Chapter 3: Perfect Incompressible Fluid Dynamics

Introduction:

Fluid dynamics (hydrodynamics) is the science that deals with the behavior of fluids in motion.

In this chapter, we will study fluids in motion by neglecting friction, i.e. by considering them as perfect (zero viscosity) and incompressible (constant density).

Steady Flow (or Permanent Flow)

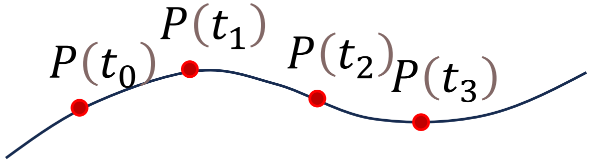
A flow is said to be steady (or permanent) when the velocity of the fluid do not change with time at any given point in space.

$$\frac{\partial V}{\partial t} = 0$$

Otherwise, the flow is said to be non-permanent or unsteady.

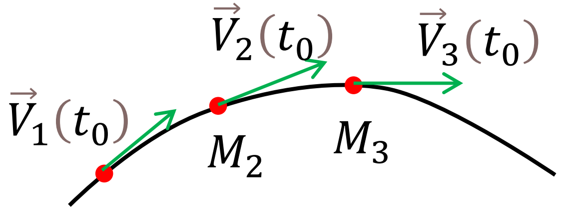
The Pathline of particle :

is the curve drawn by the successive positions occupied by the particle over time.



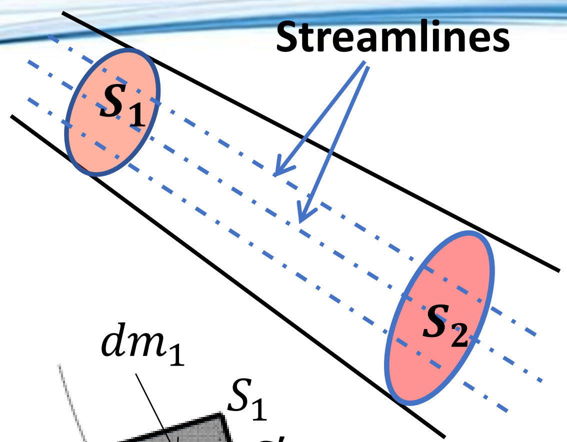
Stream line:

Is a curve tangent at each point to the velocity vector of the fluid at that instant



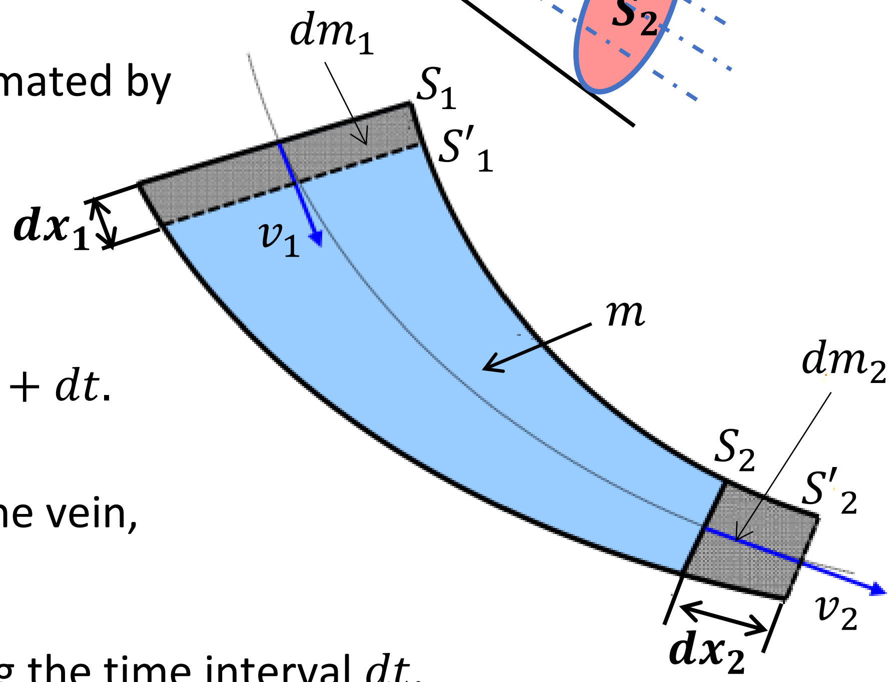
Stream tube:

