CHAPTER 1

VECTOR SPACES

Geometry predominated in Greek mathematics and it was not until Descartes in the 17th century that the link was made, thanks to the notion of reference, between geometric notions: of points on the plane or in space, of curves and algebraic ones, of pairs or triplets of real numbers and equations. This approach proved fruitful for both geometers and analysts. It offered the former all the power of analysis to deal with geometric problems and, to the latter, the representations of geometry to visualize and state the phenomena of analysis. The generalization of the geometric notions of the plane $\mathbb{R}^2$ and of space $\mathbb{R}^3$ Higher-dimensional spaces were not immediately apparent. The formalism to address this problem was lacking. It was the self-taught German mathematician Hermann Grassmann who, in the 19th century, outlined the notions of vector space and dimension. His work was difficult to understand, and it was thanks to the Italian mathematician Giuseppe Peano that these concepts were refined and took their definitive form. Linear algebra has become the framework for the study of many theories, particularly in analysis. Today, with the development of computing tools, linear algebra is more easily implemented using matrix calculus.

This chapter covers some fundamental concepts required for this document. Mastery of the material in this chapter is essential. The concepts covered are:

—Vector spaces.

—Vector subspaces.

—Basis and dimension of a vector space.

—Sum of vector subspaces.

1.1 Vector space

Throughout this chapter $\mathbb{k}$ designates the field $\mathbb{R}$ or $\mathbb{C}$. We will give the axioms that define a vector space.

Definition 1.1 ($\mathbb{k}$-vector space) *Let $(\mathbb{k}, +, \times)$ be a field. We call a vector space over the field $\mathbb{k}$ any set E equipped with an internal composition law $+$ (addition)*

$$\circledast : \begin{cases} E \times E & \to & E \\ (x, y) & \mapsto & x \circledast y \end{cases}$$

and an external composition law $\cdot$ (multiplication by a scalar)

$$. : \begin{cases} \mathbb{k} \times E & \to & E \\ (\lambda, y) & \mapsto & \lambda.x \end{cases}$$

such that

1. *$(E, \circledast)$ is a commutative group. We denote 0_E its neutral element.*
2. *For all $(\alpha, \beta) \in \mathbb{k}^2$, and for all $(x, y) \in E^2$, we have*

- $(\alpha + \beta).x = \alpha.x \circledast \beta.x$ *Axiom 1*
- $(\alpha \times \beta).x = \alpha.(\beta.x)$ *Axiom 2*
- $\alpha.(x \circledast y) = \alpha.x \circledast \alpha.y$ *Axiom 3*
- $1_{\mathbb{k}}.x = x$ *Axiom 4*

We then say that $(E, \circledast, \cdot)$ is a K-vector space. The elements of $\mathbb{k}$ are called scalars, those of E, vectors. The neutral element of $(E, \circledast)$, 0_E is called the null vector.

Example 1.1 *The commutative field $\mathbb{k}$, is a vector space on itself, for addition and the product existing on $\mathbb{k}$.*

Example 1.2 *Let $n \in \mathbb{N}^*$ be a non-zero integer, we define the Cartesian product $\mathbb{k}^n$ as follows*

If $n = 1$, $\mathbb{k}^1 = \mathbb{k}$.

If $n = 2$, $\mathbb{k}^2 = \mathbb{k} \times \mathbb{k}$ is the set of pairs formed from elements of E.

If $n = 3$, $\mathbb{k}^3 = \mathbb{k}^2 \times \mathbb{k}$ by definition, it is therefore the set of couples of the type (a, z) with a in $\mathbb{k}^2$ so of the type $a = (x, y)$ with x and y in $\mathbb{k}$. Instead of noting $((x, y), z)$ the generic element of $\mathbb{k}^3$, we denote it (x, y, z) and we talk about the triplet, (x, y, z) of $\mathbb{k}^3$.

More generally, $\mathbb{k}^n$ *will be, by rare definition, the Cartesian product of* $\mathbb{k}^{n-1}$ *by* $\mathbb{k}$*, let* $\mathbb{k}^n = \mathbb{k}^{n-1} \times \mathbb{k}$*, and for convenience, we will denote* $(x_1, x_2, ..., x_n)$ *the generic element of* $\mathbb{k}^n$*, (with each* $x_i \in \mathbb{k}$*) called n-tuple.*

Formally, we therefore define $\mathbb{k}^n$ *from the couples. On* $\mathbb{k}^n$*, we define a vector space structure by posing:*

for $X = (x_1, x_2, ..., x_n)$ *and* $Y = (y_1, y_2, ..., y_n)$ *in* $\mathbb{k}^n$*, and* λ *in* $\mathbb{k}$*,*

$$\begin{aligned} X + Y &= (x_1 + y_1, ..., x_n + y_n), \\ \lambda.X &= (\lambda x_1, \lambda x_2, ..., \lambda x_n), \end{aligned}$$

and it is easy to verify that we have a vector space structure.

Example 1.3 *The set* $\mathbb{R}_n[x]$ *polynomial functions with coefficients in* $\mathbb{R}$ *of degree* $\leq n$*, that's to say :*

$$\mathbb{R}_n[x] = \{P : \mathbb{R} \to \mathbb{R} |\ P(x) = a_0 + a_1 x + ... + a_n x^n,\ a_i \in \mathbb{R}\}$$

is a vector space over $\mathbb{R}$ *for the laws:*

$$\begin{aligned} (a_0 + a_1 x + ... + a_n x^n) + (b_0 + b_1 x + ... + b_n x^n) &: = (a_0 + b_0) + ... + (a_n + b_n) x^n \\ \lambda.(a_0 + a_1 x + ... + a_n x^n) &: = \lambda a_0 + \lambda a_1 x + ... + \lambda a_n x^n \end{aligned}$$

More generally, the set $\mathbb{R}[x]$ *polynomial functions of all possible degrees with coefficients in* $\mathbb{R}$ *is a vector space over* $\mathbb{R}$ *with the laws:*

$$\begin{aligned} \sum_k a_k x^k + \sum_k b_k x^k &: = \sum_k (a_k + b_k) x^k \\ \lambda. \sum_k a_k x^k &: = \sum_k (\lambda a_k) x^k \end{aligned} \tag{1.1}$$

(the sums only contain a finite number of non-zero terms).

