist.

Example 1.9 ***Show that*** $\lim_{z\to 0} \frac{z}{\bar{z}}$ ***does not exist.***

Solution 1.5 ***Let us examine the limit along two different paths approaching the origin.***

1. Along the real axis: ***If*** $z = x$ ***(with*** $y = 0$***), then***

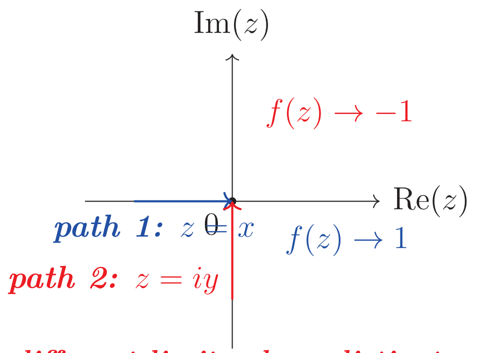
$$\frac{z}{\bar{z}} = \frac{x}{x} = 1 \quad \Rightarrow \quad \lim_{x\to 0} \frac{z}{\bar{z}} = 1$$

2. Along the imaginary axis: ***If*** $z = iy$ ***(with*** $x = 0$***), then***

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1 \quad \Rightarrow \quad \lim_{y\to 0} \frac{z}{\bar{z}} = -1$$

Since the two limits obtained by approaching the origin along different paths are distinct ($1 \neq -1$)***, the overall limit does not exist.***

$$\boxed{\lim_{z\to 0} \frac{z}{\bar{z}} \ \text{does not exist.}}$$



Two different limits along distinct paths
⇒ limit does not exist.

In the case where a complex function $f(z)$ can be expressed in terms of its real and imaginary parts as

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

then, by using the properties of real limits, one can derive a rule that allows the computation of the limit of $f(z)$ when the limits of $u(x, y)$ and $v(x, y)$ are known. In this situation, the real functions u and v are considered as functions of two real variables.

Theorem 1.1 (Real and Imaginary Parts of a Limit) ***Let***

$$f(z) = u(x,y) + iv(x,y), \quad z_0 = x_0 + iy_0, \quad \textbf{\textit{and}} \quad L = u_0 + iv_0.$$

Then,

$$\lim_{z\to z_0} f(z) = L \quad \textbf{\textit{if and only if}} \quad \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0. \end{cases}$$

Example 1.10 ***Calculate, using Theorem 1.1, the following limit:***

$$\lim_{z\to 1+i}(z^2 + i).$$

Solution 1.6 ***We first write*** $z = x + iy$***. Then,***

$$f(z) = z^2 + i = (x+iy)^2 + i = (x^2 - y^2) + i(2xy + 1).$$

Hence we can identify:

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy + 1.$$

According to Theorem 1.1, we compute the two real limits:

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1^2 - 1^2 = 0, \qquad \lim_{(x,y)\to(1,1)} v(x,y) = 2(1)(1) + 1 = 3.$$

Therefore,

$$\lim_{z\to 1+i}(z^2 + i) = 0 + 3i = 3i.$$

Theorem 1.2 (Properties of Complex Limits) ***Let*** f ***and*** g ***be two complex functions. If***

$$\lim_{z\to z_0} f(z) = L \quad \textbf{\textit{and}} \quad \lim_{z\to z_0} g(z) = M,$$

then the following properties hold:

(i) $\lim\limits_{z\to z_0} cf(z) = cL,$ ***where*** c ***is a complex constant;***

(ii) $\lim\limits_{z\to z_0} [f(z) \pm g(z)] = L \pm M;$

(iii) $\lim\limits_{z\to z_0} [f(z)\, g(z)] = L\, M;$

(iv) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M},$ ***provided that*** $M \neq 0.$

Basic results from Theorem 1.2:

$$\lim_{z\to z_0} c = c, \quad \textbf{where } c \textbf{ is a complex constant},$$

$$\lim_{z\to z_0} z = z_0.$$

Example 1.11 ***Using Theorem 1.2 and its two basic results, compute the following limits:***

(a) $\lim_{z \to i} \dfrac{(3+i)z^4 - z^2 + 2z}{z+1}$

(b) $\lim_{z \to 1+i\sqrt{3}} \dfrac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}}$

Solution 1.7 ***(a) By the properties of complex limits (Theorem 1.2):***

$$\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{(3+i)(i)^4 - (i)^2 + 2(i)}{i+1}.$$

