CHAPTER 4

SOLVING SYSTEMS OF EQUATIONS

Determinants provide an efficient and indispensable tool for discussing linear systems: they allow us to have compatibility conditions in the form of relations linking the coefficients and also provide formulas that explicitly give the solution (**Cramer's** formulas).

The essential notions covered in this chapter are:

— Determinant.
— Permutations.
— Notation convention.
— Systems of linear equations
— Definitions and interpretations.
— Matrix expression and rank of a system.
— Vector expression.
— Interpretation in terms of linear maps.
— Cramer's systems.

4.1 Determinant

4.1.1 Permutations

To define the determinant of a square matrix, we must start by introducing permutations.

A permutation is a bijection $\sigma : S \to S$, where S is a set. We usually denote a permutation in the form of a two-row array such that each element in the bottom row represents the image of the element immediately above it. Using this notation, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is a permutation such that $\sigma(1) = 3, \sigma(2) = 1$, and $\sigma(3) = 2$. (The notation for permutations should not be confused with that of matrices.) The set of all permutations of the set $\{1, 2, \ldots, n\}$ is denoted by S_n. Note that $|S_n| = n!$. The elements of S_2 are

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

and those of S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

The permutation $\sigma \in S_n$ such that $\sigma(i) = i$ for $i = 1, \ldots, n$ is the identity permutation.

Let $\sigma, \gamma \in S_n$. It is not difficult to see that $\gamma \circ \sigma \in S_n$.

Example 4.1 *Let*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

What is $\gamma \circ \sigma$?

Note that

$$(\gamma \circ \sigma)(1) = \gamma(\sigma(1)) = \gamma(3) = 1,$$
$$(\gamma \circ \sigma)(2) = \gamma(\sigma(2)) = \gamma(1) = 3,$$
$$(\gamma \circ \sigma)(3) = \gamma(\sigma(3)) = \gamma(2) = 2.$$

Thus, $\gamma \circ \sigma$ is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

If $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then $\gamma \circ \sigma$ is the identity permutation. We call γ the inverse permutation (or simply the inverse) of σ. Every permutation has an inverse.

We will see later that the definition of the determinant of an $n \times n$ matrix A is a sum of terms, each containing a product of the form $a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, for a permutation $\sigma \in S_n$, where $a_{i,j}$ denotes the (i, j) element of $\mathbf{A}$.

Recall that we use the first index for rows and the second for columns. Thus, the product $a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$ contains exactly one element from each row of A. Since $\sigma(1),\ldots,\sigma(n)$ is a permutation of the numbers $1,\ldots,n$, the product

$$a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$$

also contains exactly one element from each column of A.

Example 4.2 *Let*

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ u & v & w \end{bmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Then $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)} = a_{1,3}a_{2,1}a_{3,2} = cdv$

Exercise 4.1 *1. For each of the following permutations, give the inverse permutation.*

$$a. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \; b. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \; c. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix}$$

2. Let

$$\mathbf{A} = \begin{bmatrix} 6 & 2 & 3 \\ 0 & 1 & -1 \\ -5 & 7 & 4 \end{bmatrix}.$$

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. *What value does* $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}$ *take?*

Solution. 1. a. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

b. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

c. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix}.$$

2. $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)} = a_{1,3}a_{2,2}a_{3,1} = 3\cdot 1\cdot(-5) = -15.$ □

Before addressing the determinant, we must also introduce the notion of an inversion of a permutation.

Definition 4.1 *Let $\sigma \in S_n$. The pair (i, j) is an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$. (Not to be confused with the inverse permutation discussed earlier.) The total number of inversions of σ is denoted by $\mathrm{inv}(\sigma)$.*

Example 4.3 *Consider*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The pair $(1, 2)$ is an inversion of σ because we have $1 < 2$ and $\sigma(1) = 3 > 1 = \sigma(2)$. The other inversion is given by the pair $(1, 3)$. So $\mathrm{inv}(\sigma) = 2$.

Example 4.4 *Let $\sigma \in S_n$ for which there exist $i, j \in \{1, \ldots, n\}, i < j$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for all other indices. We want to determine $\mathrm{inv}(\sigma)$.*

Note that for each $k = i + 1, \ldots, j - 1, (i, k)$ is an inversion because $i < k$ and $\sigma(i) = j > k = \sigma(k)$.

Moreover, for each $k = i + 1, \ldots, j - 1, (k, j)$ is an inversion because $k < j$ and $\sigma(k) = k > i = \sigma(j)$.