Example 1.4 *Let* A *be a non-empty set and* E *a vector space over the field* $\mathbb{k}$*. The set* E^A *applications of* A *in* E *is equipped with a vector structure on* $\mathbb{k}$ *in the following way.*

If f *and* g *are two functions from* A *to* E*, and* λ *a scalar in* $\mathbb{k}$ *we define the function* $f + g$ *by*

$$\forall x \in A,\ (f + g)(x) = f(x) + g(x),$$

and the function $\lambda.f$ *by*

$$\forall x \in A, (\lambda.f)(x) = \lambda.f(x),$$

and here again it is a exercise to verify that we have a vector structure for the space of applications of A *in the vector space* E*. Its zero vector is the identically zero function on* A *with values in* E*,*

$$\begin{aligned} 0_{E^A} : \ A &\to E \\ x &\mapsto 0_E \end{aligned}$$

Example 1.5 *Let $E_1, ..., E_n$. n vector spaces over the commutative field $\mathbb{k}$, by proceeding by recurrence as for $\mathbb{k}^n$, we define the Cartesian product $E = E_1 \times E_2 \times ... \times E_n$ still noted $E = \prod_{i=1}^{n} E_i$, whose elements are the n-tuples $X = (X_1, ..., X_n)$, with $\forall i = 1...n$, $X_i \in E_i$.*

It is easy to verify that for the addition $+$ defined by

$$(X_1, ..., X_n) + (Y_1, ..., Y_n) = (X_1 + Y_1, ..., X_i + Y_i, ..., X_n + Y_n)$$

E is an additive group, and that with the product

$$\lambda.X = (\lambda X_1, \lambda X_2, ..., \lambda X_n),$$

we provide E with a vector structure.

1.2 Calculation rules

Proposition 1.1 *Let $(E, +, \cdot)$ be a $\mathbb{k}$-vector space. For all scalars $\alpha, \beta, \lambda \in \mathbb{k}$ and for all vectors $x, y \in E$, we have*

1. $0_{\mathbb{k}} \cdot x = 0_E$
2. $(-1) \cdot x = -x$
3. $(-\lambda) \cdot x = -(\lambda \cdot x) = \lambda \cdot (-x)$
4. $(\alpha - \beta).x = \alpha.x - \beta.x$
5. $\lambda.(x - y) = \lambda.x - \lambda.y$
6. $\lambda \cdot 0_E = 0_E$
7. $\lambda \cdot x = 0_E \Longleftrightarrow (\lambda = 0_{\mathbb{k}} \text{ or } x = 0_E)$.

Proof. 1. We have

$$\begin{aligned} 0_{\mathbb{k}} \cdot x + 0_E &= 0_{\mathbb{k}} \cdot x \text{ because } (E, +) \text{ is a group} \\ &= (0_{\mathbb{k}} + 0_{\mathbb{k}}) \cdot x \text{ because } \mathbb{k} \text{ is a field} \\ &= 0_{\mathbb{k}} \cdot x + 0_{\mathbb{k}} \cdot x. \end{aligned}$$

From where $0_{\mathbb{k}} \cdot x = 0_E$ by subtracting $0_{\mathbb{k}} \cdot x$ to the right of the two members of this equality.
2. We have :

$$\begin{aligned} x + (-1) \cdot x &= 1.x + (-1) \cdot x \text{ thanks to the axiom 4} \\ &= (1 + (-1)).x \text{ thanks to the axiom 1} \\ &= 0_{\mathbb{k}} \cdot x \text{ because } \mathbb{k} \text{ is a field} \\ &= 0_E \text{ according to 1 previous ones.} \end{aligned}$$

then $(-1).x$ is the opposite of x. We can then write : $-x = (-1).x$.

3. We have:

$$\begin{aligned}(-\lambda)x &= (-1.\lambda)x \text{ because } \Bbbk \text{ is a field}\\ &= (-1).(\lambda.x) \text{ thanks to the axiom 2}\\ &= -\lambda.x \text{ according to 2 previous ones.}\end{aligned}$$

4. We have:

$$\begin{aligned}(\alpha-\beta).x &= (\alpha+(-\beta)).x \text{ because } \Bbbk \text{ is a field}\\ &= \alpha.x + (-\beta).x \text{ thanks to the axiom 1}\\ &= \alpha.x - \beta.x \text{ according to 3 previous ones}\end{aligned}$$

5. We have:

$$\begin{aligned}\lambda.(x-y) &= \lambda.(x+(-y)) \text{ because } (E,+) \text{ is a group}\\ &= \lambda.x + \lambda.(-y) \text{ thanks to the axiom 3}\\ &= \lambda.x + \lambda.(-1y) \text{ according to 2 previous ones}\\ &= \lambda.x + (\lambda.(-1)).y \text{ thanks to the axiom 2}\\ &= \lambda.x + (-\lambda).y \text{ because } \Bbbk \text{ is a field}\\ &= \lambda.x - \lambda.y \text{ according to 3 previous ones}\end{aligned}$$

6. On a:

$$\begin{aligned}\lambda\cdot 0_E &= \lambda\cdot(x-x) \text{ car } (E,+) \text{ is a group}\\ &= \lambda\cdot x + \lambda\cdot(-x) \text{ hanks to the axiom 1}\\ &= \lambda\cdot x - \lambda\cdot x \text{ according to 3 previous ones}\\ &= 0_E \text{ car } (E,+) \text{ is a group.}\end{aligned}$$