We compute step by step:

$$i^2 = -1, \quad i^4 = 1.$$

Then:

$$\textbf{\textit{Numerator:}}\ (3+i)(1) - (-1) + 2i = 3 + i + 1 + 2i = 4 + 3i.$$

$$\textbf{\textit{Denominator:}}\ i + 1 = 1 + i.$$

Hence:

$$\frac{4+3i}{1+i} = \frac{(4+3i)(1-i)}{(1+i)(1-i)} = \frac{(4+3i)(1-i)}{2}.$$

Expanding the numerator:

$$(4+3i)(1-i) = 4 - 4i + 3i - 3i^2 = 4 - i + 3 = 7 - i.$$

Therefore:

$$\boxed{\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{7-i}{2}}.$$

(b) We apply the same rule:

$$\lim_{z \to 1+i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}} = \frac{(1+i\sqrt{3})^2 - 2(1+i\sqrt{3}) + 4}{(1+i\sqrt{3}) - 1 - i\sqrt{3}}.$$

Compute the numerator:

$$(1+i\sqrt{3})^2 = 1 + 2i\sqrt{3} + (i\sqrt{3})^2 = 1 + 2i\sqrt{3} - 3 = -2 + 2i\sqrt{3}.$$

Then:

$$-2 + 2i\sqrt{3} - 2(1+i\sqrt{3}) + 4 = -2 + 2i\sqrt{3} - 2 - 2i\sqrt{3} + 4 = 0.$$

Denominator:

$$(1+i\sqrt{3}) - 1 - i\sqrt{3} = 0.$$

So we get the indeterminate form $\frac{0}{0}$. To evaluate the limit, we simplify the expression:

$$\frac{z^2 - 2z + 4}{z - (1 + i\sqrt{3})}.$$

Factorizing the numerator:

$$z^2 - 2z + 4 = (z - (1 + i\sqrt{3}))(z - (1 - i\sqrt{3})).$$

Hence:

$$\frac{z^2 - 2z + 4}{z - (1 + i\sqrt{3})} = z - (1 - i\sqrt{3}).$$

Now taking the limit:

$$\lim_{z \to 1 + i\sqrt{3}} [z - (1 - i\sqrt{3})] = (1 + i\sqrt{3}) - (1 - i\sqrt{3}) = 2i\sqrt{3}.$$

Therefore:

$$\boxed{\lim_{z \to 1 + i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}} = 2i\sqrt{3}.}$$

1.2.4 The Complex Infinity

Unlike the real number system, which contains two distinct infinities $+\infty$ and $-\infty$, the complex number system has only one infinity. This distinction arises because the field of complex numbers $\mathbb{C}$ is not an ordered field. Recall that in the real number system, $+\infty$ and $-\infty$ serve respectively as the *upper* and *lower* *bounds* of every subset of the extended real line - that is, the set of all real numbers together with $+\infty$ and $-\infty$.

Example 1.12

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

1. Extended Complex Plane

To simplify the study of complex functions, it is often useful to extend the complex plane by adding a single point at infinity, denoted by ∞. The resulting set

$$\mathbb{C} \cup \{\infty\}$$

is called the extended complex plane.

1. Algebraic rules involving ∞

The basic operations with ∞ are defined as follows:

$$\begin{aligned} z + \infty &= \infty, \quad \text{for all } z \in \mathbb{C}, \\ z \cdot \infty &= \infty, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}, \\ \frac{z}{\infty} &= 0, \quad \text{for all } z \in \mathbb{C}, \\ \frac{z}{0} &= \infty, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

In particular, $-1 \cdot \infty = \infty$. However, the expressions $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\infty \pm \infty$, and $\frac{0}{0}$ are undefined.

2. Topological interpretation

From a topological point of view, any set of the form

$$\{ z : |z| > R \}, \quad R \geq 0,$$

is called a neighborhood of ∞.

A set $D \subset \mathbb{C} \cup \{\infty\}$ is said to contain the point at infinity if there exists a real number $M > 0$ such that D includes all points z with $|z| > M$.

Examples 1.1

- ***The open half-plane*** $\mathrm{Re}(z) > 0$ ***does not contain the point at infinity, since it includes no neighborhood of*** ∞***.***
- ***The open set***

  $$D = \{ z : |z+1| + |z-1| > 1 \}$$

  does contain the point at infinity.

3. The Structure of Infinity in the Complex Plane

When we extend the complex plane by adding the point at infinity, we obtain the extended complex plane, denoted by

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

This set is also known as the Riemann sphere.

- **The usual complex numbers are the points $z = x + iy \in \mathbb{C}$.**
- **The symbol ∞ is not a number but an additional point. It cannot be manipulated by ordinary arithmetic operations.**

4. Topological structure

To make sense of the point at infinity, we define a topology on $\widehat{\mathbb{C}}$: a neighborhood of ∞ is any set of the form

$$\{\, z \in \mathbb{C} : |z| > R \,\} \cup \{\infty\},$$

for some $R > 0$. In this sense, saying that z is "close to infinity" means that $|z|$ is very large.