Finally, the pair (i, j) is also an inversion. Thus,

$$\mathrm{inv}(\sigma) = 2(j - 1 - (i + 1) + 1) + 1 = 2(j - i) - 1,$$

which is always odd.

For example, the inversions of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

are $(2, 3), (3, 4)$ and $(2, 4)$.

Definition 4.2 *If $A = (a_{ij})$, the determinant of $\mathbf{A}$, denoted by $\det(\mathbf{A})$, is defined by*

$$\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

where $a_{i,j}$ denotes the element in the i-th row and j-th column of $\mathbf{A}$.

Calculating the determinant as defined requires the sum of $n!$ terms. Each of these terms depends on a permutation of S_n and the product of n elements of $\mathbf{A}$, and the sign depends on the parity of the number of inversions of the permutation. (The parity of an integer indicates whether the number is even or odd.)

Note that even when the size of the matrix is small, the number of terms to compute quickly becomes too high. For example, if $n = 5$, there are already $5 \times 4 \times 3 \times 2 \times 1 = 120$ terms. Fortunately, there are efficient methods for computing the determinant that do not require so many terms.

4.1.2 Determinant of small matrices

In the case of 2×2 or 3×3 matrices, there are formulas to simplify the calculations.

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The two permutations of S_2 are

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since $\operatorname{inv}(\sigma_1) = 0$ and $\operatorname{inv}(\sigma_2) = 1$, we have

$$\det(\mathbf{A}) = (-1)^{\operatorname{inv}(\sigma_1)} a_{1,\sigma_1(1)} a_{2,\sigma_1(2)} + (-1)^{\operatorname{inv}(\sigma_2)} a_{1,\sigma_2(1)} a_{2,\sigma_2(2)} = ad - bc.$$

Let

$$\mathbf{A} = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}.$$

The six permutations of S_3 are listed in the following table.

i	1	2	3
σ_i	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$
$\operatorname{inv}(\sigma_i)$	0	1	1
$(-1)^{\operatorname{inv}(\sigma_i)}$	1	-1	-1
i	4	5	6
σ_i	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
$\operatorname{inv}(\sigma_i)$	2	2	3
$(-1)^{\operatorname{inv}(\sigma_i)}$	1	1	-1

Then

$$\begin{aligned} \det(A) &= \textstyle\sum_{i=1}^{6} (-1)^{\operatorname{inv}(\sigma_i)} a_{1,\sigma_i(1)} a_{2,\sigma_i(2)} a_{3,\sigma_i(3)} \\ &= p_1q_2r_3 - p_1q_3r_2 - p_2q_1r_3 + p_2q_3r_1 + p_3q_1r_2 - p_3q_2r_1 \end{aligned}$$

4.1.3 Notation Convention

Instead of writing

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ p & q & r \end{bmatrix}\right),$$

we simply write

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}.$$

Exercise 4.2 *1. For each of the following permutations, give the number of inversions.*

$$a. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \ b. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \ c. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix}$$

2. Compute the determinant of each of the following matrices:

$$a. \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \ b. \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, \ c. \begin{bmatrix} 0 & 0 & p \\ q & 0 & 0 \\ 0 & r & 0 \end{bmatrix}$$

3. Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices such that* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by multiplying a row of* A *by the scalar* α*. Show that* $\det(\mathbf{B}) = \alpha \det(\mathbf{A})$*.*

Solution. 1. a. The number of inversions is 1, b. The number of inversions is 6, c. The number of inversions is 6.

2. a. 1, b. $ad - bc$, c. pqr.

3. Suppose $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying the i-th row by α. Then

$$\begin{aligned} \det(\mathbf{B}) &= \textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{p=1}^{n} b_{p,\sigma(p)} \\ &= \textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \left(\prod_{p \neq i} a_{p,\sigma(p)} \right) \left(\alpha a_{i,\sigma(i)} \right) \\ &= \alpha \left(\textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{p=1}^{n} a_{p,\sigma(p)} \right) \\ &= \alpha \det(\mathbf{A}). \end{aligned}$$

□

4.1.4 Special Matrices

Matrices with a zero row or column

Let A be a square matrix having a zero row or column. Then $\det(\mathbf{A}) = 0$. Indeed, note that each term in the definition $\det(\mathbf{A})$ is a product of the form $a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, for a permutation σ. Thus, each term contains exactly one element from each row and each column of $\mathbf{A}$, which implies that each term is zero, which implies that $\det(\mathbf{A}) = 0$.

