CHAPTER 2

LINEAR APPLICATIONS

In linear algebra, we are interested in applications that preserve the structure of vector spaces, that is, applications from one vector space to another that preserve linear combinations. In this chapter, which is somewhat the focus of the rest of the document, we will essentially provide basic definitions and elementary results.

The concepts covered in this chapter are:

— Definitions: Linear application, Kernel, image, and rank of a linear application.

2.1 Definitions

Definition 2.1 *Let E and E' be two vector spaces over the same field $\mathbb{k}$ and f an application of E in E'. We say that f is linear, if:*

1. $f(v+w) = f(v) + f(w)$, $\forall v, w \in E$,
2. $f(\lambda v) = \lambda f(v), \forall v \in E, \forall \lambda \in \mathbb{k}$.

The set of linear applications from E to E' is denoted $\mathcal{L}_{\mathbb{k}}(E, E')$ or, more simply, $\mathcal{L}(E, E')$.

If a linear application f from E to E (same starting and ending space) we say that f is an endomorphism of E. The set of endomorphisms of E is denoted $End_{\mathbb{k}}(E)$ or, more simply, $End(E)$.

If a linear application f is bijective, we say that f is an isomorphism of vector spaces.

Remark 2.1
1. *If f is linear, we have: $f(0) = 0$. Just take $\lambda = 0$ in $f(\lambda x) = \lambda f(x)$.*
2. *According to 1) and 2), an application $f : E \to E'$ is linear, if and only if, for all $\lambda \in \mathbb{k}$ and for all $x, y \in E$, we have*
$$f(\lambda x + y) = \lambda f(x) + f(y).$$
3. *In some references an application is also called "map".*

Example 2.1 *Let E be a $\mathbb{k}$-vector space and F a vector subspace of E. We call canonical injection of F into E, the application $i : F \to E$ defined by,*

$$\forall x \in F,\ i(x) = x,$$

So i is a linear application.

Example 2.2

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (2x+y, y-z), \end{array}$$

is a linear application.

If $v = (x, y, z)$ and $w = (x', y\prime, z')$, we have:

$$\begin{array}{rcl} f(v+w) & = & f(x+x', y+y', z+z') \\ & = & (2(x+x') + (y+y'), y+y'-z-z') \\ & = & (2x+y, y-z) + (2x'+y', y'-z') \\ & = & f(v) + f(w), \\ f(\lambda v) & = & f(\lambda x, \lambda y, \lambda z) \\ & = & (2\lambda x + \lambda y, \lambda y - \lambda z) \\ & = & \lambda(2x+y, y-z) \\ & = & \lambda f(v). \end{array}$$

As this example illustrates, the linearity of f stems from the fact that the components x, y, z in the codomain (here $\mathbb{R}^3$) all appear to the power of 1. More precisely, each component in the codomain is a homogeneous polynomial of degree 1 in x, y, z. We will examine this in more detail later.

Thus, for example, the application

$$\begin{array}{cccc} f: & \mathbb{R}^3 & \rightarrow & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (x^2-y, y+z) \end{array}$$

is not linear neither 1), nor 2) of the definition 2.1 are not satisfied because of the squared term.

Example 2.3 *Let $\mathcal{C}([0,1],\mathbb{R})$ and $\mathcal{C}^1([0,1],\mathbb{R})$ vector spaces of applications $f : [0,1] \rightarrow \mathbb{R}$ respectively continuous and continuous with continuous derivative. The application:*

$$\begin{array}{cccc} D: & \mathcal{C}^1([0,1],\mathbb{R}) & \rightarrow & \mathcal{C}([0,1],\mathbb{R}) \\ & f & \mapsto & f' \end{array}$$

is a linear application, since:

$$\begin{array}{lcll} D(f+g) & = & (f+g)' = f'+g' & = Df + Dg \\ D(\lambda f) & = & (\lambda f)' & = \lambda f' \end{array}$$

if $\lambda \in \mathbb{R}$, and f and $g \in \mathcal{C}^1([0,1],\mathbb{R})$.

Example 2.4 *Let $v_0 \neq 0_E$ a vector of E, the translation application defined by*

$$\begin{array}{cccc} t: & E & \rightarrow & E \\ & v & \mapsto & v+v_0 \end{array}$$

is not linear (note, for example, that: $t(0) = v_0 \neq 0_E$).