7. Let's suppose that $\lambda\cdot x = 0_E$, if $\lambda = 0_{\Bbbk}$ so according to 3, λ. Otherwise, if $\lambda \neq 0_{\Bbbk}$ the as $\Bbbk$ is a field λ^{-1} exists and

$$x = 1.x = \left(\lambda.\lambda^{-1}\right).x = \lambda^{-1}.(\lambda.x) = \lambda^{-1}.0_E = 0_E,$$

and so $x = 0_E$. The converse is obvious. □

1.3 Vector subspaces

Definition 1.2 (Vector subspace) *We call a vector subspace of a vector space E over the field $\Bbbk$, any subset F of E which is an additive subgroup of E is such that $\forall\lambda \in \Bbbk$, $\forall x \in F$, $\lambda.x \in F$.*

Remark 1.1 *It is then obvious that F is a vector space on $\mathbb{k}$, the conditions of the definition 1.1 being verified.*

Example 1.6 $\{0_E\}$ *and E are vector subspaces of E.*

Definition 1.3 (Linear combination) ◦ *Let $x_1, x_2, ..., x_n$ n vectors of a $\mathbb{k}$-vector space E. We call any vector a linear combination of these n vectors $x \in E$ of the form*

$$x = \lambda_1 x_1 + ... + \lambda_p x_n = \sum_{k=1}^{n} \lambda_k x_k$$

where $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$.

◦ *If A is a part of E, we call a linear combination of elements of A any linear combination of a finite number of elements of A.*

In principle, to show that F is a vector subspace, one would have to verify the eight axioms of the Definition 1.1 In fact, it is sufficient to check the "stability" of the composition laws as stated in the following proposition:

Proposition 1.2 *A part F of a vector space E on $\mathbb{k}$ is a vector subspace of E if and only if*

1. $F \neq \varnothing$
2. $\forall (\alpha, \beta) \in \mathbb{k}^2, \forall (x, y) \in F^2, \alpha.x + \beta.y \in F.$

Proof. $\Longrightarrow$) First if F is a subspace of E, as an additive subgroup F is non-empty because it contains 0, the null element of E, then if x et y are in F and α and β in the field $\mathbb{k}$, $\alpha.x$ and $\beta.y \in F$ (see definition 1.2) which is stable for addition, therefore $\alpha.x + \beta.y \in F$.

$\Longleftarrow$) If 1. and 2. are verified, with $\alpha = 1$ and $\beta = -1$ and x and y in F we have:

$$(F \neq \varnothing) \text{ et } \left(\forall (x, y) \in F^2, x - y \in F\right),$$

which already justifies that F is an additive subgroup of E, then 2. with $\beta = 0$ gives

$$\forall \lambda \in \mathbb{k}, \forall x \in F, \lambda.x \in F.$$

We have F vector subspace of E. □

Remark 1.2 *As we saw during the proof, if F is a vector subspace, then F necessarily contains the null vector.*

1.3.1 Fundamental examples of vector subspaces

1. Vector line:

Let $v \in E$, $v \neq 0$, then :

$$F = \{y \in E |\ \exists \lambda \in \Bbbk : y = \lambda v\}$$

is a vector subspace of E called a vector line generated by v.

Indeed $F \neq \emptyset$, because $v \in F$. What's more, F is stable for the laws of E, because if $x, y \in F$ (that's to say: $x = \lambda v$, $y = \mu v$), we have:

$$x + y = \lambda v + \mu v = (\lambda + \mu)v \in F$$

Likewise, if $x \in F$ (that's to say $x = \lambda v$), we have: $\mu x = \mu(\lambda v) = (\mu\lambda)v \in F$.

2. Vector plane:

Let $x_1, x_2 \in E$ then :

$$F = \{y \in E |\ \exists \lambda_1, \lambda_2 \in \Bbbk : y = \lambda_1 x_1 + \lambda_2 x_2\}$$

F is a vector subspace of E, called the subspace generated by x_1, x_2. If x_1 and x_2 are not null and x_2 does not belong to the vector line generated by x_1, F is said to be a vector plane generated by x_1 and x_2.

3. Generated subspace:

More generally, if $x_1, x_2, ..., x_p \in E$ then :

$$F = \{y \in E |\ \exists \lambda_1, ..., \lambda_p \in \Bbbk : y = \lambda_1 x_1 + ... + \lambda_p x_p\}$$

is a vector subspace of E denoted $Span\{x_1, x_2, ..., x_p\}$, said sub-space occupied by $x_1, ..., x_p$, or also space of linear combinations of $x_1, x_2, ..., x_p$. We will see later that, basically, all vector subspaces are of this type, that is to say obtained by "linear combinations" of a family of elements of E.

Remark 1.3 *Let E be the vector space of vectors with origin O. A vector line is a line passing through O. Similarly, a vector plane is a plane passing through O. More generally, a vector subspace of $\mathbb{R}^n$ can be visualized as a "p-dimensional plane" passing through O. One could give a precise meaning to the notion of "p-dimensional plane", but this is not necessary. Let us retain for the moment the fact that it must pass through O, because every vector subspace must contain the null vector. Thus, for example, a line not passing through O is not a vector subspace: the points of the line are the endpoints of the vectors from O and the null vector is not among them.*

Example 1.7 *Let*

$$F = \{(x, y, z) \in \mathbb{R}^3 |\ 3x + y + 2z = 0\}.$$

F is a vector subspace of $\mathbb{R}^3$. Indeed, let us $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in F$; we have:

$$3x_1 + y_1 + 2z_1 = 0 \text{ and } 3x_2 + y_2 + 2z_2 = 0$$

from which, by adding: $3(x_1 + x_2) + (y_1 + y_2) + 2(z_1 + z_2) = 0$, that's to say

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in F.$$

Similarly, we see that if $\lambda \in \mathbb{k}$ and $v \in F$ we have: $\lambda v \in F$.