Permutation matrices

A permutation matrix of size $n \times n$ is a matrix obtained from the identity matrix $\mathbf{I}_n$ by permuting its rows. For example,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is a permutation matrix.

As its name suggests, such a matrix directly encodes a permutation of the set $\{1, \ldots, n\}$. The interpretation is as follows: if σ is encoded by the permutation matrix, then $\sigma(i)$ is the column index of the element containing 1 in the i-th row. In the example above, the corresponding permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

because in row 1, column 2 contains 1; in row 2, column 4 contains 1; in row 3, column 1 contains 1; in row 4, column 3 contains 1.

For a given permutation matrix $\mathbf{P}$ that encodes σ, the determinant of $\mathbf{P}$ is simply $(-1)^{\text{inv}(\sigma)}$. In the example above, there are three inversions. Therefore, the determinant is $(-1)^3 = -1$.

This result can be seen directly from the definition of the determinant: each term of the sum contains a factor $(-1)^{\text{inv}(\sigma')}$which multiplies the product of $n$elements, exactly one from each row and one from each column. The only way to obtain a non-zero term is to use a permutation that extracts the non-zero element from each row. There is only one permutation for which this is the case, the one encoded by σ.

Triangular matrices

Let $\mathbf{A}$be a square upper triangular matrix (i.e., $a_{i,j} = 0$for all $i > j$). For example, the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is upper triangular.

In this case, $\det(\mathbf{A})$ is the product of the elements on the diagonal. Here, the determinant is $1 \cdot 2 \cdot 3 \cdot 1 = 6$.

Let's see why this is the case. Let $\sigma \in S_n$. We start by showing that if σ is not the identity permutation, then $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$.

Suppose that $\sigma(1) \neq 1$. Then there must be an $i \geq 2$ such that $\sigma(i) = 1$. This gives $a_{i,\sigma(i)} = 0$ since $\mathbf{A}$ is upper triangular and $i > \sigma(i)$. So, $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$ if $\sigma(1) \neq 1$.

Suppose that $\sigma(1)=1$ but $\sigma(2)\neq 2$. Then there must be an $i\neq 2$ such that $\sigma(i)=2$. But $i\neq 1$ since we already have $\sigma(1)=1$. So, $i\geq 3$. This again gives $a_{i,\sigma(i)}=0$, since $i>\sigma(i)$. So, $\prod_{i=1}^{n} a_{i,\sigma(i)}=0$ if $\sigma(1)=1$ and $\sigma(2)\neq 2$.

We can continue in this manner to show that if $\sigma(i)=i$ and $\sigma(i+1)\neq i+1$, then $\prod_{i=1}^{n} a_{i,\sigma(i)}=0$. Thus, the only term in $\det(\mathbf{A})$ that can be non-zero is the one for which $\sigma(i)=i$ for all $i=1,\ldots,n$, which implies that $\det(\mathbf{A})=a_{1,1}a_{2,2}\cdots a_{n,n}$.

Using a similar argument, we can conclude that the determinant of a lower triangular matrix (a matrix where all elements below the diagonal are zero) is also given by the product of the elements on the diagonal.

Exercise 4.3 *1. Compute the determinant of each of the following matrices:*

$$a.\begin{bmatrix}2&3\\0&2\end{bmatrix},\ b.\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}\ \text{and } c.\begin{bmatrix}2-i&0\\3&1+i\end{bmatrix}.$$

Solution. a. 4, b. adf, c. $(2-i)(1+i)=3+i$. □

4.1.5 Fundamental Properties

Proposition 4.1 *Let A, $B\in S^{n\times n}$, S a ring. Then*

$$\det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})$$

An immediate and useful consequence of the previous proposition is that

$$\det\left(\mathbf{A}^{k}\right)=\det(\mathbf{A})^{k}$$

for every natural number k.

Example 4.5 *Let $\mathbf{A},\mathbf{B}\in\mathbb{R}^{3\times 3}$ such that $\det(\mathbf{A})=3$ and $\det(\mathbf{B})=-2$. Then $\det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})=-6$.*

Example 4.6 *Let*

$$\mathbf{A}=\begin{bmatrix}-2&1\\-1&1\end{bmatrix}.$$

Then $\det\left(\mathbf{A}^{100}\right)=\det(\mathbf{A})^{100}=(-1)^{100}=1$.

