CHAPTER 3

MATRICES

In the case where $\mathcal{L}_{\mathbb{k}}(E, E')$ is finite-dimensional, say of dimension r, by choosing a basis we can associate to every linear map f an r-tuple of elements of $\mathbb{k}$, the components of f. For reasons we will see later, these components are arranged not on a line but in a table with a certain number of rows and columns, which is called the matrix associated with the linear map f.

The notions covered in this chapter are:
— Matrices associated with linear maps.
— Product of two matrices.
— Matrix of the inverse of a map.
— Calculation of the inverse of a matrix.
— Change of basis.
— Rank of a matrix.

Definition 3.1 *A matrix of type (p,n) with coefficients in $\mathbb{k}$ is a table A of pn elements of $\mathbb{k}$ arranged in p rows and n columns:*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{pmatrix} \quad \text{or, abbreviated} : A = (a_{ik}), \text{ or also} : A = \|a_{ik}\|.$$

The set of matrices with p rows and n columns is denoted $\mathcal{M}_{p,n}(\mathbb{k})$. If $n = p$, $\mathcal{M}_{n,n}(\mathbb{k})$ is denoted: $\mathcal{M}_n(\mathbb{k})$.

Example 3.1

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}) \quad \begin{pmatrix} 1 & 2-i & 3+i \\ 0 & 1+i & i \\ -i & 2 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{C})$$

Note that in the notation we have just adopted, a_{ik} denotes the element in the i-th row and the k-th column.

Another notation that we will also use later is the notation by columns:

$$A = \|c_1, \dots, c_n\|, \quad \text{where} \quad c_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{pk} \end{pmatrix} \quad \text{is the } k^{\text{th}} \text{ column}$$

On the set $\mathcal{M}_{p,n}(\mathbb{k})$ we define the operations:

- Addition: if $A = (a_{ik}), B = (b_{ik})$, we denote $C = A + B$ the matrix (c_{ik}) such that:

$$c_{ik} = a_{ik} + b_{ik}, \quad \forall i, k$$

- Multiplication by a scalar: if $A = (a_{ik})$ and $\lambda \in \mathbb{k}$ we denote λA the matrix (λa_{ik}) that is-to say the matrix obtained by multiplying all elements by λ.

Example 3.2

$$\begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 7 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

$$5 \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 10 & -5 & 0 & 15 \\ 5 & 10 & 5 & -5 \end{pmatrix}.$$

It is easy to see that, endowed with these operations, $\mathcal{M}_{p,n}(\mathbb{k})$ is a vector space over $\mathbb{k}$. The neutral element is the matrix whose all elements are zero, called the zero matrix, denoted 0 . The opposite of the matrix (a_{ik}) is the matrix $(-a_{ik})$.

Proposition 3.1 *We have:*

$$\dim_{\mathbb{k}} \mathcal{M}_{p,n}(\mathbb{k}) = pn$$

Proof. One easily verifies that the pn matrices, called *elementary matrices*

$$E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \ldots, E_{ik} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \leftarrow \text{i}^{\text{th}} \text{ row}, \ldots$$

$$E_{pn} = \begin{pmatrix} 0 & \cdots & 0 & \\ 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \cdots & 1 \end{pmatrix},$$

form a basis of $\mathcal{M}_{p,n}(\mathbb{k})$ called the canonical basis. □

3.1 Matrix of a linear map

Let E and E' be two vector spaces over $\mathbb{k}$, of dimension n and p respectively, and $f : E \longrightarrow E'$ a linear map. Let us choose a basis $\{e_1, \ldots, e_n\}$ of E and a basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ of E'. The images by f of the vectors $e_1, \ldots, e_n$ are decomposed on the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:

$$\begin{aligned} f(e_k) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ f(e_k) &= a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots + a_{p2}\varepsilon_p \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p. \end{aligned}$$

Definition 3.2 *We call the matrix of f in the bases $\{e_1, \ldots, e_n\}, \{\varepsilon_1, \ldots, \varepsilon_p\}$ the matrix denoted $M(f)_{e_i,\varepsilon_j}$ belonging to $\mathcal{M}_{p,n}(\mathbb{k})$ whose columns are the components of the vectors $f(e_1), \ldots, f(e_n)$ in the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:*

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pn} \end{pmatrix}.$$

We will also use the notation: $\|f(e_1), \ldots, f(e_n)\|_{\varepsilon_j}$.