Example 1.8 *We have:*

$$G = \{(x, y, z) \in \mathbb{R}^3 |\ x + 4y + z = 1\},$$

is not a vector subspace of $\mathbb{R}^3$ because $0_{\mathbb{R}^3} = (0, 0, 0) \notin G$ $(0 + 4.0 + 0 \neq 1)$.

Proposition 1.3 *Let F and G be two vector subspaces of E.*

1. *$F \cap G$ is a vector subspace of E.*
2. *$F \cup G$ is not in general a vector subspace of E.*
3. *The complementary $E \backslash F$ of a vector subspace F is not a vector subspace of E.*

Proof. **1.** We have first $F \cap G \neq \varnothing$, because $0_E \in F \cap G$.

Let $x, y \in F \cap G$, we have: $x, y \in F$ then $x + y \in F$. Likewise, if $x, y \in G$, $x + y \in G$ and consequently $x + y \in F \cap G$.

If $\lambda \in \mathbb{k}$ and $x \in F \cap G$, we have: $x \in F$, then $\lambda x \in F$, and $x \in G$, then $\lambda x \in G$, from where: $\lambda x \in F \cap G$.

2. This is because in general $F \cup G$ is not stable by the sum. For example, let $E = \mathbb{R}^2$, F the vector line generated by $(1, 0)$ and G the vector line generated by $(0, 1)$. We have : $(1, 0) \in F$ then $(1, 0) \in F \cup G$. $(0, 1) \in G$ then $(0, 1) \in F \cup G$ but: $w = (1, 0) + (0, 1) = (1, 1) \notin F \cup G$.

3. $E \backslash F$ does not contain 0_E, so it is not a vector subspace (Remark 1.2). □

1.4 Basis (in finite dimension)

Definition 1.4 (Generating family) *A family of vectors $\{v_1, ..., v_p\}$ of a vector space E is said to be generating, if $E = Span\{v_1, ..., v_p\}$, which means that everything $\forall x \in E$, decomposes on vectors v_i, or even that everything $x \in E$ is a linear combination of vectors v_i,*

$$x = \lambda_1 v_1 + ... + \lambda_p v_p = \sum_{k=1}^{p} \lambda_k v_k.$$

Remark 1.4 *Such a (finite) family does not always exist. Consider for example* $\mathbb{R}[x]$ *awith the vector space structure defined by the laws (1.1) and let* $\{P_1, ..., P_n\}$ *a finite family of polynomials. It cannot be generating because, by performing linear combinations, we will only obtain polynomials of* $\deg < Sup\{deg\ of\ (P_i)\}$.

Example 1.9 *In* $\mathbb{R}^2$*, let* $v_1 = (1, 1)$ *and* $v_2 = (1, -1)$*. Let us show that* $\{v_1, v_2\}$ *is generating. Let* $x = (a, b) \in \mathbb{R}^2$ *with* a, b *arbitrary: it is a question of showing that it exists* $x_1, x_2 \in \mathbb{R}$ *such that* $x = x_1v_1 + x_2v_2$*, that's to say:*

$$x = (a, b) = (x_1, x_1) + (x_2, -x_2) = (x_1 + x_2, x_1 - x_2)$$

This means that $\forall (a, b) \in \mathbb{R}^2$, $\exists\ x_1, x_2 \in \mathbb{R}$ *checking the system:*

$$\begin{cases} x_1 + x_2 = & a \\ x_1 - x_2 = & b, \end{cases}$$

By solving, we find in fact:

$$x_1 = \frac{a+b}{2}\ et\ x_2 = \frac{a-b}{2},$$

solution defined for arbitrary a, b*. Therefore* $\{v_1, v_2\}$ *is generative.*

Definition 1.5 *A vector space is said to be finite-dimensional if there exists a finite generating family; otherwise, it is said to be infinite-dimensional.*

Definition 1.6 (Linearly independent family) *Let* $\{v_1, ..., v_p\}$*, a finite family of elements of* E*. We say that it is free, if:*

$$\lambda_1 v_1 + ... + \lambda_p v_p = 0 \Longrightarrow \lambda_1 = \lambda_2 = ... = \lambda_p = 0.$$

We also say that vectors $v_1, ..., v_p$ *are linearly independent. A family that is not free is said to be linked (its vectors are also said to be linked or linearly dependent).*

Example 1.10 *In* $\mathbb{R}^3$*, vectors* $v_1 = (1, 1, -1)$*,* $v_2 = (0, 2, 1)$ *and* $v_3 = (0, 0, 5)$ *are linearly independent. Indeed, suppose that there exist real numbers* λ_1*,* λ_2*,* λ_3 *so that* $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^3}$*, that's to say:*

$$\lambda_1(1, 1, -1) + \lambda_2(0, 2, 1) + \lambda_3(0, 0, 5) = 0_{\mathbb{R}^3}$$

We obtain

$$(\lambda_1, \lambda_2, -\lambda_1 + \lambda_2 + 5\lambda_3) = 0_{\mathbb{R}^3},$$

then

$$\begin{cases} \lambda_1 = & 0 \\ \lambda_2 = & 0 \\ -\lambda_1 + \lambda_2 + 5\lambda_3 = & 0 \end{cases}$$

which immediately gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Example 1.11 *In $\mathcal{F}(\mathbb{R},\mathbb{R})$ the family $\{\sin,\cos,\exp\}$ is linearly independent. Let $\alpha,\beta,\gamma\in\mathbb{R}$ such that $\alpha\sin+\beta\cos+\gamma\exp=0$. Then for all $x\in\mathbb{R}$, $\alpha\sin(x)+\beta\cos(x)+\gamma\exp(x)=0$, which is also written*

$$\alpha\frac{\sin(x)}{\exp(x)}+\beta\frac{\cos(x)}{\exp(x)}+\gamma=0.$$

It is easily shown using the gendarmes' theorem that

$$lim_{+\infty}\frac{\sin(x)}{\exp(x)}=lim_{+\infty}\frac{\cos(x)}{\exp(x)}=0.$$

We deduce that $\gamma=0$. We then have $\forall x\in\mathbb{R}$, $\alpha\sin(x)+\beta\cos(x)+\gamma\exp(x)=0$. If we do $x=0$ we obtaint $\beta=0$ and if we do $x=\pi/2$,it comes that $\alpha=0$. It was then clearly shown that $\alpha=\beta=\gamma=0$.