The determinant also satisfies the following properties:

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by adding a multiple of one row of $\mathbf{A}$ to another row of $\mathbf{A}$, then $\det(\mathbf{B})=\det(\mathbf{A})$.

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by exchanging two rows of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
- $\det\left(\mathbf{A}^\top\right) = \det(\mathbf{A})$.
- If $\mathbf{A}$ is invertible, then $\det\left(\mathbf{A}^{-1}\right) = \det(\mathbf{A})^{-1}$.
- If α is a scalar and $\mathbf{A}$ is of size $n \times n$, $\det(\alpha\mathbf{A}) = \alpha^n \det(\mathbf{A})$.

Example 4.7 *Let* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. *Note that* $\det(\mathbf{A}) = 3$.
- $\det\left(\mathbf{A}^\top\right) = \det(\mathbf{A}) = 3$.
- *Note that* $\mathbf{A}$ *is invertible. So,* $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})} = \frac{1}{3}$.
- $\det(2\mathbf{A}) = 2^2 \det(\mathbf{A}) = 4 \cdot 3 = 12$ *since* $\mathbf{A}$ *is* 2×2.
- $\det(-\mathbf{A}) = \det((-1)\mathbf{A}) = (-1)^2 \det(\mathbf{A}) = \det(\mathbf{A}) = 3$. *Note that in this case,* $\det(-\mathbf{A}) = \det(\mathbf{A})$.

Exercise 4.4 *1. Prove that if* $\mathbf{A}$ *is invertible, then* $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$.

2. Prove that if $\mathbf{A} \in \mathbb{R}^{4\times 4}$, then $\det(-\mathbf{A}) = \det(\mathbf{A})$.

Solution. 1. Note that, in this case,

$$\det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A}) = \det\left(\mathbf{A}^{-1}\mathbf{A}\right) = \det(\mathbf{I}) = 1$$

hence $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$.
2. $\det(-\mathbf{A}) = \det((-1)\mathbf{A}) = (-1)^4 \det(\mathbf{A}) = \det(\mathbf{A})$. □

4.1.6 Determinant of a block matrix

Let $\mathbf{A}$ be a square matrix that can be written in block form as

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix},$$

where $\mathbf{B}$ and $\mathbf{D}$ are square matrices. Then $\det(\mathbf{A}) = \det(\mathbf{B})\det(\mathbf{D})$.

For example, the determinant of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is $1 \cdot 1 \cdot 2 \cdot 3 = 6$.

4.1.7 Computing the determinant by cofactor expansion

Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\mathbf{A}_{i,j}$ be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and the j-th column from $\mathbf{A}$. The (i, j)-cofactor of $\mathbf{A}$ is defined by

$$C_{i,j} = (-1)^{i+j} \det\left(\mathbf{A}_{i,j}\right).$$

The determinant of $\mathbf{A}$ can be computed by expanding along any row or any column. For example, expanding along the i-th row gives

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{i,j} C_{i,j}.$$

Similarly, expanding along the j-th column gives

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{i,j} C_{i,j}.$$

Example 4.8 *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}.$$

We can compute $\det(\mathbf{A})$ *by expanding along the first row. Then*

$$\begin{aligned}
&\det(\mathbf{A}) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} \\
&= 1 \cdot (-1)^{1+1} \det\left(\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}\right) + 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) + 3 \cdot (-1)^{1+3} \det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right) \\
&= 1 \cdot (24 - 0) + 2 \cdot (-1)(0 - 5) + 3 \cdot (0 - 4) \\
&= 24 + 10 - 12 = 22.
\end{aligned}$$

We can also expand along the second column. Then

$$\begin{aligned}
&\det(\mathbf{A}) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2} \\
&= 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) + 4 \cdot (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}\right) + 0 \cdot C_{3,2} \\
&= 2 \cdot (-1)(0 - 5) + 4 \cdot (6 - 3) + 0 \\
&= 10 + 12 = 22.
\end{aligned}$$

Exercise 4.5 *1. Compute the determinant of*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

by expanding along the first row.

2. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

by first using the property of block matrices.