Example 2.5 *Let E be a $\mathbb{k}$-vector space of finite-dimension n and $(e_1, e_2, ..., e_n)$ a basis of E. Then the application f defined by,*

$$\begin{array}{cccc} t: & \mathbb{k}^n & \rightarrow & E \\ & (\alpha_1, ..., \alpha_n) & \mapsto & f((\alpha_1, ..., \alpha_n)) = \sum_{i=1}^n \alpha_i e_i. \end{array}$$

is an isomorphism of vector spaces. We therefore deduce that any $\mathbb{k}$-vector space of finite dimension n is isomorphic to $\mathbb{k}^n$.

Proposition 2.1 *i) Let E, F and G be three vector spaces on the same field $\mathbb{k}$. $f : E \rightarrow F$ and $g : F \rightarrow G$ two linear applications. Then $g \circ f$ is a linear application.*

ii) Let $f : E \to F$ be a an isomorphism of vector spaces, then $f^{-1} : F \to E$ is also an isomorphism of vector spaces.

iii) Two finite-dimensional vector spaces of the same dimension are isomorphic.

Proof. *i*) Let $x \in E, y \in E$ and $\alpha \in \mathbb{k}$, then we have

$$\begin{aligned}(g \circ f)(x+y) &= g(f(x+y)) \\ &= g(f(x)+f(y)) \quad (\text{ car } f \text{ est linéaire }) \\ &= g(f(x))+g(f(y)) \quad (\text{ car } g \text{ est linéaire }) \\ &= (g \circ f)(x)+(g \circ f)(y).\end{aligned}$$

We also have

$$\begin{aligned}(g \circ f)(\alpha x) &= g(f(\alpha x)) \\ &= g(\alpha f(x)) \quad (\text{ car } f \text{ est linéaire }) \\ &= \alpha g(f(x)) \quad (\text{ car } g \text{ est lincaire }) \\ &= \alpha(g \circ f)(x).\end{aligned}$$

Then $g \circ f$ is linear.

ii) Let's suppose that $f : E \longrightarrow F$ is an isomorphism of vector spaces. Let $x \in F, y \in F$ and $\alpha \in \mathbb{k}$. Let $a \in E$ and $b \in E$, such that $f(a) = x$ and $f(b) = y$. As f is linear, then we have $f(a+b) = x+y$, so we will have

$$f^{-1}(x+y) = f^{-1}(f(a+b)) = a+b = f^{-1}(x)+f^{-1}(y).$$

We also have

$$f^{-1}(\alpha x) = f^{-1}(\alpha f(a)) = f^{-1}(f(\alpha a)) = \alpha a = \alpha f^{-1}(x).$$

Then f^{-1} is linear

iii) Let E and F be two vector spaces of the same dimension n, then according to the previous example, E and F are isomorphic to $\mathbb{k}^n$. Then if $\varphi : E \longrightarrow \mathbb{k}^n$ and $\psi : F \longrightarrow \mathbb{k}^n$ are two isomorphisms of vector spaces, then $\psi^{-1} \circ \varphi : E \longrightarrow F$ is an isomorphism of vector spaces. □

Proposition 2.2 *Let E and E' be two $\mathbb{k}$-vector spaces. For f and g two elements of $L_{\mathbb{k}}(E, E')$ and for α element of $\mathbb{k}$, we define $f+g$ and $\alpha.f$, by*

$$\forall x \in E, \quad (f+g)(x) = f(x)+g(x) \text{ and } (\alpha \cdot f)(x) = \alpha \cdot f(x).$$

Then $(L_{\mathbb{k}}(E, E'), +, \cdot)$ is a $\mathbb{k}$-vector space.