If there is no possible ambiguity, we will also write $M(f)$ instead of $M(f)_{e_i,\varepsilon_j}$, but it is clear that the matrix associated with f depends on the choice of bases of E and E'.

In the case where f is an endomorphism, we can choose the same basis in E considered as the starting and arrival space. In this case, we will denote $M(f)_{e_i}$ instead of $M(f)_{e_i,e_j}$.

Proposition 3.2 *Let E and E' be two vector spaces over $\mathbb{k}$ of dimension n and p respectively, $\{e_i\}$ and $\{\varepsilon_j\}$ bases of E and E'. Then the map:*

$$\begin{array}{rccc} M: & \mathcal{L}_K(E,E') & \longrightarrow & \mathcal{M}_{p,n}(K) \\ & f & \longmapsto & M(f)_{e_i,\varepsilon_j} \end{array}$$

is a vector space isomorphism, that is:

$$\begin{cases} M(f+g) = M(f) + M(g) \\ M(\lambda f) = \lambda M(f), \end{cases}$$

and M is bijective.

In particular: $\dim_{\mathbb{k}} \mathcal{L}(E,E') = np$.

Proof. We indeed have:

$$\begin{aligned} M(f+g)_{e_i,\varepsilon_j} &= \|(f+g)(e_1),\ldots,(f+g)(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1)+g(e_1),\ldots,f(e_n)+g(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1),\ldots,f(e_n)\|_{\varepsilon_j} + \|g(e_1),\ldots,g(e_n)\|_{\varepsilon_j} \end{aligned}$$

by the definition of matrix addition, that is:

$$M(f+g)_{e_i,\varepsilon_j} = M(f)_{e_i,\varepsilon_j} + M(g)_{e_i,\varepsilon_j}.$$

Similarly, if $\lambda \in \mathbb{k}$:

$$\begin{aligned} M(\lambda f)_{e_i,\varepsilon_j} &= \|(\lambda f)(e_1),\ldots,(\lambda f)e_n\|_{\varepsilon_j} \\ &= \|\lambda f(e_1),\ldots,\lambda f(e_n)\|_{e_j} = \lambda \|f(e_1),\ldots,f(e_n)\|_{\varepsilon_j} \end{aligned}$$

So M is linear.

On the other hand M is surjective. Indeed, let:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pn} \end{pmatrix} \in \mathcal{M}_{pn}(K)$$

and $f \in \mathcal{L}(E,F)$ defined as follows. First set:

$$\begin{gathered} f(e_1) = a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f(e_n) = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{gathered}$$

Then, extend f by linearity to E, that is, if:

$$x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E,$$

set:

$$f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n).$$

It is easy to verify that f is linear and that $A = M(f)_{e_i,\varepsilon_j}$. Finally M is injective. Let indeed $f \in \operatorname{Ker} M$:

$$M(f) = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This means that $f(e_1) = 0, \ldots, f(e_n) = 0$. So, if $x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E$, we will have $f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n) = 0$, that is $f = 0$. By proposition 2.4, f is injective. □

Example 3.3 *Let E be of dimension n and:*

$$\begin{array}{llll} \mathrm{id}_E : & E & \longrightarrow & E \\ & x & \mapsto & x \end{array}$$

Consider a basis $\{e_i\}$. We have: $id_E(e_i) = e_i$. So:

$$M\left(\underset{E}{\mathrm{id}}\right)_{e_i} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \quad (1 \textit{ is the unit element of } \mathbb{k})$$

This matrix is denoted I_n or simply I and is called the unit matrix of $\mathcal{M}_n(\mathbb{k})$.

$f(e_1) = a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \quad f(e_n) = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p f(e_n) = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots +$

Example 3.4 *Let $\{\epsilon_1, \epsilon_2\}$ be the canonical basis of $\mathbb{R}^2$ and $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$. Consider the linear map:*

$$\begin{array}{llll} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (x-y, z-y) \end{array}$$

We have

$$\begin{aligned} f(e_1) &= f(1,0,0) = (1,0) = \epsilon_1 \\ f(e_2) &= f(0,1,0) = (-1,-1) = -\epsilon_1 - \epsilon_2 \\ f(e_3) &= f(0,0,1) = (0,1) = \epsilon_2 \end{aligned}$$

so

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Example 3.5 *Consider the map*

$$\begin{array}{rccc} w: & \mathbb{R}^n & \rightarrow & \mathbb{R} \\ & (x_1, ..., x_n) & \mapsto & a_1x_1 + ... + a_nx_n \end{array}$$