Solution. 1. We have:

$$\begin{aligned}\det(\mathbf{A}) &= 1\cdot\det\left(\begin{bmatrix}5 & 6\\ 8 & 9\end{bmatrix}\right) - 2\cdot\det\left(\begin{bmatrix}4 & 6\\ 7 & 9\end{bmatrix}\right) + 3\cdot\det\left(\begin{bmatrix}4 & 5\\ 7 & 8\end{bmatrix}\right)\\ &= 1\cdot(45-48) - 2\cdot(36-42) + 3\cdot(32-35) = -3+12-9 = 0.\end{aligned}$$

2. The matrix is of the form $\begin{bmatrix}\mathbf{B} & \mathbf{0}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}$, where $\mathbf{B} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $\mathbf{C} = \begin{bmatrix}5 & 6\\ 9 & 10\end{bmatrix}$, and $\mathbf{D} = \begin{bmatrix}7 & 8\\ 11 & 12\end{bmatrix}$. Then $\det(\mathbf{A}) = \det(\mathbf{B})\det(\mathbf{D}) = (4-6)\cdot(84-88) = (-2)\cdot(-4) = 8.$ □

4.1.8 The adjugate matrix and Cramer's rule

Let $\mathbf{A}$ be an $n\times n$ matrix. The adjugate matrix of $\mathbf{A}$ is the $n\times n$ matrix whose (i,j)-th element is $C_{j,i}$, where $C_{j,i}$ is the (j,i)-cofactor of $\mathbf{A}$.

Note the reversal of indices: the element in the (i,j)-th position of the adjugate matrix is the (j,i)-cofactor of $\mathbf{A}$.

The adjugate matrix of $\mathbf{A}$ is denoted by $\operatorname{adj}(\mathbf{A})$.

It can be shown that

$$\mathbf{A}\cdot\operatorname{adj}(\mathbf{A}) = \operatorname{adj}(\mathbf{A})\cdot\mathbf{A} = \det(\mathbf{A})\mathbf{I}.$$

If $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}).$$

This gives a formula for the inverse of $\mathbf{A}$.

Example 4.9 *Let*

$$\mathbf{A} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Then $\operatorname{adj}(\mathbf{A}) = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$, *and* $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$.

Example 4.10 *Let*

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 1 & 0 & 6\end{bmatrix}.$$

We have already computed $\det(\mathbf{A}) = 22$. *The cofactors are:*

$$C_{1,1} = (-1)^{1+1} \det\left(\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}\right) = 24,$$
$$C_{1,2} = (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) = 5,$$
$$C_{1,3} = (-1)^{1+3} \det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right) = -4,$$
$$C_{2,1} = (-1)^{2+1} \det\left(\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}\right) = -12,$$
$$C_{2,2} = (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}\right) = 3,$$
$$C_{2,3} = (-1)^{2+3} \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) = 2,$$
$$C_{3,1} = (-1)^{3+1} \det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = -2,$$
$$C_{3,2} = (-1)^{3+2} \det\left(\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}\right) = -5,$$
$$C_{3,3} = (-1)^{3+3} \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\right) = 4.$$

Then

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

So,

$$\mathbf{A}^{-1} = \frac{1}{22} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

Now, consider the system of linear equations

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix. If $\det(\mathbf{A}) \neq 0$, then the system has a unique solution given by

$$x_j = \frac{\det\left(\mathbf{A}_j(\mathbf{b})\right)}{\det(\mathbf{A})},$$

where $\mathbf{A}_j(\mathbf{b})$ is the matrix obtained by replacing the j-th column of $\mathbf{A}$ with the column vector $\mathbf{b}$.

This is known as Cramer's rule.

Example 4.11 *Consider the system*

$$x + 2y + 3z = 6$$
$$4y + 5z = 7$$
$$x + 6z = 8$$

In matrix form, this is $\mathbf{Ax} = \mathbf{b}$*, where*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}.$$

We have already computed $\det(\mathbf{A}) = 22$*. Then*

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 6 & 2 & 3 \\ 7 & 4 & 5 \\ 8 & 0 & 6 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 6 & 2 & 3 \\ 7 & 4 & 5 \\ 8 & 0 & 6 \end{bmatrix}\right)$ *by expanding along the second column:*

$$\begin{aligned} & 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}\right) + 4 \cdot (-1)^{2+2} \det\left(\begin{bmatrix} 6 & 3 \\ 8 & 6 \end{bmatrix}\right) + 0 \cdot C_{3,2} \\ = \; & 2 \cdot (-1)(42 - 40) + 4 \cdot (36 - 24) = 2 \cdot (-1) \cdot 2 + 4 \cdot 12 = -4 + 48 = 44. \end{aligned}$$

So, $x = \frac{44}{22} = 2$.