Proposition 1.4 *A family $\{v_1,...,v_p\}$ is linked if and only if at least one of the vectors v_i is written as a linear combination of the other vectors in the family.*

Proof. $\Longrightarrow$) : If $\{v_1,...,v_p\}$ is linearly dependent family, there exists λ_1,..., λ_p not all null such that $\lambda_1v_1+...+\lambda_pv_p=0$. If, for example $\lambda_1\neq0$, we can write:

$$v_1=-\frac{\lambda_2}{\lambda_1}v_2+...+\frac{-\lambda_p}{\lambda_1}v_p$$

$\Longleftarrow$) : suppose for example, that v_1 is a linear combination of vectors $\lambda_2v_2,...,\lambda_pv_p$, so there exists $\mu_2,...,\mu_p\in\mathbb{k}$ such thqt $v_1=\lambda_2v_2+...+\lambda_pv_p$, that's to say:

$$v_1-\lambda_2v_2-...-\lambda_pv_p=0.$$

So there exists a linear combination of vectors $\{v_1,...,v_p\}$ which is zero, without the coefficients all being zero. So the family is linked (linearly dependent). □

Proposition 1.5 *Let $\{v_1,...,v_p\}$ a linearly independent family and x any vector in the space generated by the vectors v_i (that is, x is a linear combination of the v_i). So the decomposition of x on the v_i is unique.*

Proof. Let

$$\begin{aligned} x &= \lambda_1v_1+...+\lambda_pv_p,\\ x &= \beta_1v_1+...+\beta_pv_p, \end{aligned}$$

two decompositions of x. Taking the difference we find:

$$(\lambda_1-\beta_1)\,v_1+...+(\lambda_p-\beta_p)\,v_p=0,$$

Since the family is free, we have $(\lambda_1-\beta_1)=...=(\lambda_p-\beta_p)=0$, that's to say: $\lambda_1=\beta_1,...,\lambda_p=\beta_p$. □

Definition 1.7 (Basis) *A family that is both generating and linearly independent is called a basis.*

Proposition 1.6 *A familY $\{v_1, ..., v_p\}$ is a basis of E if and only if every $x \in E$ decomposes in a unique way on v_i. That's to say :*
$\forall x \in E$ there is a unique one n-scalars $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$ so that:

$$x = \lambda_1 v_1 + ... + \lambda_n v_n.$$

Proof. The existence of decomposition for all $x \in E$ is equivalent to the fact that the family is generative; uniqueness to the fact that the family is linearly independent. □

Example 1.12 (Canonical basis of $\mathbb{R}^n$) *Let the vectors be:*

$$e_1 = (1, 0, ..., 0), ..., e_i = (0, ..., \underbrace{1}_{i^e rang}, ..., 0), ..., e_n = (0, 0, ..., 1).$$

We already know that they form a generating family. Let us show that it is free. We have:

$$\lambda_1 e_1 + ... + \lambda_i e_i + ... + e_n v_n = 0_{\mathbb{R}^n},$$

that's to say:

$$\lambda_1(1, 0, ..., 0) + ... + \lambda_i(0, ..., \underbrace{1}_{i^e rang}, ..., 0) + ... + \lambda_n(0, 0, ..., 1) = 0_{\mathbb{R}^n},$$

then

$$(\lambda_1, \lambda_2, ..., \lambda_n) = 0_{\mathbb{R}^n}.$$

Therefore $\{e_1, ..., e_n\}$ is a basis of $\mathbb{R}^n$, called canonical basis.

Example 1.13 (*Canonical basis of $\mathbb{R}_n[x]$*) *The family $\mathcal{B} = \{1, x, ..., x^n\}$ is a basis of $\mathbb{R}_n[x]$, indeed, any $P(x) = a_0 + a_1 x + ... + a_n x^n$, $a_i \in \mathbb{R}$; B is therefore generative. Moreover :*

$$\lambda_0 1 + \lambda_1 x + ... + \lambda_n x^n = 0 \Longrightarrow \lambda_0 = \lambda_1 = ... = \lambda_n = 0.$$

Example 1.14 *Let $F = \{(x, y, z) \in \mathbb{R}^3 |\ 2x + y + 2z = 0\}$. Find a basis of F. We saw that F is a vector subspace of $\mathbb{R}^3$, we have: $(x, y, z) \in F \Leftrightarrow y = -2x - 2z$ then:*

$$u \in F \Leftrightarrow u = (x, -2x - 2z, z) \Leftrightarrow u = x(1, -2, 0) + z(0, -2, 1).$$

Therefore the vectors $v_1 = (1, 2, 0), v_2 = (0, -3, 1)$ form a generating family of F. On the other hand:

$$\lambda_1 v_1 + \lambda_2 v_2 = 0 \Leftrightarrow \lambda_1(1, -2, 0) + \lambda_2(0, -2, 1) = (0, 0, 0),$$

which is equivalent to $\lambda_1 = \lambda_2 = 0$. Then $\{v_1, v_2\}$ is linearly independent and therefore it is a basis of F.

Proposition 1.7 *We have :*

1. *$\{x\}$ is a linearly independent family $\Longleftrightarrow x \neq 0_E$.*
2. *Any family containing a generating family is generating.*
3. *Any subfamily of a linearly independent family is linearly independent.*
4. *Any family containing a linearly dependent family is linearly dependent.*
5. *Any family $\{v_1, ..., v_p\}$ one of the vectors of which v_i is null, is linearly dependent.*

Proof.