Proof. Just check that $L_{\mathbb{k}}(E, E')$ is a vector subspace of E'^E the $\mathbb{k}$-vector space of all applications of E to E'. □

Theorem 2.1 *Let E and E' be two $\Bbbk$-vector spaces of finite dimension. Then $L_{\Bbbk}(E,E')$ is of finite dimension and we have*

$$\dim\left(L_K(E,E')\right) = \dim(E) \times \dim(E').$$

Proof. Let $m = \dim(E), n = \dim(E'), (e_1, e_2, \ldots, e_m)$ be a basis of E and $(e'_1, e'_2, \ldots, e'_n)$ a basis of E'. For $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$, where for everything $p \in \mathbb{N}^*$, we pose $\mathbb{N}_p = \{1, 2, \ldots, p\}$, we define the application $f_{ij} : E \longrightarrow E'$ by,

$$\forall x \in E, \quad f_{ij}(x) = x_j e'_i \quad \text{where } x = \sum_{j=1}^{m} x_j e_j.$$

Then, $B = \{f_{ij} : (i,j) \in \mathbb{N}_m \times \mathbb{N}_n\}$ forms a basis of $L_K(E,E')$. Indeed,
Let $f \in L_{\Bbbk}(E,E')$, so for each $j \in \mathbb{N}_{m,}$, we have $f\left(e_j\right) = \sum_{i=1}^{n} \alpha_{ij} e'_i$.
So for any $x \in E$ with $x = \sum_{j=1}^{m} x_j e_j$, we have

$$\begin{aligned} f(x) &= \sum_{j=1}^{m} x_j f\left(e_j\right) = \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} \alpha_{ij} e'_i\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} x_j e'_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} f_{ij}(x). \end{aligned}$$

Then B is a finite generating part of $L_K(E,E')$.
It is easy to verify that B is a linearly independent part, noting that

$$\forall k \in \mathbb{N}_m, \quad f_{ij}\left(e_k\right) = \begin{cases} e'_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

$\operatorname{Card}(B) = \operatorname{Card}\left(\mathbb{N}_m \times \mathbb{N}_n\right) = m \times n$, then, $\dim\left(L_K(E,E')\right) = m \times n$. □

2.2 Kernel, Image and Rank

Proposition 2.3 *Let E and F be two $\Bbbk$-vector spaces and $f : E \longrightarrow F$ a linear application. Then,*

i) The image by f of a vector subspace of E is a vector subspace of F. In particular, $f(E)$ is a vector subspace of F, called the image of f and denoted $\operatorname{Im} f$. Sa dimension est appelée rang de f et est notée

$$rg\ f = \dim\left(\operatorname{Im} f\right).$$

ii)The inverse image by f of a vector subspace of F is a vector subspace of E. In particular, $f^{-1}\left(\{0_F\}\right)$ is a vector subspace of E, called the kernel of f and denoted $\ker(f)$.

Proof. i) Let A be a vector subspace of E. Then, $f(A) \neq \emptyset$, since $0_F = f(0_E)$, then $0_F \in f(A)$.

If $x \in A$, $y \in A$ and $\alpha \in K$, we have

$$f(x) + f(y) = f(x+y) \text{ and } \alpha \cdot f(x) = f(\alpha \cdot x).$$

as $x + y \in A$ and $\alpha \cdot x \in A$, therefore $f(x) + f(y) \in f(A)$ and $\alpha \cdot f(x) \in f(A)$.

ii) Now let B be a vector subspace of F. Then $f^{-1}(B) \neq \emptyset$, since $f(0_E) = 0_F$ and $0_F \in B$, then $0_E \in f^{-1}(B)$.

If $x \in f^{-1}(B), y \in f^{-1}(B)$ and $\alpha \in K$, therefore we have $f(x) \in B$ and $f(y) \in B$ and as B is a vector subspace of F and f linear, then $f(x+y) \in f(B)$ and $f(\alpha \cdot x) \in f(B)$, then $x + y \in f^{-1}(B)$ and $\alpha \cdot x \in f^{-1}(B)$. Let us remember that

$$z \in f^{-1}(B) \Longleftrightarrow f(z) \in B.$$

□

Remark 2.2 *Let E and F be two $\mathbb{k}$-vector spaces and $f : E \longrightarrow F$ a linear application. So*

1.
$$\ker(f) = \{x \in E : f(x) = 0_F\}$$
$$x \in \ker(f) \Longleftrightarrow f(x) = 0_F.$$

2.
$$\text{Imf} = \{f(x) : x \in E\}$$
$$y \in \text{Imf} \Longleftrightarrow \exists x \in E : y = f(x).$$

Proposition 2.4 *SLet E and F be two $\mathbb{k}$-vector spaces and $f : E \longrightarrow F$ a linear application. So*

i) f is injetive $\Longleftrightarrow \ker(f) = \{0_E\}$.

ii) f is surjetive $\Longleftrightarrow \operatorname{Im} f = F$.