Similarly,

$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 1 & 6 & 3 \\ 0 & 7 & 5 \\ 1 & 8 & 6 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 1 & 6 & 3 \\ 0 & 7 & 5 \\ 1 & 8 & 6 \end{bmatrix}\right)$ *by expanding along the first column:*

$$\begin{aligned} & 1 \cdot (-1)^{1+1} \det\left(\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}\right) + 0 + 1 \cdot (-1)^{3+1} \det\left(\begin{bmatrix} 6 & 3 \\ 7 & 5 \end{bmatrix}\right) \\ = \; & 1 \cdot (42 - 40) + 1 \cdot (30 - 21) = 2 + 9 = 11. \end{aligned}$$

So, $y = \frac{11}{22} = \frac{1}{2}$.

Finally,

$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 1 & 2 & 6 \\ 0 & 4 & 7 \\ 1 & 0 & 8 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 1 & 2 & 6 \\ 0 & 4 & 7 \\ 1 & 0 & 8 \end{bmatrix}\right)$ *by expanding along the second column:*

$$\begin{aligned} & 2\cdot(-1)^{1+2}\det\left(\begin{bmatrix} 0 & 7 \\ 1 & 8 \end{bmatrix}\right) + 4\cdot(-1)^{2+2}\det\left(\begin{bmatrix} 1 & 6 \\ 1 & 8 \end{bmatrix}\right) + 0 \\ = \ & 2\cdot(-1)(0-7) + 4\cdot(8-6) = 2\cdot(-1)\cdot(-7) + 4\cdot 2 = 14 + 8 = 22. \end{aligned}$$

So, $z = \frac{22}{22} = 1.$

Thus, the solution is $x = 2, y = \frac{1}{2}, z = 1.$

Exercise 4.6 *1. Use Cramer's rule to solve the system:*

$$\begin{aligned} 2x - y &= 5 \\ x + 3y &= -1 \end{aligned}$$

Solution. We have $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $\det(\mathbf{A}) = 6 - (-1) = 7.$

Then

$$x = \frac{\det\left(\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}\right)}{7} = \frac{15-1}{7} = \frac{14}{7} = 2,$$

$$y = \frac{\det\left(\begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}\right)}{7} = \frac{-2-5}{7} = \frac{-7}{7} = -1.$$

So, the solution is $x = 2, y = -1.$ □

4.2 Systems of Linear Equations

4.2.1 Definitions and Interpretations

A system of m linear equations in n unknowns is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are given numbers (or elements of a field), and $x_1, \ldots, x_n$ are the unknowns.

The system is said to be homogeneous if $b_1 = b_2 = \cdots = b_m = 0$; otherwise, it is non-homogeneous.

A solution of the system is an n-tuple $(s_1, \ldots, s_n)$ such that when we substitute $x_1 = s_1, \ldots, x_n = s_n$, all the equations are satisfied.

The system is said to be consistent if it has at least one solution; otherwise, it is inconsistent.

4.2.2 Matrix Expression and Rank of a System

The system can be written in matrix form as

$$\mathbf{Ax} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix $\mathbf{A}$ is called the coefficient matrix.

The augmented matrix of the system is the matrix $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$.

The rank of the system is the rank of the coefficient matrix $\mathbf{A}$.

The system is consistent if and only if the rank of $\mathbf{A}$ is equal to the rank of the augmented matrix.

If the system is consistent, then the number of free parameters in the general solution is $n - \text{rank}(\mathbf{A})$.

4.2.3 Vector Expression

The system can also be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

where $\mathbf{a}_j$ is the j-th column of $\mathbf{A}$.

This expresses $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$.

Thus, the system is consistent if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$.

4.2.4 Interpretation in Terms of Linear Maps

If we consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{Ax}$, then the system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the image of T.

The kernel of T is the set of solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

4.2.5 Cramer's Systems

A system $\mathbf{Ax} = \mathbf{b}$ is called a Cramer system if $\mathbf{A}$ is a square matrix and $\det(\mathbf{A}) \neq 0$.

In this case, the system has a unique solution given by Cramer's rule.

Exercise 4.7 *1. Determine whether the following system is consistent:*

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + 4z &= 5 \\ 4x + 5y + 6z &= 9 \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 9 \end{array}\right].$$

We reduce it to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

The last row gives $0 = 2$, which is impossible. So, the system is inconsistent. □