1. $\Longleftarrow$) According to Proposition 1.1 (7), $\lambda \cdot x = 0_E \Longleftrightarrow (\lambda = 0_{\mathbb{k}}$ or $x = 0_E)$. Then, if $x \neq 0$, $\lambda x = 0$ implies that $\lambda = 0$, which means that $\{x\}$ is a linearly independent family.

   $\Longrightarrow$) Suppose that $\{x\}$ is a linearly independent family. So, according to the definition of linearly independent family, if $\lambda x = 0$ we necessarily have $\lambda = 0$, which means, always according to the proposition 1.1 (7), that $x \neq 0$.

2. Let $\{v_1, ..., v_p\}$ a generative family and $x = \lambda_1 v_1 + ... + \lambda_p v_p$ an arbitrary element of E. We can also write:

   $$x = \lambda_1 v_1 + ... + \lambda_p v_p + 0w_1 + ... + 0w_q, \quad w_1, ..., w_q \in E.$$

   Then any $x \in E$ is a linear combination of $v_1, ..., v_p, w_1, ..., w_q$.

3. Let $\mathcal{T} = \{v_1, ..., v_p\}$ is a linearly independent family and $\mathcal{T}'$ a subfamily of $\mathcal{T}$. If we are going to change the numbering, we can assume that $\mathcal{T}\prime = \{v_1, ..., v_k\}$ (with $k < p$). If $\mathcal{T}\prime$ was linearly dependent, one of the vectors $v_1, ..., v_k$ would be a linear combination of the others. There would therefore exist an element of $\mathcal{T}$ which would be written as a linear combination of certain elements of $\mathcal{T}$. But this is impossible because $\mathcal{T}$ is linearly independent family (see Proposition 1.4).

4. Let $\mathcal{F} = \{v_1, ..., v_p\}$ a linearly dependent family and $\mathcal{G} = \{v_1, ..., v_p, w_1, ..., w_q\}$. According to proposition 1.4, one of v_i is a linear combination of the others. Now, vectors v_i belong to $\mathcal{G}$; so one of the elements of $\mathcal{G}$ is a linear combination of the others, and therefore $\mathcal{G}$ is linearly dependent.

5. Obvious from 4., because it is a family containing $\{0\}$, and $\{0\}$ is a linearly dependent, from 1.

□

1.5 Dimension of a vector space

Definition 1.8 (Dimension of a vector space) *If $E = \{0\}$, we say that E is of dimension 0 and we denote $\dim E = 0$. Otherwise, if E is a vector space on $\mathbb{k}$ of finite dimension not reduced to $\{0\}$,we call dimension of E the cardinality of a basis of E and we denote it $\dim_{\mathbb{k}} E$.*

Example 1.15 $\dim \mathbb{R}^n = n$, $\dim \mathbb{R}_n[x] = n + 1$.

Remark 1.5 *A family f of at least $n+1$ vectors in a space E of dimension n is always linearly dependent. Indeed, if it were linearly independent then we would have a linearly independent family f of cardinality greater than that of any basis $\mathcal{B}$ of E. So, $\mathcal{B}$ is a generating family of E and the cardinality of a linearly independent family is always smaller than that of a generating family.*

1.6 Dimension of a vector subspace

Proposition 1.8 *Let E a finite-dimensional vector space n and F a vector subspace of E. We have:*

1. *F is of finite dimension and $\dim F \leq \dim E$.*
2. *$(\dim F = \dim E) \Leftrightarrow F = E$*

Remark 1.6 *To verify that two vector subspaces F and G are equal*
- *We show that $F \subset G$.*
- *We show that $\dim F = \dim G$.*

Example 1.16 *Let $E = \mathbb{R}^4$ and*

$$\begin{aligned} F &= Vect((1,1,\alpha,3),(0,1,1,2)), \\ G &= \{(x,y,z,t) \in \mathbb{R}^4 |\ x - y + z = 0, x + 2y - t = 0\}. \end{aligned}$$

Let us look at what condition on $\alpha \in \mathbb{R}$ do we have $F = G$?

Let us first assume that $F = G$. Then $(1,1,\alpha,3) \in G$ and must satisfy in particular the equation $x - y + z = 0$. We then find that $\alpha = 0$.

Let us show that if $\alpha = 0$ then $F = G$. We know that $F = Vect((1,1,0,3),(0,1,1,2))$. The vectors $(1,1,0,3),(0,1,1,2)$ generate F and they are non-collinear so they form a linearly independent family. Therefore, it is a basis of F and $\dim F = 2$. Moreover

$$\begin{aligned} G &= \{(x,y,z,t) \in E |\ x - y + z = 0, x + 2y - t = 0\} \\ &= \{(x,y,y-x,x+2y) |\ x,y \in \mathbb{R}\} \\ &= Vect((1,0,-1,1),(0,1,1,2)), \end{aligned}$$

we then show in the same way as before that $\dim G = 2$. *What's more* $(1,1,0,3)$ *and* $(0,1,1,2)$ *satisfies the system*

$$\begin{cases} x - y + z = 0 \\ x + 2y - t = 0 \end{cases}$$

then $(1,1,0,3),(0,1,1,2) \in G$ *and as* G *is a subspace,*

$$F = Vect((1,1,\alpha,3),(0,1,1,2) \subset G.$$

Finally, as $\dim F = \dim G$, *we obtain* $F = G$.