A Stream tube is any surface formed by streamlines resting on a closed contour



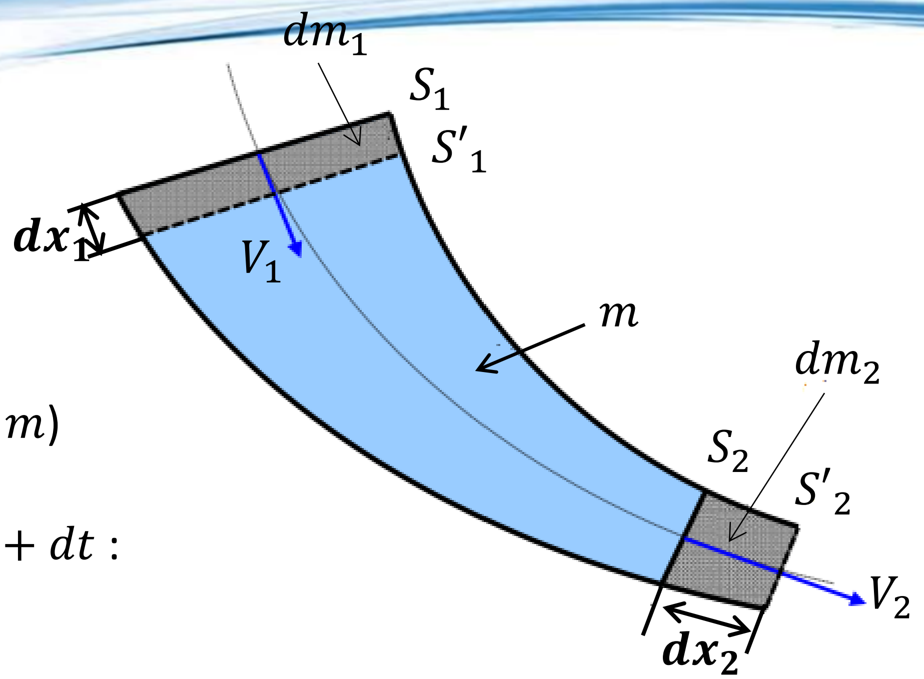
Equation of continuity (mass conservation):

Let us consider a vein (conduit) of an incompressible fluid, with a density ρ animated by a permanent flow. We refer to:



- S_1 and S_2 respectively the inlet and outlet cross-section of the fluid at time t,
- S'_1 and S'_2 respectively the inlet and outlet section of the fluid at time $t' = t + dt$.
- $\vec{v}_1$ and $\vec{v}_2$ are the velocity vectors of flow through the sections S_1 and S_2 of the vein,
- $\boldsymbol{dx_1}$and $\boldsymbol{dx_2}$ *are* respectively the movements of the sections S_1 and S_2 during the time interval dt,
- dm_1 is the incoming elementary mass between the sections S_1 and S'_1,
- dm_2 is the outgoing elementary mass between the sections S_2 and S'_2,
- m is the total mass between S'_1 et S_2,

- dV_1 : Elementary volume between sections S_1 et S'_1,
- dV_2 : Elementary volume between sections S_2 et S'_2,



- At time t : the fluid between S_1 and S_2 has a mass equal to $(dm_1 + m)$
- At time $t + dt$: the fluid between S'_1 and S'_2 has a mass equal to $(dm_2 + m)$

Let us apply the principle of conservation of the mass of fluid between t and $t + dt$:

$$dm_1 + m = dm_2 + m \quad \Longrightarrow dm_1 = dm_2 \quad \Longrightarrow \rho_1 dV_1 = \rho_2 dV_2 \quad \Longrightarrow \rho_1 S_1 dx_1 = \rho_2 S_2 dx_2$$