The definition of limit at infinity becomes:

$$\lim_{z \to \infty} f(z) = L \quad \iff \quad \forall \varepsilon > 0,\ \exists R > 0,\ |z| > R \Rightarrow |f(z) - L| < \varepsilon.$$

Geometric structure (Riemann sphere) The extended complex plane can be visualized as a sphere in $\mathbb{R}^3$, called the Riemann sphere. We imagine a sphere of radius 1 centered at $(0, 0, 1)$ in space. Each point $z = x + iy$ of the complex plane (lying on the plane $z = 0$) is mapped to the sphere by ***stereographic projection.*** The point ∞ corresponds to the North Pole of the sphere.

Thus, topologically:

$$\widehat{\mathbb{C}} \cong S^2.$$

5. Partial algebraic structure

Arithmetic operations involving ∞ are defined by convention:

$$\begin{aligned} z + \infty &= \infty, \quad z \in \mathbb{C}, \\ z \cdot \infty &= \infty, \quad z \neq 0, \\ \frac{z}{\infty} &= 0, \quad z \in \mathbb{C}, \\ \frac{z}{0} &= \infty, \quad z \neq 0. \end{aligned}$$

However, the following expressions are undefined:

$$0 \cdot \infty, \quad \infty - \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}.$$

Therefore, $\widehat{\mathbb{C}}$ is not a field but a compact topological space endowed with a useful partial algebra.

6. Summary

Aspect	Description
Set	$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
Nature of ∞	A point added, not a number
Topology	Neighborhoods of ∞: $\{z : |z| > R\} \cup \{\infty\}$
Geometry	Topologically equivalent to a sphere (Riemann sphere)
Algebra	Partial arithmetic: $z + \infty = \infty,\ z/\infty = 0$, etc.

1.2.5 Continuity of Complex Functions

The definition of continuity for a complex function is analogous to that of a real function. In other words, a complex function is continuous at a point if its value approaches the function value at that point as the variable approaches it.

Definition 1.28 (Continuity at a point) ***Let*** $f : D \subset \mathbb{C} \to \mathbb{C}$ ***be a complex function and let*** $z_0 \in D$***. We say that*** f ***is continuous at*** z_0 ***if***

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Equivalently, using the ε–δ definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } |z - z_0| < \delta, \text{ then } |f(z) - f(z_0)| < \varepsilon.$$

Remark 1.10 ***This definition is identical in form to that of real-valued functions, but the variable*** z ***now approaches*** z_0 ***from*** all possible directions ***in the complex plane.***

1. Criteria for Continuity

Just as in the real case, for a complex function f to be continuous at a point, the following three conditions must be satisfied.

Definition 1.29 (Continuity criteria) ***A complex function*** f ***is continuous at a point*** $z_0 \in \mathbb{C}$ ***if and only if the following conditions hold:***

(i) The limit $\lim_{z \to z_0} f(z)$ ***exists.***

(ii) The function f ***is defined at*** z_0***.***

(iii) The limit equals the function value:

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If a complex function does not satisfy one or more of these conditions at a point, it is said to be discontinuous at that point.

2. Branches of Multivalued Functions

A ***branch*** of a multivalued function $F(z)$ defined on a domain D is any complex function $f_i(z)$ that is defined and continuous on a certain subdomain $D_i \subset D$ and that assigns ***exactly one value*** of $F(z)$ to each point $z \in D_i$.
In the definition of the branches $f_i(z)$ of a multivalued function, a point that is common to the different cuts used to define the domain D_i of a branch is called a Branch Point.

Example 1.13 ***The function***

$$f(z) = \frac{1}{1+z^2}$$

is discontinuous at $z = i$ ***and*** $z = -i$***, since it is not defined at these points.***

Example 1.14 (Continuity verification) ***(a) Verify whether the function***

$$f(z) = z^2 - iz + 2$$

is continuous at $z_0 = 1 - i$.

(b) Show that the function

$$f(z) = \sqrt{z}$$

is discontinuous at $z_0 = -1$.