1.7 Sum of vector subspaces

Definition 1.9 (Sum of two vector subspaces) *Let* F *and* G *two vector subspaces of a* $\Bbbk$-*vector space* E. *We call the sum of* F *and* G *and we denote by* $F+G$ *the vector subspace of* E *given by*

$$F + G = \{x + y \mid (x,y) \in F \times G\}.$$

Remark 1.7 *The subset* $F+G$ *is indeed a vector subspace of* E. *Indeed,* $F+G \subset E$ *because* E *is stable for addition. Moreover,* $F+G$ *is non-empty because* F *and* G *are. Finally, if* $u = x + y \in F + G$ *and* $u' = x' + y' \in F + G$ *with* $x, x' \in F$ *and* $y, y' \in G$ *then,for* $\alpha, \beta \in \Bbbk$.

$$\alpha u + \beta u' = \underbrace{\alpha x + \beta x'}_{\in F} + \underbrace{\alpha y + \beta y'}_{\in G} \in F + G$$

because F *and* G *are vector subspaces.*

Proposition 1.9 *Let* F *and* G *two vector subspaces of a* $\Bbbk$-*vector space* E. *Then* $F+G$ *is the smallest vector subspace of* E *container* $F \cup G$.

Proof. We proved in the previous remark that $F+G$ is a vector subspace of E. It contains F and G because 0_E is an element of F and G and therefore $F = F + 0_E \subset F + G$ and $G = 0_E + G \subset F + G$. Moreover, if we consider a subspace H of E which contains $F \cup G$ so let's show that $F + G \subset H$. Let $x + y \in F + G$ with $x \in F$ and $y \in G$. As $F \cup G \subset H$, we also have $x, y \in H$ and since H is a vector subspace, it follows that $x + y \in H$. Then $F + G \subset H$ and $F + G$ is the smallest vector subspace of E containing F and G. □

Example 1.17 *In* $\mathcal{F}(\mathbb{R},\mathbb{R})$, *Let* $F = Vect(\sin)$ *and* $G = Vect(\exp)$ *then:*

$$F + G = Vect(\sin, \exp) = \{x \mapsto \alpha \sin(x) + \beta \exp(x) \mid \alpha, \beta \in \mathbb{R}\}$$

Proposition 1.10 *Let A and B be two parts of a $\mathbb{k}$-vector space E, then*

$$Vect(A) + Vect(B) = Vect(A \cup B).$$

Example 1.18 *In space $\mathbb{R}^3$, we consider the parties $F = \{(x, 0, 0) \mid x \in \mathbb{R}\}$ and $G = \{(x, x, 0) \mid x \in \mathbb{R}\}$. Let us show that these are vector subspaces of $\mathbb{R}^3$ and determine the subspace $F + G$. We have $F = Vect(1, 0, 0)$ and $G = Vect(1, 1, 0)$ then F and G are vector subspaces of $\mathbb{R}^3$. Moreover $F + G = Vect((1, 0, 0), (1, 1, 0))$ and we recognize that $F + G$ is the vector plane of $\mathbb{R}^3$ generated by $(1, 0, 0)$ and $(1, 1, 0)$.*

1.7.1 Direct sum, supplementary subspaces

Definition 1.10 (Direct sum) *We say that two vector subspaces F and G of E are in direct sum if $F \cap G = \{0_E\}$. We then note $F \oplus G$ their sum.*

In other words:

$$\zeta = F \oplus G \Leftrightarrow \begin{cases} \zeta = F + G \\ et \\ F \cap G = \{0_E\} \end{cases}$$

Example 1.19 *In $\mathbb{C}$, the vector subspaces $F = \mathbb{R}$ and $G = i\mathbb{R}$ are in direct sum:*

Let $x \in F \cap G$, then $x \in F$ thus x is real, and $x \in G$ then x is pure imaginary. x is a complex that is both real and pure imaginary, so $x = 0_E$.

Example 1.20 *In $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$, we consider*

$$F = \{f \in E \mid f(0) = 0\}\} \text{ et } G = Vect(x \mapsto 1).$$

It is clear that F and G are vector subspaces of E. G is the set of constant applications of $\mathbb{R}$ in $\mathbb{R}$.

Let $f \in F \cap G$. $f \in F$ then $f(0) = 0$. Likewise $f \in G$ so there exists $a \in \mathbb{R}$ such that $f(x) = a$. But $a = f(0) = 0$ then $f = 0_E$. Thus F and G are in direct sum.

Proposition 1.11 *Let F and G be two vector subspaces of the $\mathbb{k}$-espace vectoriel E. F and G are in direct sum if and only if $\forall x \in F + G, \exists!(x_1, x_2) \in F \times G : x = x_1 + x_2$ (that is, the decomposition of x is unique).*

Proof. $\Longrightarrow$) Suppose that F and G are in direct sum and that $x \in F + G$. By definition, there exists $x_1 \in F$ and $x_2 \in G$ such that $x = x_1 + x_2$. Let's assume that there exists $x'_1 \in F$ and $x'_2 \in G$ such that we still have $x = x'_1 + x'_2$. As $x = x_1 + x_2 = x'_1 + x'_2$, we have the equality : $x_1 - x'_1 = x_2 - x'_2$. Let us denote this vector by y. As F and G are vector subspaces of E,

$y = x_1 - x'_1 \in F$ and $y = x_2 - x'_2 \in G$. Therefore $y \in F \cap G$. But F and G being in direct sum, we have $F \cap G = \{0_E\}$ then $y = 0$. Therefore, $x_1 = x'_1$ and $x_2 = x'_2$ and then uniqueness.

