4.3 Exercises

Exercise 1

Without calculating them, explain why the following determinants are zero:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & 10 \\ 0 & 2 & 5 \\ 0 & 1 & 1 \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} -1 & 2 & 3 \\ 1 & -2 & 5 \\ 1 & -2 & 2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -3 \\ 3 & 1 & 2 \end{vmatrix}, \ \Delta_4 = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_5 = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}, \ \Delta_6 = \begin{vmatrix} 1 & \cos 2x & 2\cos^2 x \\ 1 & -\cos 2x & 2\sin^2 x \\ \cos x \sin x & \cos x \sin x & \sin 2x \end{vmatrix}$$

Exercise 2
Show that:

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 0 \end{vmatrix} = 60$$

Exercise 3
The numbers $119, 153$ and 289 are all divisible by 17. Show, without expanding it, that the determinant

$$\begin{vmatrix} 1 & 1 & 9 \\ 1 & 5 & 3 \\ 2 & 8 & 9 \end{vmatrix}$$

is divisible by 17.
Exercise 4
Let $(a, b, c) \in \mathbb{R}^3$. Consider the polynomials $P_a = (X - a)^2$, $P_b = (X - b)^2$ and $P_c = (X - c)^2$. Determine for which values of (a, b, c) the family $\mathcal{P} = (P_a, P_b, P_c)$ forms a basis of $\mathbb{R}_2[X]$?
Exercise 5
Show that

$$\begin{aligned} \Delta &= \begin{vmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{vmatrix} \\ &= -2\sin(a-b)\sin(b-c)\sin(c-a) \end{aligned}$$

Exercise 6
Let

$$P_1 = 2X^2 - X + 1,\ P_2 = X^2 + 2X \text{ and } P_3 = X^2 - 1$$

Show that the family $\mathcal{P} = (P_1, P_2, P_3)$ is a basis of $\mathbb{R}_2[X]$.
Exercise 7
Under what condition on the real number a does the family $e = (e_1, e_2, e_3)$:

$$e_1 = (a, 1, 1) \quad e_2 = (1, a, 1) \quad e_3 = (1, 1, a)$$

form a basis of $\mathbb{R}^3$?
Exercise 8
Solve in $\mathbb{R}^3$ the systems:

$$1. \begin{cases} x - y + z = 1 \\ 3y - z = 2 \\ 2z = 8 \end{cases} \quad 2. \begin{cases} x - y + 2z = 1 \\ 2x - 3y + z = 4 \\ x - 3y - 4z = 5 \end{cases}$$

3. $\begin{cases} x+2y+3z=1 \\ -x-3y+5z=2 \\ x+y+z=-1 \end{cases}$ 4. $\begin{cases} y+3z=0 \\ x+2y+6z=2 \\ 7x+3y+9z=14 \end{cases}$

5. $\begin{cases} x+2y+z=2 \\ 2x+y+z=-1 \\ x-3y+2z=-1 \end{cases}$ 6. $\begin{cases} 2x-y+3z=1 \\ x+y-z=2 \\ x-2y+4z=1 \end{cases}$

7. $\begin{cases} 2x-y+3z=0 \\ x+y+2z=0 \end{cases}$ 8. $\begin{cases} x+y-z=1 \\ 2x+2y-2z=2 \\ -x-y+z=-1 \end{cases}$

Exercise 9
Without attempting to solve the following systems, discuss the nature of their solution set:

$$\begin{cases} x+y-z=0 \\ x-y=0 \\ x+y+z=0 \end{cases}, \begin{cases} x+3y+2z=1 \\ 2x-2y=2 \\ x+y+z=2 \end{cases} \text{ and } \begin{cases} x+3y+2z=1 \\ 2x-2y=2 \\ x+y+z=3 \end{cases}$$

Exercise 10
Discuss, depending on the value of m, the dimension of the solution space of the following systems:
1.

$$\begin{cases} x+my+z=0 \\ mx+y+mz=0 \end{cases}$$

2.

$$\begin{cases} x+y+mz=0 \\ x+my+z=0 \\ mx+y+z=0 \end{cases}$$

4.4 Solutions

Exercise 1
1. A column of Δ_1 is zero, so $\Delta_1 = 0$.
2. The first two columns of Δ_2 are proportional, so $\Delta_2 = 0$.
3. The first column of Δ_3 is the sum of the other two, so $\Delta_3 = 0$.
4. Δ_4 being triangular, the determinant Δ_4 is equal to the product of its diagonal terms. One

of these being zero, Δ_4 is also zero.
5.

$$\Delta_5 \overset{C_3 \to C_2 + C_3}{=} \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

and the first and third columns of Δ_5 are proportional. It follows that $\Delta_5 = 0$.
6. The last column of Δ_6 is the sum of the other two, so $\Delta_6 = 0$.

Exercise 2

Easy by column permutations.