By endowing $\mathbb{R}^n$ *with the canonical basis* $\{e_1, ..., e_n\}$ *and* $\mathbb{R}$ *with the canonical basis* $\{1\}$*, we have:*

$$\begin{array}{l} f(e_1) = f(1, ..., 0) = a_1 = a_1\epsilon_1 \\ f(e_2) = f(0, 1, ..., 0) = a_2 = a_2\epsilon_2 \\ .... \\ f(e_n) = f(0, 0, ..., 1) = a_n = a_n\epsilon_n \end{array}$$

Then

$$M(w)_{e_i,\varepsilon_j} = (a_1, ..., a_n).$$

3.2 Product of two matrices

In the previous paragraph, we defined the operations of addition and multiplication by a scalar on matrices. By virtue of proposition 3.2, these operations correspond to the analogous operations on linear maps, that is we have:

$$M(f + g) = M(f) + M(g)$$
$$M(\lambda f) = \lambda M(f).$$

We will now define a new operation, the product of matrices. As we will see (cf. proposition 3.5), it corresponds to the composition of maps, in the sense that:

$$M(f \circ g) = M(f) \cdot M(g).$$

First, note that the composition of maps cannot be done for any pair of maps, but only if the arrival space of g is included in the starting space of f. This situation will be found for matrices: the product can only be performed between matrices of a certain type.

Definition 3.3 *We call the product of matrices the map:*

$$\begin{array}{ccc} \mathcal{M}_{p,n}(K) \times \mathcal{M}_{n,q}(K) & \longrightarrow & \mathcal{M}_{p,q}(K) \\ (a_{ji}), (b_{mk}) & \longmapsto & (c_{jk}) \end{array}$$

where:

$$c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k} + ... + a_{jn}b_{nk}.$$

In other words, the element c_{jk} of the j^{th} row and k^{th} column of the product $C = AB$ is the sum of the products of the elements of the j-th row of A by the elements of the same rank of the k^{th} column of B. Briefly, we say that the product of two matrices is performed rows by columns. Here is the diagram of this definition.

Remark 3.1 *The product AB can only be performed if the number of columns of A is equal to the number of rows of B.*

Example 3.6 *Let*

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

We have:

$$\times \underbrace{\begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 4 & 1 & 1 & -2 \\ 4 & 4 & 3 & 0 \end{pmatrix}}_{C=AB}$$

Example 3.7 *Let*

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

we have:

$$\times \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 11 \\ 7 \\ 6 \end{pmatrix}}_{AB}$$

The following remarks are important:

Remark 3.2 *1. We can have $AB = 0$ without A or B being zero.*

2. $AB = AC$ with $A \neq 0$ does not necessarily imply $B = C$ (that is, in general we cannot "simplify" by A, even if $A \neq 0$).

3. In general we have $AB \neq BA$ (that is: multiplication between matrices is not commutative).

Example 3.8 *Let*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

By performing the products, we find:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(which shows 1.)}$$

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{so } BA \neq AB (\text{ which shows 3.})$$

and $AB = AC$ (which shows 2. since we have $B \neq C$).

Proposition 3.3 *Multiplication is associative, that is:*

$$A(BC) = (AB)C, \quad (\forall A \in \mathcal{M}_{p,n}, \forall B \in \mathcal{M}_{n,q}, \forall C \in \mathcal{M}_{q,m}).$$

Multiplication is distributive on the left and right with respect to addition, that is:

$$A(B + C) = AB + AC, \quad (A + D)B = AB + DB, \quad (\forall A, D \in \mathcal{M}_{p,n}, \forall B, C \in \mathcal{M}_{n,q})$$

Proof. Left as an exercise. □

Note finally that multiplication is an internal law on the set $\mathcal{M}_n(K)$ of square matrices of order n, that is it is a map:

$$\mathcal{M}_n(K) \times \mathcal{M}_n(K) \longrightarrow \mathcal{M}_n(K).$$

One immediately verifies that the matrix I_n is the neutral element of multiplication, that is: $\forall A \in \mathcal{M}_n(K): \quad I_n A = A I_n = A.$

3.3 Matrix of a vector

Definition 3.4 *Let E be a vector space of dimension n, $\{e_1, \ldots, e_n\}$ a basis of E and $x = x_1e_1 + \cdots + x_ne_n$ a vector of E. We call the matrix of x in the basis $\{e_i\}$ the column matrix of the components of x in the basis $\{e_i\}$:*

$$M(x)_{e_i} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

(also denoted $M(x)$).