For an incompressible fluid, we have : $\rho_1 = \rho_2 = \rho \quad \Longrightarrow S_1 dx_1 = S_2 dx_2$

Dividing by dt, we get:

$$S_1 \frac{dx_1}{dt} = S_2 \frac{dx_2}{dt} \quad \Longrightarrow S_1 v_1 = S_2 v_2$$

The continuity equation represents the law of mass conservation

Concept of flow rate:

- The flow rate is a quantity of fluid flowed during time t.
- This quantity can be defined by a volume or a mass.

1. **Mass flow rate:** The mass flow of a fluid vein is the limit of the ratio $\frac{dm}{dt}$ when dt tends to 0.

$$q_m = \frac{dm}{dt}$$

q_m is the mass of fluid per unit time (Kg/s) which passes through any straight section of the pipe.

dm : The elementary mass in (kg) that passes through the section during a time interval dt .

dt : Time interval in (s)

Taking into account the ***continuity equation*** ***(****$dm_1 = dm_2$****)*** , we obtain: $q_m = \frac{dm}{dt} = \frac{dm_1}{dt} = \frac{dm_2}{dt} = \rho S_1 \frac{dx_1}{dt} = \rho S_2 \frac{dx_2}{dt}$

$$\Longrightarrow q_m = \rho S_1 v_1 = \rho S_2 v_2$$

In general, the mass flow rate is given by: $$\boldsymbol{q_m = \rho\, S\, v}$$

$\boldsymbol{\rho}$: density (Kg/m3) S : Fluid Vein Section (m^2) v : Fluid velocity through the cross-section S (m/s)

2. Volume flow rate: The volume flow rate q_v is the elementary volume dV of fluid that passes through an elementary straight surface dS during a time interval dt

$$q_v = \frac{dV}{dt}$$

We can also write the volume flow as a function of the flow velocity:

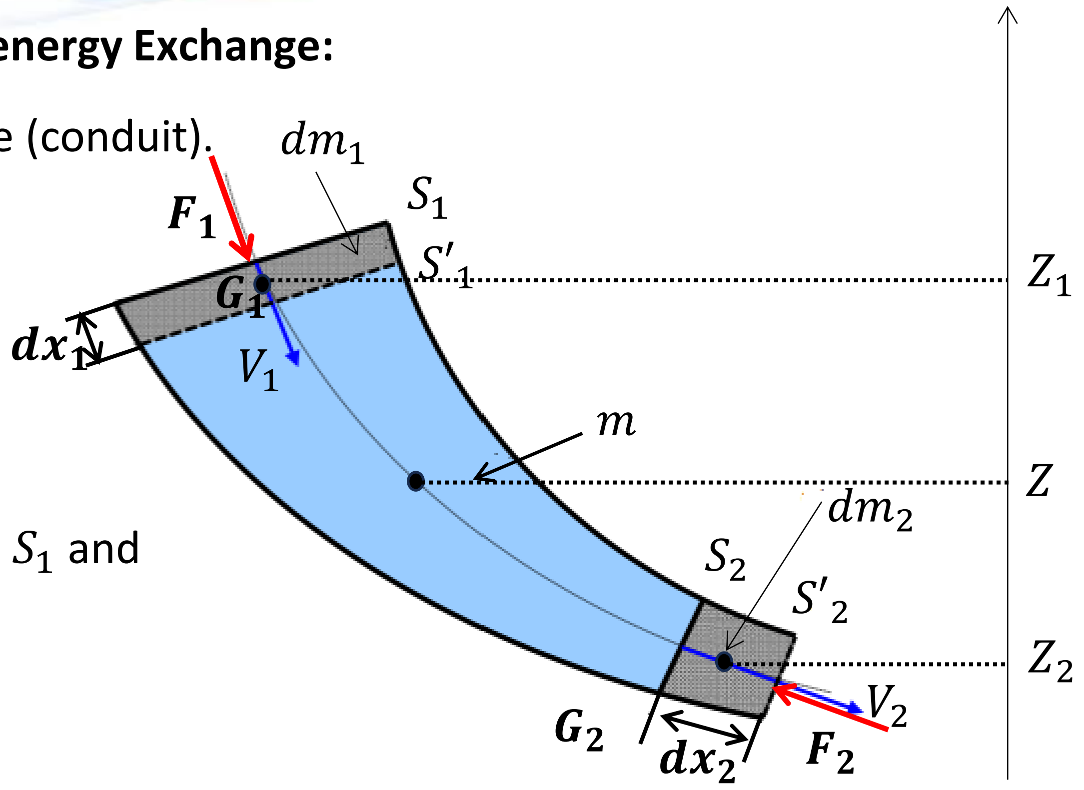
$$q_v = \frac{dV}{dt} = S\frac{dx}{dt} = Sv$$