$\Longleftarrow$) Let $x \in F \cap G$. There are then two pairs of $F \times G$ allowing to decompose x into a vector of F and a vector of G:$(x, 0)$ et $(0, x)$. By hypothesis, they are equal: $(x, 0) = (0, x)$. Therefore $x = 0$ and the subspaces F and G are in direct sum. □

Definition 1.11 (Supplementary subspaces) *Let E be a vector space and F, G two vector subspaces of E. We say that F and G are supplementary (or that G is a supplementary of F), if $E = F \oplus G$.*

Proposition 1.12 *Let E be a vector space and F, G two vector subspaces of E. Then $E = F \oplus G$ (F et G are supplementary) if and only if for any basis $\mathcal{B}_1$ of F and for any basis $\mathcal{B}_2$ of G, $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a basis of E.*

Proof. $\Longleftarrow$) Let $\mathcal{B}_1 = \{v_\alpha\}_{\alpha \in A}$ and $\mathcal{B}_2 = \{w_\beta\}_{\beta \in B}$ basis of F and G respectively and suppose that $\{v_\alpha, w_\beta\}_{(\alpha,\beta) \in A \times B}$ is a basis of E. Then all $x \in E$ is written in a unique way:

$$x = \lambda_1 v_{\alpha_1} + ... + \lambda_p v_{\alpha_p} + \mu_1 w_{\beta_1} + ... + \mu_q w_{\beta_q},$$

that is to say everything $x \in E$ is written in a unique way $x = x_1 + x_2$ with $x_1 \in F$ and $x_2 \in G$, then $E = F \oplus G$.

$\Longrightarrow$) If $E = F \oplus G$, any $x \in E$ decomposes in a unique way on F and G and, therefore, on the family $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$. We deduce that $\mathcal{B}$ is a basis of E. □

Corollary 1.1 *Let E be a vector space. For any vector subspace F, there always exists a supplementary. The supplementary of F is not unique, but if E is finite-dimensional, all supplementary spaces of F have the same dimension.*

Theorem 1.1 *Let E be a finite-dimensional vector space and F, G two vector subspaces of E. Then*

$$\dim (F + G) \Leftrightarrow \dim F + \dim G - \dim F \cap G.$$

Proof. Let's suppose that $\dim F = p$, $\dim G = q$ and $\dim F \cap G = r$. Note that, since $F \cap G$ is a vector subspace of F and G, we have $r < p$ and $r < q$. Consider a basis $\{a_1, ..., a_r\}$ of $F \cap G$. Since the family $\{a_1, ..., a_r\}$ is linearly independent, we can complete it in a base of F and also in a base of G. We can therefore construct: a basis of F of the type $\{a_1, ..., a_r, e_{r+1}, ..., e_p\}$ and a basis of G of the type $\{a_1, ..., a_r, f_{r+1}, ..., f_q\}$.

We know that any vector of $F+G$ is written as the sum of a vector of F, and a vector of G and therefore it is of the form:

$$\begin{aligned} x &= \lambda_1 a_1 + ... + \lambda_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \\ & \quad \mu_1 a_1, ... + \mu_r a_r + \mu_{r+1} f_{r+1}, ... + \mu_q f_q, \end{aligned}$$

that is, by posing $\tau_i = \lambda_i + \mu_i$, for $i = 1, ..., r$:

$$x = \tau_1 a_1 + ... + \tau_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \mu_{r+1} f_{r+1}, ... + \mu_q f_q. \tag{1.2}$$

Therefore, the family $\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$ egene-in-gender $F+G$. Let us show that it is free. Let us consider a null linear combination:

$$\underbrace{\tau_1 a_1 + ... + \tau_r a_r}_{\alpha \in F \cap G} + \underbrace{\lambda_{r+1} e_{r+1} + ... + \lambda_p e_p}_{\beta \in F} + \underbrace{\mu_{r+1} f_{r+1}, ... + \mu_q f_q}_{\gamma \in G} = 0.$$

We have $\alpha + \beta + \gamma = 0$, then $\gamma = -(\alpha + \beta)$. Thus $\gamma \in g$ and $\alpha + \beta \in F$, then $\gamma \in F \cap G$. Therefore, γ can be written as a linear combination of the a_i:

$$\mu_{r+1} f_{r+1}, ... + \mu_q f_q = \delta_1 a_1 + ... + \delta_r a_r.$$

But $\{a_1, ..., a_r, \mu_{r+1} f_{r+1}, ... + \mu_q f_q\}$ is a basis of G so all coefficients of this linear combination must be null. In particular, $\mu_{r+1} = 0, ..., \mu_q = 0$. Likewise $\lambda_{r+1} = 0, ..., \lambda_q = 0$. From (1.2) we then deduce that:

$$\tau_1 a_1 + ... + \tau_r a_r = 0$$

Now the family $\{a_1, ..., a_r\}$ is linearly independent, so $\tau_1 = 0, ..., \tau_r = 0$. So the family

$$\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$$

is linearly independent and therefore it is a basis of $F+G$. We deduce that:

$$\begin{aligned} \dim (F+G) &= r + (p-r) + (q-r) = p + q - r \\ &= \dim E + \dim G + \dim F \cap G. \end{aligned}$$

□

Corollary 1.2 *Let E be a finite-dimensional vector space and F, G two vector subspaces of E. Then*

$$E = F \oplus G \Leftrightarrow \begin{cases} F \cap G = \{0_E\} \\ \dim E = \dim F + \dim G \end{cases}$$

Example 1.21 *In* $E = \mathbb{C}$, $F = Vect(\{1\})$ *et* $G = Vect(\{i\})$. *We know that* F *and* G *are vector subspaces of* E. *Si* $x \in F \cap G$ *then* x *is both real and pure imaginary therefore* $x = 0$ *and* $F \cap G = \{0_E\}$. *Moreover*

$$F + G = Vect(1) + Vect(i) = Vect(1, i) = \mathbb{C}$$

then $E = F + G$. *We then showed that* F *and* G *are supplementary.*

Example 1.22 *In* $\mathbb{R}^3$, *Let* π *a vector plane and* v *a vector not contained in this plane. We have:*

$$\mathbb{R}^3 = \pi \oplus Vect(v)$$

because if $\{e_1, e_2\}$ *is a basis of* π, *then* $\{e_1, e_2, v\}$ *is a basis of* $\mathbb{R}^3$.