Proof. i) $\Longrightarrow$) Suppose that f is injective and that $x \in \ker(f)$. We have $f(x) = 0_F$ and as f is linear, then $f(0_E) = 0_F$, then $f(x) = f(0_E)$ and since f is injective, then $x = 0_E$. So, $\ker(f) = \{0_E\}$.

$\Longleftarrow$) Suppose that $\ker(f) = \{0_E\}$.

Let $x \in E$ and $y \in E$, such that $f(x) = f(y)$, do we have $x = y$?

As f is linear and $f(x) = f(y)$, so we have $f(x-y) = 0_F$, then $x - y \in \ker(f)$, then as $\ker(f) = \{0_F\}$, then we have $x = y$ and consequently, f is injective.

ii) Trivial, because an application $f : E \longrightarrow F$ is surjective, if and only if, $f(E) = F$. □

Example 2.6 *Let:*

$$D: \begin{array}{ccc} \mathbb{R}[x] & \longrightarrow & \mathbb{R}[x] \\ P & \mapsto & P'. \end{array}$$

The kernel of D is formed by the constant polynomials. On the other hand, $\operatorname{Im} D = \mathbb{R}[x]$, *because if $P \in \mathbb{R}[x], Q(x) := \int_0^x P(t)dt$ is a polynomial and we have $Q' = P$ that's to say $DQ = P$.*

Example 2.7 *Let:*

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3(x,y,z) \longmapsto (x',y',z') \quad \textit{where :} \quad \begin{cases} x' = x + y - z \\ y' = 2x + y - 3z \\ z' = 3x + 2y - 4z. \end{cases}$$

$ker(f)$ is the set of triplets $(x,y,z) \in \mathbb{R}^3$ which satisfies the system:

$$\begin{cases} x + y - z = 0 \\ 2x + y - 3z = 0 \\ 3x + 2y - 4z = 0. \end{cases}$$

It is easy to find $x = 2\lambda, y = -\lambda, z = \lambda$; that's to say $\operatorname{Ker} f$ is the vector line generated by the vector $(2,-1,1)$.

Regarding $\operatorname{Im} f$, we have :

$(x',y',z') \in \operatorname{Im} f$, if and only if, it exists $(x,y,z) \in \mathbb{R}^3$ satisfies the system:

$$2x + y - 3z = y' 3x + 2y - 4z = z'$$

It is therefore a question of knowing for what values of x', y', z'. This system is compatible. By scaling, we find :

$$\begin{cases} x + y - z = x' \\ -y - z = y' - 2x' \\ -y - z = z' - 3x' \end{cases} \Longrightarrow \begin{cases} x - y - z = x' \\ y + z = 2x' - y' \\ 2x' - y' + z' - 3x' = 0, \end{cases}$$

the compatibility condition is $2x' - y' + z' - 3x' = 0$ so, $x' + y' - z' = 0$. The image of f is therefore the plan of $\mathbb{R}^3$ of the équation $x' + y' - z' = 0$.

Proposition 2.5 *Let $f \in \mathcal{L}(E,E')$ and $\{v_i\}_{i\in I}$ a family of vectors of E.*

1. *If f is injective and the family of $E\,\{v_i\}_{i\in I}$ is linearly independent, then the family $\{f(v_i)\}_{i\in I}$ of E' is linearly independent.*

2. *If f is surjective and the family $\{v_i\}_{i\in I}$ is generator of E then the family $\{f(v_i)\}_{i\in I}$ is generator of E'.*

In particular if f is bijective the image of a basis of E is a basis of E'.