Solution 1.8 ***(a) The function*** $f(z) = z^2 - iz + 2$ ***is a polynomial in*** z***. Since polynomials are continuous everywhere in*** $\mathbb{C}$***,*** f ***is continuous at*** $z_0 = 1 - i$***.***

(b) Why $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$

We must first fix what we mean by $\sqrt{z}$***. The complex square root is naturally*** **multi-valued*****:***

$$\sqrt{z} = \pm\sqrt{r}\, e^{i\theta/2} \quad \textit{when } z = re^{i\theta} \ (r > 0).$$

To obtain a single-valued function we choose a branch. The usual **principal branch** ***of the square root is defined by***

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}, \qquad \theta \in (-\pi, \pi],$$

which forces a branch cut along the negative real axis $(-\infty, 0]$***. On this branch the function is single-valued and continuous on*** $\mathbb{C}\setminus(-\infty, 0]$***, but it is*** **not** ***defined on the cut itself; in particular it is not defined at*** $z_0 = -1$***.***

There are two natural ways to conclude non-continuity at $z_0 = -1$***:***

(A) The function is not defined at $z_0 = -1$ ***on the principal branch. By the continuity criteria, a function that is not defined at*** z_0 ***cannot be continuous there.***

(B) The one-sided limits (approaching from two sides of the cut) are different. This shows there is no possible single-limit value at -1 ***even if we try to take limits from different directions.***

We illustrate (B) by taking two explicit approach paths that tend to -1 ***from opposite sides of the negative real axis.***

Path 1 (approach from above). Take

$$z_1(t) = e^{i(\pi - t)}, \qquad t > 0,\ t \to 0^+.$$

Then $z_1(t) \to -1$ ***as*** $t \to 0^+$***. Using the principal branch (argument*** $\theta = \pi - t \in (-\pi, \pi]$***) we get***

$$\sqrt{z_1(t)} = \sqrt{1}\, e^{i(\pi - t)/2} = e^{i\pi/2}\, e^{-it/2} \xrightarrow[t\to 0^+]{} e^{i\pi/2} = i.$$

Path 2 (approach from below). Take

$$z_2(t) = e^{i(-\pi+t)}, \qquad t > 0,\ t \to 0^+.$$

Then $z_2(t) \to -1$ ***as*** $t \to 0^+$***. On the principal branch the argument*** $\theta = -\pi + t$ ***(note*** $-\pi + t$ ***is just inside the interval*** $(-\pi, \pi]$***), so***

$$\sqrt{z_2(t)} = \sqrt{1}\, e^{i(-\pi+t)/2} = e^{-i\pi/2}\, e^{it/2} \xrightarrow[t\to 0^+]{} e^{-i\pi/2} = -i.$$

Thus the limits along the two paths exist but are different:

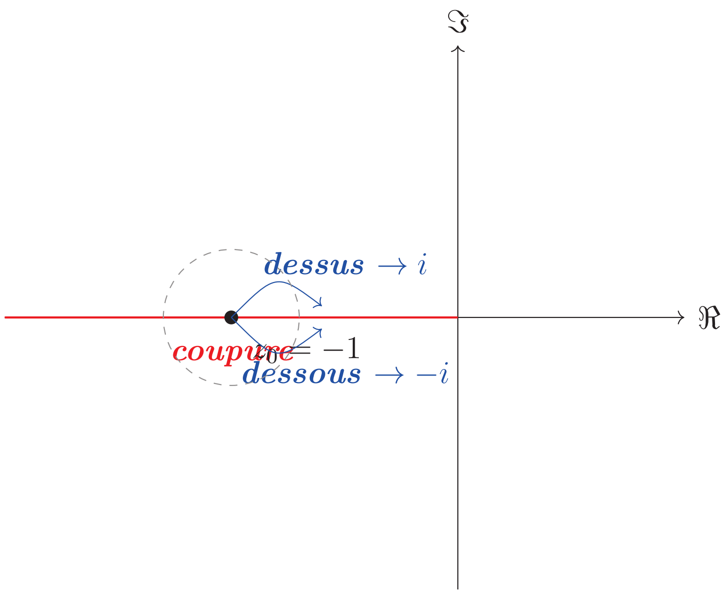
$$\lim_{z\to -1 \text{ from above}} \sqrt{z} = i, \qquad \lim_{z\to -1 \text{ from below}} \sqrt{z} = -i.$$

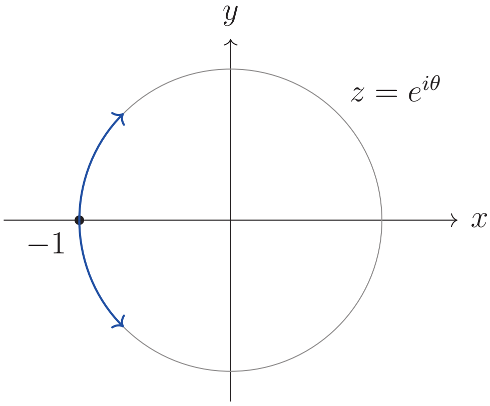
Because these one-sided limits are unequal, the two-dimensional limit $\lim_{z\to -1}\sqrt{z}$ ***does not exist. Hence*** $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$ ***(on the principal branch).***

