3.9 Exercises

Exercise 1.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

1. Compute A^2 and A^3. Compute $A^3 - A^2 + A - I$.
2. Express A^{-1} in terms of A^2, A and I.
3. Express A^4 in terms of A^2, A and I.

Exercise 2.

Let A be the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute $(A-2I)^3$, then deduce that A is invertible and determine A^{-1} in terms of I, A and A^2.

Exercise 3.

Let $\beta = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$.

Let u be the linear map that associates to a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ the vector

$$u(x) = (x_2 - 2x_3, 2x_1 - x_2 + 4x_3, x_1 - x_2 + 3x_3)$$

1. Determine the matrix A of u in the canonical basis.
2. Determine a basis (a, b) of $\ker(u - Id)$.
3. Give a vector c such that $\ker(u) = \text{vect}(c)$.
4. Show that $\beta' = (a, b, c)$ is a basis of $\mathbb{R}^3$.
5. Determine the matrix D of u in the basis β'.
6. Show that $\text{Im}(u) = \ker(u - Id)$
7. Show that $\ker(u) \oplus \text{Im}(u) = \mathbb{R}^3$.

Exercise 4.

Let $u : \mathbb{R}_2[X] \to \mathbb{R}[X]$ be defined by $u(P) = P + (1 - X)P'$

Let $\beta = (1, X, X^2)$ be the canonical basis of $\mathbb{R}_2[X]$

1. Show that u is an endomorphism of $\mathbb{R}_2[X]$.
2. Determine the matrix of u in β.
3. Determine the kernel and the image of u.

3.10 Solutions

Exercise 1.

1. We have

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 - A^2 + A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

2. $A^3 - A^2 + A - I = O \Leftrightarrow A\left(A^2 - A + I\right) = I$ so $A^{-1} = A^2 - A + I$
3. $A^3 = A^2 - A + I$ so

$$A^4 = A\left(A^2 - A + I\right) = A^3 - A^2 + A = \left(A^2 - A + I\right) - A^2 + A = I$$

Exercise 2. We have:

$$A - 2I = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}$$

$$(A-2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(A-2I)^3 = (A-2I)(A-2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Which implies that

$$A^3 - 3 \times 2A^2 + 3 \times 2^2 A - 2^3 I = 0$$

Because A and I commute.
Which is equivalent to

$$A^3 - 6A^2 + 12A - 8I = 0$$

Or again

$$A^3 - 6A^2 + 12A = 8I$$

Then by dividing by 8 and factoring A

$$A\left(\frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I\right) = I$$

Which shows that A is invertible and that

$$A^{-1} = \frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I$$

Exercise 3.
1. The coordinates of $u(x)$ in the canonical basis are

$$\begin{pmatrix} x_2 - 2x_3 \\ 2x_1 - x_2 + 4x_3 \\ x_1 - x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

So the matrix of u in the canonical basis is

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix}$$

2. Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the coordinates of a vector $x = (x_1, x_2, x_3)$ in the canonical basis

$$\begin{aligned} x &= (x_1, x_2, x_3) \in \ker(u - Id) \Leftrightarrow (A - I)X = 0 \\ &\Leftrightarrow \begin{pmatrix} -1 & 1 & -2 \\ 2 & -2 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ 2x_1 - 2x_2 + 4x_3 = 0 \Leftrightarrow x_1 - x_2 + 2x_3 = 0 \Leftrightarrow x_1 = x_2 - 2x_3 \\ x_1 - x_2 + 2x_3 = 0 \end{cases} \end{aligned}$$

So $x = (x_2 - 2x_3, x_2, x_3) = x_2(1, 1, 0) + x_3(-2, 0, 1)$
Let $a = (1, 1, 0)$ and $b = (-2, 0, 1)$, (a, b) is a family of two non-proportional vectors, so linearly independent, which span $\ker(u - Id)$, it is a basis of $\ker(u - Id)$.

3. Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the coordinates of a vector $x = (x_1, x_2, x_3)$ in the canonical basis

$$\begin{aligned} x = (x_1, x_2, x_3) &\in \ker(u) \Leftrightarrow AX = 0 \\ &\Leftrightarrow \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0 \\ x_1 - x_2 + 3x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 2x_3 \\ 2x_1 - 2x_3 + 4x_3 = 0 \\ x_1 - 2x_3 + 3x_3 = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} x_2 = 2x_3 \\ 2x_1 + 2x_3 = 0 \\ x_1 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 2x_3 \end{cases} \end{aligned}$$

So $x = (-x_3, 2x_3, x_3)$, if we let $c = (-1, 2, 1)$ then $\ker(u) = \text{Vect}(c)$

4. We have:

$$\begin{aligned} \det(a, b, c) &= \begin{vmatrix} 1 & -2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \\ &= -2 - (-2 + 1) = -1 \neq 0 \end{aligned}$$

By expanding along the first column, so (a, b, c) is a basis of $\mathbb{R}^3$.

5. $u(a) - a = 0_{\mathbb{R}^3} \Rightarrow u(a) = a$, similarly $u(b) = b$ and $u(c) = 0_{\mathbb{R}^3}$ so

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6. According to the matrix of u in the basis β', $\text{Im}(u) = \text{Vect}(a, b) = \ker(u - Id)$

7. By the rank-nullity theorem

$$\dim(\ker u) + \dim(\text{Im}\, u) = \dim\left(\mathbb{R}^3\right)$$

It remains to show that the intersection of $\ker(u)$ and $\text{Im}(u)$ is the zero vector.

$$\begin{aligned} x &\in \ker(u) \cap \text{Im}(u) \Leftrightarrow \begin{cases} x \in \text{Im}(u) \\ x \in \ker(u) \end{cases} \Leftrightarrow \begin{cases} x \in \ker(u) \\ x \in \ker(u - Id) \end{cases} \\ &\Leftrightarrow \begin{cases} u(x) = 0_{\mathbb{R}^3} \\ u(x) - x = 0_{\mathbb{R}^3} \end{cases} \Leftrightarrow \begin{cases} u(x) = 0_{\mathbb{R}^3} \\ u(x) = x \end{cases} \Leftrightarrow x = 0_{\mathbb{R}^3} \end{aligned}$$

We thus have $\ker(u) \oplus \operatorname{Im}(u) = \mathbb{R}^3$

Exercise 4.

1. We have:

$$u(\alpha P + \beta Q) = \alpha P + \beta Q + (1-X)(\alpha P + \beta Q)'$$
$$= \alpha P + \beta Q + (1-X)(\alpha P' + \beta Q')$$
$$= \alpha (P + (1-X)P') + \beta (Q + (1-X)Q') = \alpha u(P) + \beta u(Q)$$

So u is a linear map

$$d^\circ P \leq 2 \Rightarrow d^\circ u(P) \leq 2$$

It goes from $\mathbb{R}_2[X]$ to $\mathbb{R}_2[X]$ so it is an endomorphism of $\mathbb{R}_2[X]$.

2. We have:

$$u(1) = 1 + (1-X) \times 0 = 1$$
$$u(X) = X + (1-X) \times 1 = 1$$
$$u(X^2) = X^2 + (1-X) \times 2X = 2X - X^2$$
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

3. We have: $P \in \ker(u)$

$$\begin{aligned} u(P) &= 0 \Leftrightarrow u\left(aX^2 + bX + c\right) = au\left(X^2\right) + bu(X) + cu(1) \\ &= a\left(2X - X^2\right) + b + c = 0 \\ &\Leftrightarrow -aX^2 + 2aX + b + c = 0 \Leftrightarrow \begin{cases} a = 0 \\ c = -b \end{cases} \\ P &= bX - b = b(X-1) \end{aligned}$$

So $\ker(u)$ is the vector line spanned by the polynomial $X - 1$.

$$\begin{aligned} \operatorname{Im}(u) &= \operatorname{Vect}\left(u(1), u(X), u\left(X^2\right)\right) = \operatorname{Vect}\left(1, 1, 2X - X^2\right) \\ &= \operatorname{Vect}\left(1, 2X - X^2\right) \end{aligned}$$

These two polynomials are not proportional so they form a linearly independent family (and generating) of $\operatorname{Im}(u)$ so a basis of $\operatorname{Im}(u)$.