Proof. 1. Let's assume the family $\{v_i\}_{i\in I}$ is linearly independent and let f is injective. For any extracted family $\{v_{\alpha_1}, \ldots, v_{\alpha_q}\}$, the relation

$$\lambda_1 f(v_{\alpha_1}) + \cdots + \lambda_q f(v_{\alpha_q}) = 0$$

implies that $f(\lambda_1 v_{\alpha_1} + \cdots + \lambda_q v_{\alpha_q}) = 0$, so, $\lambda_1 v_{\alpha_1} + \cdots + \lambda_q v_{\alpha_q} \in \operatorname{Ker} f$. But, $\operatorname{Ker} f = \{0\}$, then

$$\lambda_1 v_{\alpha_1} + \cdots + \lambda_q v_{\alpha_q} = 0$$

and since the family $\{v_i\}_{i\in I}$ is linearly independent, we have $\lambda_1 = 0, \ldots, \lambda_q = 0$. Then the family $\{f(v_i)\}_{i\in I}$ is linearly independent.
2. Let $y \in E'$ any; since f is surjective, there exists $x \in E$ such that $y = f(x)$. On the other hand, the family $\{v_i\}_{i\in I}$ is a generator, so x is of the form

$$x = \lambda_1 v_{\alpha_1} + \cdots + \lambda_p v_{\alpha_p},$$

thus : $f(x) = \lambda_1 f(v_{\alpha_1}) + \cdots + \lambda_p f(v_{\alpha_p})$ .y is therefore a linear combination of elements of the family $\{f(v_i)\}_{i\in I}$ and, since it is chosen arbitrarily in E', the family $\{f(v_i)\}_{i\in I}$ is generator. □

Theorem 2.2 *Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.*

Proof. Indeed, if there exists an isomorphism $f : E \longrightarrow E'$, the image by f of a basis of E is a basis of E', then E and E' have the same dimension. Conversely, suppose that $\dim E = \dim E'$ and let $\{e_1, \ldots, e_n\}, \{e'_1, \ldots, e'_n\}$ two bases respectively of E and E'. Consider the application $f : E \longrightarrow E'$ constructed as follows:
- for $k = 1, \ldots, n$ we pose: $f(e_k) = e'_k$;
- if $x = \sum_{k=1}^{n} x_k e_k$ we pose : $f(x) = \sum_{k=1}^{n} x_k f(e_k) = \sum_{k=1}^{n} x_k e'_k$,

(in other words, we define f on the basis of E and extend it by linearity over the whole of E). We easily verify that f is linear and bijective (the verification is left as an exercise). □

Remark 2.3 *As can be seen from the proof, the isomorphism of E onto E' depends on the choice of bases in E and in E and in general there is no canonical isomorphism.*

Proposition 2.6 *Let E and F be two K-vector spaces and $f : E \longrightarrow F$ a linear application. So* *a*) For any vector subspace G of E, we have

$$f^{-1}(f(G)) = G + \ker(f).$$

b) f eis injective, if and only if, for any vector subspace G of E, we have

$$f^{-1}(f(G)) = G.$$

c) For any vector subspace H of F, we have

$$f\left(f^{-1}(H)\right) = H \cap \operatorname{Im} f.$$

d) f is surjective, if and only if, for any vector subspace H of F, we have

$$f\left(f^{-1}(H)\right) = H.$$

Proof. *a*)Let $x \in E$, then we have

$$\begin{aligned} x \in f^{-1}(f(G)) &\Longleftrightarrow f(x) \in f(G) \\ &\Longleftrightarrow \exists a \in G : f(x) = f(a) \\ &\Longleftrightarrow \exists a \in G : f(x-a) = 0 \\ &\Longleftrightarrow \exists a \in G : x - a \in \ker(f) \\ &\Longleftrightarrow \exists a \in G, \exists b \in \ker(f) : x = a + b \\ &\Longleftrightarrow x \in G + \ker(f). \end{aligned}$$

b) $\Longrightarrow$) If we assume that f is injective, then $\ker(f) = \{0\}$, so, according to a), for any vector subspace G of E,

$$f^{-1}(f(G)) = G.$$

$\Longleftarrow$) If we assume that for any vector subspace G of E, we have $f^{-1}(f(G)) = G$, so in particular, we have

$$f^{-1}\left(f\left(\{0_E\}\right)\right) = \{0_{E^3}\}.$$

But, $f\left(\{0_E\}\right) = \{f\left(0_E\right)\} = \{0_F\}$, then

$$\ker(f) = f^{-1}\left(\{0_F\}\right) = \{0_E\}.$$

c) $\subset$) Let $y \in f\left(f^{-1}(H)\right)$. so there exists $x \in f^{-1}(H)$, such that $y = f(x)$. As $x \in f^{-1}(H)$, then $f(x) \in H$, thus $y \in H \cap \operatorname{Im} f$.