Remark 1.11
- ***The non-existence of a single-valued, continuous square root on any domain that encircles the origin is a consequence of the branch point at*** 0***: analytic continuation around*** 0 ***changes the sign of*** $\sqrt{z}$***. Any branch cut must connect*** 0 ***to*** ∞***; if the chosen cut passes through*** -1 ***(as the usual negative-real-axis cut does), then*** -1 ***lies on the cut and the principal branch is not defined there and shows the discontinuity described above.***

- ***If one chooses a different branch cut that*** does not ***pass through*** -1***, then one can define a single-valued branch of*** $\sqrt{z}$ ***that is continuous in a neighborhood of*** -1***. So the discontinuity at*** -1 ***is not an intrinsic property of the point*** -1 ***alone but depends on the chosen branch (equivalently: on the domain where the single-valued branch is defined). The usual statement -***$\sqrt{z}$ ***is discontinuous at*** -1***- refers to the common principal branch with cut along*** $(-\infty, 0]$***.***

Conclusion. On the principal branch (argument in $(-\pi, \pi]$***) the square root is not defined at*** -1 ***and the limits from the two sides of the negative real axis give*** i ***and*** $-i$ ***respectively. Therefore*** $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$***.***





3. Properties of Continuous Functions

As for limits, we study the continuity of a complex function expressed in terms of its real and imaginary parts:

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

Theorem 1.3 (Real and Imaginary Parts of a Continuous Function)

$$\textit{Let } f(z) = u(x, y) + iv(x, y) \quad \textit{and } z_0 = x_0 + iy_0.$$

Then the complex function f is continuous at z_0 if and only if the two real functions u and v are continuous at the point (x_0, y_0).

Example 1.15 ***Using Theorem 1.3, show that the function***

$$f(z) = \overline{z}$$

is continuous on $\mathbb{C}$.

Theorem 1.4 (Properties of Continuous Functions) ***If f and g are two complex functions continuous at z_0, then the following functions are also continuous at the same point:***

1. ***$cf(z)$, where c is a complex constant,***
2. ***$f(z) \pm g(z)$,***
3. ***$f(z) \cdot g(z)$,***
4. ***$\dfrac{f(z)}{g(z)}$, provided $g(z_0) \neq 0$.***

- ***Polynomial functions are continuous over the entire complex plane $\mathbb{C}$.***
- ***Unlike complex polynomial functions, complex rational functions***

$$f(z) = \frac{p(z)}{q(z)}$$

are not always continuous on the entire complex plane $\mathbb{C}$, but they are continuous only on their domain of definition, that is, where $q(z) \neq 0$.

- Boundedness Property:
If a complex function f is continuous on a closed and bounded region Γ, then f is bounded on Γ. Thus, there exists a real constant $M > 0$ such that

$$|f(z)| \leq M \quad \textbf{for all } z \in \Gamma.$$

1.3 Solved Exercises

Exercise 1.5 ***Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$. Compute the following:***
(a) $z_1 + z_2$, ***(b)*** $z_1 - z_2$, ***(c)*** $z_1 \cdot z_2$, ***(d)*** $\dfrac{z_1}{z_2}$, ***(e)*** $\overline{z_1}$, ***(f)*** $|z_1|$, ***(g)*** $\sqrt{z_1}$,
(h) Solve $z^2 + 1 = 0$.

Solution 1.9 ***(a) Sum:***

$$z_1 + z_2 = (3 + 4i) + (1 - 2i) = 4 + 2i.$$

(b) Difference:

$$z_1 - z_2 = (3 + 4i) - (1 - 2i) = 2 + 6i.$$

(c) Product:

$$z_1 \cdot z_2 = (3 + 4i)(1 - 2i) = 11 - 2i.$$

(d) Quotient:

$$\frac{z_1}{z_2} = \frac{3 + 4i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = -1 + 2i.$$

(e) Conjugate:

$$\overline{z_1} = 3 - 4i.$$

(f) Modulus:

$$|z_1| = \sqrt{3^2 + 4^2} = 5.$$

(g) **Square root of z_1 using the polar identification method:**

Step 1: Write z_1 in polar form:

$$z_1 = 3 + 4i \implies \rho = |z_1| = \sqrt{3^2 + 4^2} = 5, \quad \phi = \arctan\frac{4}{3} \approx 0.927 \ \textbf{\textit{radians}}.$$

Step 2: Let $z = r(\cos\theta + i\sin\theta)$ such that $z^2 = z_1$. Then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta) = \rho(\cos\phi + i\sin\phi).$$

Step 3: Identification gives:

$$r^2 = \rho \implies r = \sqrt{\rho} = \sqrt{5} \approx 2.236,$$

$$2\theta = \phi + 2k\pi \implies \theta = \frac{\phi}{2} + k\pi, \quad k = 0, 1.$$