Remark 3.3 *This definition is consistent with the definition of the matrix associated with a linear map. Indeed, we can identify any vector of E with a linear map from K into E: to every x of E we associate the linear map*

If we write the matrix of f in the basis $\varepsilon = 1$ of K and $\{e_i\}$ of E, we have:

$$f(\varepsilon) = x = x_1 e_1 + \cdots + x_n e_n,$$

so:

$$M(f)_{\varepsilon, e_i} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

$$\begin{array}{rccc} f: & \mathbb{k} & \longrightarrow & E \\ & \lambda & \longmapsto & \lambda x. \end{array}$$

Proposition 3.4 *Let E, F be two vector spaces over K, $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_p\}$ two bases of E and F respectively. For any map $f \in \mathcal{L}_K(E, F)$ and for any $x \in E$, we have:*

$$M(f(x))_{\varepsilon_j} = M(f)_{e_i, \varepsilon_j} M(x)_{e_i},$$

or more briefly:

$$M(f(x)) = M(f)M(x).$$

Proof. Let

$$M(f)_{e_i, \varepsilon_j} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \ldots & a_{pn} \end{pmatrix},$$

which means that: $f(e_j) = a_{1j}\varepsilon_1 + \cdots + a_{pj}\varepsilon_p = \sum_{k=1}^{p} a_{kj}\varepsilon_k$. We have:

$$\begin{aligned} f(x) &= f(x_1 e_1 + \cdots + x_n e_n) = \sum_{j=1}^{n} x_j f(e_j) \\ &= \sum_{j=1}^{n} x_j \sum_{k=1}^{p} a_{kj}\varepsilon_k = \sum_{k=1}^{p} \underbrace{\left(\sum_{j=1}^{n} a_{kj} x_j \right)}_{y_k} \varepsilon_k \\ &= \sum_{k=1}^{p} y_k \varepsilon_k. \end{aligned}$$

So:

$$M(f(x))_{\varepsilon_j} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix},$$

with

$$y_k = \sum_{j=1}^{n} a_{kj} x_j.$$

On the other hand:

$$M(f)_{e_i,\varepsilon_j}\cdot M(x)_{e_i} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{pj}x_j \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$$

so:

$$M(f)_{e_i,\varepsilon_j}\cdot M(x)_{e_i} = M(f(x))_{\varepsilon_j}.$$

□

Example 3.9 *Let the plane $\mathbb{R}^2$ referred to its canonical basis. Determine the image of the vector $x = (3, 2)$ by the rotation with center O and angle $\pi/6$.*

We have:

$$M(f(x)) = M(f)\cdot M(x) = \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2}\,\frac{\sqrt{3}}{2} & \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}-2}{2} \\ \frac{3+2\sqrt{3}}{2} \end{pmatrix}.$$

3.4 Product of matrices

Proposition 3.5 *Let E, F, G be three vector spaces over K of dimensions n, p, q respectively. Let $\{e_i\}, \{\varepsilon_j\}, \{\eta_k\}$ be bases of E, F, G respectively. Let $f \in \mathcal{L}_K(E,F)$ and $g \in \mathcal{L}_K(F,G)$. Then:*

$$M(g\circ f)_{e_i,\eta_k} = M(g)_{\varepsilon_j,\eta_k} M(f)_{e_i,\varepsilon_j}.$$

Proof. Let

$$M(f)_{e_i,\varepsilon_j} = (a_{ji}), \quad M(g)_{\varepsilon_j,\eta_k} = (b_{kj}).$$

This means that:

$$f(e_i) = \sum_{j=1}^{p} a_{ji}\varepsilon_j, \quad g(\varepsilon_j) = \sum_{k=1}^{q} b_{kj}\eta_k.$$

Then:

$$\begin{aligned}(g\circ f)(e_i) &= g(f(e_i)) = g\left(\sum_{j=1}^{p} a_{ji}\varepsilon_j\right)\\ &= \sum_{j=1}^{p} a_{ji} g(\varepsilon_j) = \sum_{j=1}^{p} a_{ji}\sum_{k=1}^{q} b_{kj}\eta_k\\ &= \sum_{k=1}^{q}\left(\sum_{j=1}^{p} b_{kj}a_{ji}\right)\eta_k.\end{aligned}$$