$\supset$) Let $y \in H \cap \operatorname{Im} f$, then we have

$$\begin{aligned} y \in H \cap \operatorname{Im} f &\Longrightarrow y \in H \text{ and } y \in \operatorname{Im} f \\ &\Longrightarrow y \in H \text{ and } \exists x \in E, : y = f(x) \\ &\Longrightarrow x \in f^{-1}(H) \text{ and } f(x) \in H \\ &\Longrightarrow f(x) \in f\left(f^{-1}(H)\right) \\ &\Longrightarrow y \in f\left(f^{-1}(H)\right). \end{aligned}$$

d) $\Longrightarrow$) Suppose f is surjective, then $\operatorname{Im} f = F$, so for any subspace H of E, we have

$$f\left(f^{-1}(H)\right) = H \cap F = H.$$

$\Longleftarrow$) Suppose that for any vector subspace H of F we have

$$f\left(f^{-1}(H)\right) = H,$$

so in particular, we will have $f\left(f^{-1}(F)\right) = F$. But, $f^{-1}(F) = E$, then $f(E) = F$, consequently, f is surjective. □

In the case where the spaces E and E' are finite-dimensional, the dimensions of the kernel and the image of f are related by the relation given in the following theorem, one of the most important in Linear Algebra:

Theorem 2.3 (Rank theorem) *Let E and E' be two finite-dimensional vector spaces and $f : E \longrightarrow E'$ a linear application. We then have:*

$$\dim E = \operatorname{rg} f + \dim(\operatorname{Ker} f).$$

Proof. Let's suppose $\dim E = n$, $\dim\operatorname{Ker} f = r$ and show that $\dim(\operatorname{Im} f) = n - r$. Let $\{w_1, \ldots, w_r\}$ be a basis of $\operatorname{Ker} f$, and $\{v_1, \ldots, v_{n-r}\}$ a family of vectors such that $\{w_1, \ldots, w_r, v_1, \ldots, v_{n-r}\}$ be a basis of E. Let $\mathcal{B} = \{f(v_1), \ldots, f(v_{n-r})\}$. We show that $\mathcal{B}$ is a basis of $\operatorname{Im} f$.

- $\mathcal{B}$ generates $\operatorname{Im} f$. Let $y = f(x) \in \operatorname{Im} f$. As $x \in E$, x is of the form $x = a_1 w_1 + \cdots + a_r w_r + b_1 v_1 + \cdots + b_{n-r} v_{n-r}$. Then we have :

$$y = a_1 f(w_1) + \cdots + a_r f(w_r) + b_1 f(v_1) + \cdots + b_{n-r} f(v_{n-r}),$$

$$= b_1 f(v_1) + \cdots + b_{n-r} f(v_{n-r})$$

which shows that $\mathcal{B}$ generates $\operatorname{Im} f$.

- $\mathcal{B}$ is linearly independent. Suppose that $\lambda_1 f(v_1) + \cdots + \lambda_{n-r} f(v_{n-r}) = 0$; we will obtain

$$f(\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r}) = 0,$$

then:

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} \in \operatorname{Ker} f.$$

Therefore, there is $a_1, \ldots, a_r \in \Bbbk$ such that:

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} = a_1 w_1 + \cdots + a_r w_r,$$

so,

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} - a_1 w_1 - \cdots - a_r w_r = 0.$$

Since the family $\{v_1, \ldots, v_{n-r}, w_1, \ldots, w_r\}$ is linearly independent, the coefficients of this linear combination are all zero; in particular : $\lambda_1 = 0, \ldots, \lambda_{n-r} = 0$, so, $\mathcal{B}$ is linearly independent. $\square$

To show that a linear map is bijective, it is necessary to show that it is injective and surjective; however, in the finite-dimensional case, if the dimension of the starting space and that of the arrival space are the same, it is sufficient to demonstrate one of the two properties - either injectivity or surjectivity, hence this important corollary.