Exercise 3

Let Δ be this determinant. We have:

$$\begin{aligned} 1000\Delta &= \begin{vmatrix} 100 & 10 & 9 \\ 100 & 50 & 3 \\ 200 & 80 & 9 \end{vmatrix} \overset{C_1 \to C_1 + C_2 + C_3}{=} \begin{vmatrix} 119 & 10 & 9 \\ 153 & 50 & 3 \\ 289 & 80 & 9 \end{vmatrix} \\ &= 17 \begin{vmatrix} a & 10 & 9 \\ b & 50 & 3 \\ c & 80 & 9 \end{vmatrix} \end{aligned}$$

where a, b and c denote the quotient of $119, 153$ and 289 by 17, respectively. We obtain: $\Delta = 17m$ with

$$m = \begin{vmatrix} a & 10 & 9 \\ b & 50 & 3 \\ c & 80 & 9 \end{vmatrix}$$

which is an integer. Since 17 is coprime with 1000, applying Gauss's lemma, 17 divides Δ.

Exercise 4

The matrix of the family $\mathcal{P}$ in the canonical basis $(1, X, X^2)$ of $\mathbb{R}_2[X]$ is

$$M = \begin{pmatrix} a^2 & b^2 & c^2 \\ -2a & -2b & -2c \\ 1 & 1 & 1 \end{pmatrix}$$

Using Vandermonde determinants, we find

$$\det M = -2(b-a)(c-a)(c-b).$$

The family $\mathcal{P}$ forms a basis of $\mathbb{R}_2[X]$ if and only if the scalars a, b and c are pairwise distinct.

Exercise 5

We expand along the first row and recognize the addition formulas:

$$\begin{aligned}
\Delta &= \cos(a-b)[\cos(b+c)\sin(c+a)-\cos(c+a)\sin(b+c)] \\
&- \cos(b-c)[\cos(a+b)\sin(c+a)-\cos(c+a)\sin(a+b)] \\
&+ \cos(c-a)[\cos(a+b)\sin(b+c)-\cos(b+c)\sin(a+b)] \\
&= \cos(a-b)\sin(a-b)+\cos(b-c)\sin(b-c) \\
&+ \cos(c-a)\sin(c-a) = \tfrac{1}{2}(\sin 2(a-b)+\sin 2(b-c)+\sin 2(c-a))
\end{aligned}$$

then we use the two formulas

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

and

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}:$$

$$\begin{aligned}
\Delta &= \tfrac{1}{2}(2\sin(a-c)\cos(a+c-2b)+\sin 2(c-a)) \\
&= \sin(a-c)\cos(a+c-2b)+\sin(c-a)\cos(c-a) \\
&= \sin(a-c)(\cos(a+c-2b)-\cos(c-a)) \\
&= -2\sin(a-b)\sin(b-c)\sin(c-a)
\end{aligned}$$

Exercise 6

Letting $e = (X^2, X, 1)$ be the canonical basis of $\mathbb{R}_2[X]$, we have:

$$\underset{e}{\mathrm{Mat}}(\mathcal{P}) = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

which is invertible. The family $\mathcal{P}$ is therefore a basis of $\mathbb{R}_2[X]$.

Exercise 7

The family e forms a basis of $\mathbb{R}^3$ if and only if

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \neq 0.$$

This determinant is equal to: $(a-1)^2(a+2)$.

The family e is therefore free if and only if $a \neq 1$ and $a \neq -2$.

Exercise 8

1. Working backwards, we find successively: $z = 4; y = 2; x = -1$.

2.

$$\begin{cases} x - y + 2z = 1 \\ 2x - 3y + z = 4 \\ x - 3y - 4z = 5 \end{cases} \iff \begin{cases} x - y + 2z = 1 \\ -y - 3z = 2 \\ -2y - 6z = 4 \end{cases}$$

The last two equations are equivalent. The system is of rank 2 and consistent. Taking z as a parameter, the solution set is $\{(-5z-1,-3z-2,z) \mid z \in \mathbb{K}\}$.
3. Cramer's system: $\left\{\left(-\frac{5}{2},1,\frac{1}{2}\right)\right\}$.
4. System of rank 2 and consistent. $\{(2,-3z,z) \mid z \in \mathbb{K}\}$.
5. Cramer's system: $\{(-2,1,2)\}$.
6. System of rank 2 but not consistent. No solution.
7. $\left\{\begin{array}{r} 2x-y+3z=0 \\ x+y+2z=0 \end{array}\right.$ i.e., $\left\{\begin{array}{r} 2x-y+3z=0 \\ 3x+5z=0 \end{array}\right.$. The system is of rank 2, so consistent. Taking z as a parameter, the solution set is $\left\{\left(-\frac{5}{3}z,-\frac{1}{3}z,z\right) \mid z \in \mathbb{K}\right\}$.
8. The system is clearly of rank 1 and consistent (we have three times the same equation). The solution set is the plane with equation $x+y-z=1$.

Exercise 9

First system: Matrix of rank 3 (invertible) so a unique solution $(0,0,0)$.
Second system: Matrix of rank 2, system not consistent, no solution.
Third system: Matrix of rank 2, system not consistent, no solution.

Exercise 10

1. If $m=1$ or $m=-1$, then the system is of rank 1. The solution space is of dimension 2. Otherwise the system is of rank 2 and the solution space is of dimension 1. In the latter case, the solution set is $\{(x,0,-x), x \in \mathbb{R}\}$.
2. If $m=1$, then the system is of rank 1. The solution space is of dimension 2. If $m=-2$, then the system is of rank 2. The solution space is of dimension 1. Otherwise the system is a Cramer system. The solution space is of dimension 0.