Step 4: Compute the two roots: For $k = 0$***:***

$$\theta_0 = \frac{\phi}{2} \approx 0.464 \ \textbf{\textit{radians}}, \qquad z_0 = r(\cos\theta_0 + i\sin\theta_0).$$

For $k = 1$***:***

$$\theta_1 = \frac{\phi}{2} + \pi \approx 3.606 \ \textbf{\textit{radians}}, \qquad z_1 = r(\cos\theta_1 + i\sin\theta_1).$$

- Square root of z_1 **using the algebraic method with modulus:**

Let $z = x + iy$ ***such that*** $z^2 = z_1 = 3 + 4i$***.***
Step 1: Expand z^2***:***

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xy\, i.$$

Step 2: Identify the real and imaginary parts with $z_1 = 3 + 4i$***:***

$$\begin{cases} x^2 - y^2 = 3, \\ 2xy = 4. \end{cases}$$

Step 3: Add the modulus condition:

$$x^2 + y^2 = |z|^2 = |z_1| = \sqrt{3^2 + 4^2} = 5.$$

Step 4: Solve the system:

$$\begin{cases} x^2 - y^2 = 3, \\ x^2 + y^2 = 5. \end{cases}$$

Add the two equations:

$$2x^2 = 8 \implies x^2 = 4 \implies x = \pm 2.$$

Then from $x^2 + y^2 = 5$***:***

$$y^2 = 5 - 4 = 1 \implies y = \pm 1.$$

Step 5: Determine compatible signs from $2xy = 4$***:***

$$2 \cdot 2 \cdot 1 = 4 \quad \textbf{\textit{and}} \quad 2 \cdot (-2) \cdot (-1) = 4.$$

Step 6: Conclude the two square roots:

$$\boxed{z = 2 + i \quad \textbf{\textit{and}} \quad z = -2 - i}.$$

(h) Solve $z^2 + 1 = 0$***:***

$$z^2 = -1 \implies z = \pm i.$$

Exercise 1.6 ***Express the complex number*** $z = -1 + i\sqrt{3}$ ***in polar form and exponential form.***

Solution 1.10 ***Step 1: Compute the modulus r of z:***

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$

Step 2: Compute the argument θ of z:

$$\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right).$$

Since z is in the second quadrant, we have

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Step 3: Write z in polar form:

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Step 4: Write z in exponential form:

$$z = re^{i\theta} = 2e^{i\frac{2\pi}{3}}.$$

Exercise 1.7 ***(a) Given the set of points defined by $|z-1| = 2$, describe the set geometrically.***

(b) Given the set of points defined by $|z-1| \leq 2$, describe the set geometrically.

(c) Given the set of points defined by $1 \leq |z-1| \leq 2$, describe the set geometrically.

Solution 1.11 ***Part (a): Circle***

Step 1: Write z in Cartesian form:

$$z = x + iy, \quad x, y \in \mathbb{R}.$$

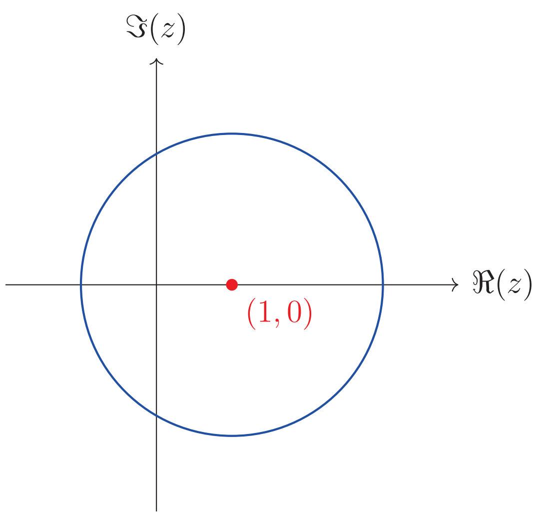
Step 2: Express the modulus:

$$|z-1| = |(x+iy)-1| = \sqrt{(x-1)^2 + y^2}.$$

Step 3: Apply the condition $|z-1| = 2$:

$$\sqrt{(x-1)^2 + y^2} = 2 \implies (x-1)^2 + y^2 = 4.$$

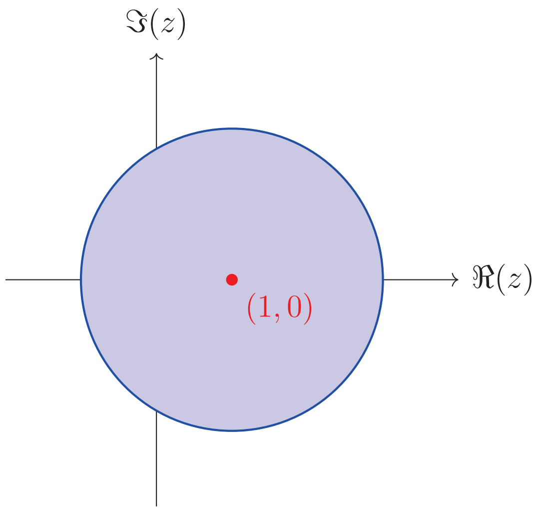
Step 4: Geometric description: - Circle with center $(1,0)$ and radius 2.