So, the element of the k^{th} row and i^{th} column of $M(g\circ f)_{e_i,\eta_k}$ is:

$$\sum_{j=1}^{p} b_{kj}a_{ji},$$

which is the element of the k^{th} row and i^{th} column of the product $M(g)_{\varepsilon_j,\eta_k}\cdot M(f)_{e_i,\varepsilon_j}$. □

Example 3.10 *Let $f:\mathbb{R}^2\to\mathbb{R}^3$ and $g:\mathbb{R}^3\to\mathbb{R}^2$ defined by:*

$$f(x,y) = (x-y, x+y, 2x)$$
$$g(x,y,z) = (x+2y-z, 2x-y+z)$$

Let us determine the matrix of $g\circ f$ in the canonical bases.

We have:

$$M(f)_{can} = \begin{pmatrix} 1 & -1\\ 1 & 1\\ 2 & 0\end{pmatrix},\quad M(g)_{can} = \begin{pmatrix} 1 & 2 & -1\\ 2 & -1 & 1\end{pmatrix}$$

$$M(g\circ f)_{can} = M(g)_{can}\cdot M(f)_{can}$$

$$= \begin{pmatrix} 1 & 2 & -1\\ 2 & -1 & 1\end{pmatrix}\begin{pmatrix} 1 & -1\\ 1 & 1\\ 2 & 0\end{pmatrix} = \begin{pmatrix} 1+2-2 & -1+2-0\\ 2-1+2 & -2-1+0\end{pmatrix} = \begin{pmatrix} 1 & 1\\ 3 & -3\end{pmatrix}.$$

3.5 Matrix of the inverse of a map

Proposition 3.6 *Let E and F be two vector spaces over K of the same dimension n. Let $f\in\mathcal{L}_K(E,F)$ be an isomorphism. Then the matrix $M(f)_{e_i,\varepsilon_j}$ is invertible, and we have:*

$$\left(M(f)_{e_i,\varepsilon_j}\right)^{-1} = M(f^{-1})_{\varepsilon_j,e_i}.$$

Proof. We have: $f\circ f^{-1} = \mathrm{id}_F$ and $f^{-1}\circ f = \mathrm{id}_E$. By proposition 3.5, we have:

$$M(f)_{e_i,\varepsilon_j}M(f^{-1})_{\varepsilon_j,e_i} = M(f\circ f^{-1})_{\varepsilon_j,\varepsilon_j} = M(\mathrm{id}_F)_{\varepsilon_j} = I_n$$
$$M(f^{-1})_{\varepsilon_j,e_i}M(f)_{e_i,\varepsilon_j} = M(f^{-1}\circ f)_{e_i,e_i} = M(\mathrm{id}_E)_{e_i} = I_n.$$

So $M(f)_{e_i,\varepsilon_j}$ is invertible and its inverse is $M(f^{-1})_{\varepsilon_j,e_i}$. □

3.6 Inverse of a matrix

Definition 3.5 *A matrix $A \in \mathcal{M}_n(K)$ is said to be invertible if there exists a matrix $B \in \mathcal{M}_n(K)$ such that:*

$$AB = BA = I_n.$$

This matrix B is then unique, it is called the inverse of A and is denoted A^{-1}.

Example 3.11 *The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible and its inverse is $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.*

Proposition 3.7 *Let E be a vector space of dimension n, $\{e_i\}$ a basis of E and $f \in \mathcal{L}_K(E,E)$ an endomorphism. Then f is an isomorphism if and only if the matrix $M(f)_{e_i}$ is invertible.*

Proof. If f is an isomorphism, then by proposition 3.6, $M(f)_{e_i}$ is invertible and $M(f)_{e_i}^{-1} = M(f^{-1})_{e_i}$.

Conversely, if $M(f)_{e_i}$ is invertible, let $g \in \mathcal{L}_K(E,E)$ such that $M(g)_{e_i} = M(f)_{e_i}^{-1}$. Then:

$$M(f \circ g)_{e_i} = M(f)_{e_i} M(g)_{e_i} = M(f)_{e_i} M(f)_{e_i}^{-1} = I_n$$
$$M(g \circ f)_{e_i} = M(g)_{e_i} M(f)_{e_i} = M(f)_{e_i}^{-1} M(f)_{e_i} = I_n$$

so $f \circ g = g \circ f = \mathrm{id}_E$, that is f is an isomorphism and $f^{-1} = g$. □

3.7 Change of basis

Let E be a vector space of dimension n and $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ two bases of E. The change of basis is the passage from the first basis to the second. It is determined by the matrix of the vectors of the new basis $\{e'_i\}$ in the old basis $\{e_i\}$, that is:

$$M(\underset{E}{\mathrm{id}})_{e'_i, e_i} = \|e'_1, \ldots, e'_n\|_{e_i}.$$

This matrix is called the change of basis matrix from $\{e_i\}$ to $\{e'_i\}$.