Corollary 2.1 *Let $f \in \mathcal{L}(E, E')$, E, E' being two vector spaces of the same finite dimension (in particular, for example, if $f \in$ End E, with E of finite dimension). Then the following properties are equivalent:*

1. *f is injective.*
2. *f is surjective.*
3. *f is bijective.*

Proof. It is sufficient, of course, to show that 1 is equivalent to 2. As we have seen, f is injective if and only if Ker $f = \{0\}$. Since $\dim E = \operatorname{rg} f + \dim(\ker f)$, f is injective if and only if $\dim E = \operatorname{rg} f$, so, $\dim E = \dim(\operatorname{Im} f)$. Now, by hypothesis, $\dim E = \dim E'$, then f is injective if and only if $\dim(\operatorname{Im} f) = \dim E'$. Since $\operatorname{Im} f \subset E'$ this is equivalent to $\operatorname{Im} f = E'$, then f is surjective. $\square$

Remark 2.4 *This result is false in infinite dimension, a counterexample: the application:*

$$D: \begin{array}{ccc} \mathbb{R}[x] & \longrightarrow & \mathbb{R}[x] \\ P & \mapsto & P' \end{array}$$

is surjective and not injective.

Theorem 2.4 *Let E be a K-finite-dimensional vector space and $u : E \longrightarrow E$ an endomorphism of E. Then the following propositions are equivalent:*

i) $E = \ker u \oplus \operatorname{Im} u$.
ii) $\operatorname{Im} u = \operatorname{Im} u^2$.
iii) $\ker u = \ker u^2$.
iv) $\ker u \cap \operatorname{Im} u = \{0\}$.

Proof. *i)* $\Longrightarrow$ *ii)* Let's suppose that $E = \ker u \oplus \operatorname{Im} u$ and show that $\operatorname{Im} u = \operatorname{Im} u^2$. To do this, let us first note that everything $u \in L_K(E)$, we have $\operatorname{Im} u^2 \subseteq \operatorname{Im} u$. So, it is enough to show that $\operatorname{Im} u \subseteq \operatorname{Im} u^2$. For this, let $y \in \operatorname{Im} u$, so there exists $x \in E$, such that $y = u(x)$. Since $E = \ker u \oplus \operatorname{Im} u$, then $x = x_1 + u(x_2)$, with $x_1 \in \ker u$, then $y = u^2(x)$, consequently, $y \in \operatorname{Im} u^2$.

ii) $\Rightarrow$ *iii)* Let's suppose that $\operatorname{Im}(u) = \operatorname{Im} u^2$ and show that $\ker u = \ker u^2$. For this, let us also note that for any $u \in L_K(E)$, we have $\ker u \subseteq \ker u^2$. So, it is enough to show that $\ker u^2 \subseteq \ker u$. According to the rank theorem, we have

$$\dim(E) = \dim(\ker u) + \dim(\operatorname{Im} u) = \dim\left(\ker u^2\right) + \dim\left(\operatorname{Im} u^2\right).$$

Since $\operatorname{Im} u = \operatorname{Im} u^2$ then $\dim(\ker u) = \dim(\ker u^2)$, consequently, we will have $\ker u = \ker u^2$.

iii) $\Longrightarrow$ *iv)* Let's suppose that $\ker u = \ker u^2$ and show that $\ker u \cap \operatorname{Im} u = \{0\}$. Let $y \in E$, so we have

$$\begin{aligned} & y \in \ker u \cap \operatorname{Im} u \Leftrightarrow u(y) = 0 \text{ and } \exists x \in E : y = u(x) \\ & \Rightarrow u^2(x) = u(y) = 0 \qquad \Rightarrow x \in \ker u^2 \\ & \Rightarrow x \in \ker u \qquad \Rightarrow u(x) = 0 \qquad \Rightarrow y = 0 \end{aligned}$$

iv $\Rightarrow$ *i)* Trivial, because we know that

$$E = \ker u \oplus \operatorname{Im} u \Leftrightarrow \begin{cases} \dim(\ker u) + \dim(\operatorname{Im} u) = \dim(E) \\ \text{ et} \\ \ker u \cap \operatorname{Im} u = \{0\}. \end{cases}$$

□