Part (b): Disk
Step 1: Condition $|z-1| \leq 2$ ***implies:***

$$(x-1)^2 + y^2 \leq 4.$$

Step 2: Geometric description: - Closed disk (including the boundary) with center $(1,0)$ ***and radius*** 2***.***



Part (c): Ring (Annulus)
Step 1: Condition $1 \leq |z-1| \leq 2$ ***in Cartesian form:***

$$z = x + iy \implies 1 \leq \sqrt{(x-1)^2 + y^2} \leq 2 \implies 1 \leq (x-1)^2 + y^2 \leq 4.$$

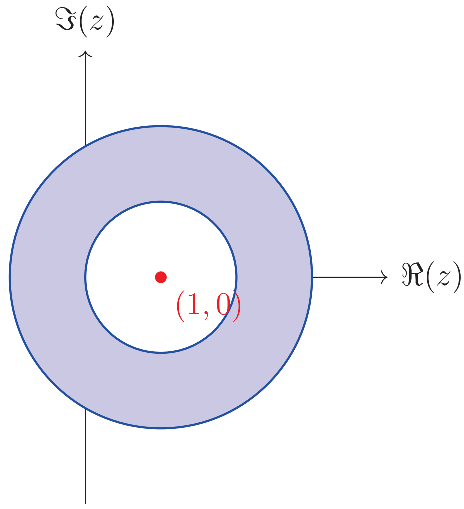
Step 2: Explanation of the left-hand side:

$$1 \leq |z-1| \implies (x-1)^2 + y^2 \geq 1.$$

- This represents all points whose distance from the center $(1,0)$ ***is at least*** 1***. - Geometrically, it is the **exterior of the inner circle** of radius 1.***
Step 3: Explanation of the right-hand side:

$$|z-1| \leq 2 \implies (x-1)^2 + y^2 \leq 4.$$

- Restricts points to **inside the outer circle** of radius 2.
Step 4: Geometric description: - The set is a **ring (annulus)** centered at $(1,0)$***, with inner radius*** 1 ***and outer radius*** 2***.***



Exercise 1.8 ***Complex functions: evaluation and separation of real and imaginary parts***

(A) Evaluate the following complex functions at the given points:

- ***(i)*** $f(z) = z\overline{z} + 2z - i$, $\quad z = \{1+2i, -2+i, 3-3i\}$
- ***(ii)*** $f(z) = |z|^2 + 3\Re(zi) - z$, $\quad z = \{2-3i, 1+i, -1+2i\}$
- ***(iii)*** $f(z) = \ln|z| + i\,\mathrm{Arg}(z)$, $\quad z = \{2, 2i, -1+i\}$
- ***(iv)*** $f(z) = x^2 - y^2 + i(2x+y)$, $\quad z = \{1+i, 2-3i, -2+2i\}$, ***where*** $z = x+iy$
- ***(v)*** $f(z) = e^{2z}$, $\quad z = \{i\pi, 1-i, \ln 3 + i\pi/4\}$

(B) Find the real part u ***and imaginary part*** v ***of the following functions:***

- ***(i)*** $f(z) = 5z + 2 - 3i$
- ***(ii)*** $f(z) = -2z + \overline{z} + i$
- ***(iii)*** $f(z) = \frac{z}{\overline{z}+1}$
- ***(iv)*** $f(z) = e^{z+i}$

(C) Express the real part u ***and imaginary part*** v ***in terms of*** r ***and*** θ ***for the following functions:***

- ***(i)*** $f(z) = \overline{z}$
- ***(ii)*** $f(z) = z^3$
- ***(iii)*** $f(z) = e^{2z}$
- ***(iv)*** $f(z) = z - \frac{1}{z}$

Solution 1.12 ***Part (A): Evaluation of complex functions***

(i) $f(z) = z\overline{z} + 2z - i$

- $z = 1+2i$: $z\overline{z} = (1+2i)(1-2i) = 1+4 = 5$, $2z = 2+4i$, $-i = -i$
  $f(1+2i) = 5 + (2+4i) - i = 7+3i$

- $z = -2+i$: $z\overline{z} = (-2+i)(-2-i) = 4+1 = 5$, $2z = -4+2i$, $-i = -i$ $f(-2+i) = 5 + (-4+2i) - i = 1+i$
- $z = 3-3i$: $z\overline{z} = 9+9 = 18$, $2z = 6-6i$, $-i = -i$ $f(3-3i) = 18+6-6i-i = 24-7i$