If we denote $P = (p_{ij})$ this matrix, we have:

$$e'_j = p_{1j}e_1 + p_{2j}e_2 + \cdots + p_{nj}e_n = \sum_{i=1}^{n} p_{ij}e_i.$$

Proposition 3.8 *The change of basis matrix is invertible. Its inverse is the matrix of the identity map from the basis $\{e'_i\}$ to the basis $\{e_i\}$, that is:*

$$P^{-1} = M(\underset{E}{\mathrm{id}})_{e_i, e'_i}.$$

Proof. By proposition 3.6, we have:

$$M(\underset{E}{\mathrm{id}})^{-1}_{e'_i,e_i} = M(\overset{-1}{\underset{E}{\mathrm{id}}})_{e_i,e'_i} = M(\underset{E}{\mathrm{id}})_{e_i,e'_i}.$$

□

Let us now see how the components of a vector and the matrix of an endomorphism are transformed by a change of basis.

Proposition 3.9 *Let $x \in E$ and $X = M(x)_{e_i}, X' = M(x)_{e'_i}$. Then:*

$$X = PX', \quad \text{that is: } X' = P^{-1}X.$$

Proof. We have:

$$M(x)_{e_i} = M(\underset{E}{\mathrm{id}}(x))_{e_i} = M(\underset{E}{\mathrm{id}})_{e'_i,e_i} M(x)_{e'_i} = PX'.$$

□

Proposition 3.10 *Let $f \in \mathcal{L}_K(E,E)$ and $A = M(f)_{e_i}, A' = M(f)_{e'_i}$. Then:*

$$A' = P^{-1}AP.$$

Proof. We have:

$$\begin{aligned} A' &= M(f)_{e'_i} = M(\mathrm{id}_E \circ f \circ \mathrm{id}_E)_{e'_i,e'_i} \\ &= M(\mathrm{id}_E)_{e_i,e'_i} M(f)_{e_i} M(\mathrm{id}_E)_{e'_i,e_i} = P^{-1}AP. \end{aligned}$$

□

3.8 Rank of a matrix

Definition 3.6 *Let $A \in \mathcal{M}_{p,n}(K)$. We call the rank of A the dimension of the subspace of K^p generated by the columns of A. We denote it $\mathrm{rg}(A)$.*

Proposition 3.11 *Let E and F be two vector spaces over K of dimensions n and p respectively, $\{e_i\}$ and $\{\varepsilon_j\}$ bases of E and F respectively. Let $f \in \mathcal{L}_K(E,F)$. Then:*

$$\mathrm{rg}(f) = \mathrm{rg}\left(M(f)_{e_i,\varepsilon_j}\right).$$

Proof. We have:

$$\mathrm{rg}(f) = \dim \mathrm{Im}(f) = \dim \left[\mathrm{vect}\left(f(e_1), \ldots, f(e_n)\right)\right].$$

But the components of the vectors $f(e_1), \ldots, f(e_n)$ in the basis $\{\varepsilon_j\}$ are the columns of $M(f)_{e_i,\varepsilon_j}$. So the subspace generated by the columns of $M(f)_{e_i,\varepsilon_j}$ is isomorphic to $\mathrm{Im}(f)$, hence the result. □

Proposition 3.12 *Let* $A \in \mathcal{M}_{p,n}(K)$ *and* $B \in \mathcal{M}_{n,q}(K)$. *Then:*

$$\mathrm{rg}(AB) \leq \min\left(\mathrm{rg}(A), \mathrm{rg}(B)\right).$$

Proof. Let $f : K^n \to K^p$ and $g : K^q \to K^n$ be the linear maps whose matrices in the canonical bases are A and B respectively. Then $AB = M(f \circ g)$. But:

$$\mathrm{rg}(f \circ g) \leq \mathrm{rg}(f) \quad \text{and} \quad \mathrm{rg}(f \circ g) \leq \mathrm{rg}(g),$$

so:

$$\mathrm{rg}(AB) \leq \min\left(\mathrm{rg}(A), \mathrm{rg}(B)\right).$$

□