(ii) $f(z) = |z|^2 + 3\Re(iz) - z$

- $z = 2-3i$: $|z|^2 = 4+9 = 13$, $iz = i(2-3i) = 2i+3$, $\Re(iz) = 3$, $3\Re(iz) = 9$, $-z = -2+3i$ $f(2-3i) = 13+9-2+3i = 20+3i$
- $z = 1+i$: $|z|^2 = 2$, $iz = i(1+i) = i-1$, $\Re(iz) = -1$, $3\Re(iz) = -3$, $-z = -1-i$ $f(1+i) = 2-3-1-i = -2-i$
- $z = -1+2i$: $|z|^2 = 1+4 = 5$, $iz = i(-1+2i) = -i-2$, $\Re(iz) = -2$, $3\Re(iz) = -6$, $-z = 1-2i$ $f(-1+2i) = 5-6+1-2i = 0-2i = -2i$

(iii) $f(z) = \ln|z| + i\,\mathrm{Arg}(z)$

- $z = 2$: $|z| = 2$, $\mathrm{Arg}(z) = 0$ **?** $f(2) = \ln 2 + i0 = \ln 2$
- $z = 2i$: $|z| = 2$, $\mathrm{Arg}(z) = \pi/2$ **?** $f(2i) = \ln 2 + i\pi/2$
- $z = -1+i$: $|z| = \sqrt{2}$, $\mathrm{Arg}(z) = 3\pi/4$ **?** $f(-1+i) = \frac{1}{2}\ln 2 + i3\pi/4$

(iv) $f(z) = x^2 - y^2 + i(2x+y)$

- $z = 1+i$, $x = 1, y = 1$ **?** $f = 1-1+i(2+1) = 0+3i = 3i$
- $z = 2-3i$, $x = 2, y = -3$ **?** $f = 4-9+i(4-3) = -5+i1 = -5+i$
- $z = -2+2i$, $x = -2, y = 2$ **?** $f = 4-4+i(-4+2) = 0-2i = -2i$

(v) $f(z) = e^{2z}$

- $z = i\pi$: $e^{2i\pi} = \cos(2\pi) + i\sin(2\pi) = 1$
- $z = 1-i$: $e^{2(1-i)} = e^2 e^{-2i} = e^2(\cos 2 - i\sin 2)$
- $z = \ln 3 + i\pi/4$: $e^{2(\ln 3 + i\pi/4)} = e^{2\ln 3}e^{i\pi/2} = 9e^{i\pi/2} = 9i$

Part (B): Real and imaginary parts u and v

(i) $f(z) = 5z+2-3i = 5(x+iy)+2-3i = (5x+2)+i(5y-3) \implies u = 5x+2, v = 5y-3$

(ii) $f(z) = -2z + \overline{z} + i = -2(x+iy) + (x-iy) + i = -x - iy + i = -x + i(-y+1) \implies u = -x, v = -y+1$

(iii) $f(z) = \frac{z}{\overline{z}+1} = \frac{x+iy}{x-iy+1} \cdot \frac{x+1+iy}{x+1+iy} = \frac{x(x+1)+y^2+iy}{(x+1)^2+y^2} \implies u = \frac{x(x+1)+y^2}{(x+1)^2+y^2}, v = \frac{y}{(x+1)^2+y^2}$

(iv) $f(z) = e^{z+i} = e^{x+iy+i} = e^x e^{i(y+1)} = e^x(\cos(y+1) + i\sin(y+1)) \implies u = e^x\cos(y+1), v = e^x\sin(y+1)$

Part (C): Real and imaginary parts in terms of r and θ

(i) $f(z) = \overline{z} = r(\cos\theta - i\sin\theta) \implies u = r\cos\theta, v = -r\sin\theta$

(ii) $f(z) = z^3 = r^3e^{i3\theta} = r^3(\cos 3\theta + i\sin 3\theta) \implies u = r^3\cos 3\theta, v = r^3\sin 3\theta$

(iii) $f(z) = e^{2z} = e^{2re^{i\theta}} = e^{2r\cos\theta}(\cos(2r\sin\theta)+i\sin(2r\sin\theta)) \implies u = e^{2r\cos\theta}\cos(2r\sin\theta), v = e^{2r\cos\theta}\sin(2r\sin\theta)$

(iv) $f(z) = z - 1/z = re^{i\theta} - (1/r)e^{-i\theta} = (r - 1/r)\cos\theta + i(r + 1/r)\sin\theta \implies u = (r - 1/r)\cos\theta, v = (r + 1/r)\sin\theta$