3. Relationship between mass flow and volume flow:

$$q_m = \rho S v = \rho q_v$$

Bernoulli's theorem (Conservation of energy)- Case of a Flow without energy Exchange:

- Let be a permanent flow of an incompressible perfect fluid in a pipe (conduit).
- The two sections S_1 and S_2 delimit a certain mass of fluid at time *t.*
- we denote Z_1 and Z_2 respectively the heights of the centres of gravity of the masses dm_1 and dm_2.
- F_1 and F_2 are the norms of the fluid pressure forces acting at sections S_1 and S_2 respectively



Let us apply the kinetic energy theorem to dm:

$$\Delta E_C(dm) = \sum W_{\vec{F}_{ext}}$$ ***(The variation in kinetic energy is equal to the sum of the work of the external forces)***

The variation of kinetic energy: $\Delta E_C(dm) = \frac{1}{2} dm({v_2}^2 - {v_1}^2)$

The work of the force of gravity: $W_P = dm.g.(Z_1 - Z_2)$

The work of the inner forces is zero because the fluid is perfect ($\mu = 0$)

Work of pressure forces: **On section S_1**: $W_1 = F_1.dx_1 = P_1.S_1.v_1dt$

On section S_2: $W_2 = -F_2.dx_2 = -P_2.S_2.v_2dt$

On the side surface: $W_{sd} = 0$

$$\Rightarrow \frac{1}{2}dm({v_2}^2 - {v_1}^2) = P_1.S_1.v_1dt - P_2.S_2.v_2dt + dm.g.(Z_1 - Z_2)$$

$$\Rightarrow \frac{1}{2}\rho.dm({v_2}^2 - {v_1}^2) = P_1.\rho.S_1.v_1dt - P_2.\rho.S_2.v_2dt + dm.\rho.g.(Z_1 - Z_2)$$

According to the equation for the conservation of mass flow: $dm = \rho.S_1.v_1dt = \rho.S_2.v_2dt$

$$\Rightarrow \frac{1}{2}\rho.dm({v_2}^2 - {v_1}^2) = P_1.dm - P_2.dm + dm.\rho.g.(Z_1 - Z_2) \quad \Rightarrow \frac{1}{2}\rho({v_2}^2 - {v_1}^2) = P_1 - P_2 + \rho g(Z_1 - Z_2)$$

We arrive at Bernoulli's equation:

$$P_1 + \rho g Z_1 + \frac{1}{2}\rho {v_1}^2 = +P_2 + \rho g Z_2 + \frac{1}{2}\rho {v_2}^2$$

$$\Rightarrow P + \rho g Z + \frac{1}{2}\rho v^2 = Cte$$

The terms of this equation are energies per unit volume $[J/m^3]$, These are also pressure terms $[Pa]$.

$$\Longrightarrow P + \rho g Z + \frac{1}{2}\rho v^2 = Cte$$

Other forms of Bernoulli's theorem:

- Dividing the expression by ρg we get: $$Z + \frac{v^2}{2g} + \frac{P}{\rho g} = Cte \quad (in\ [m])$$

All terms are unit heights per meter [m] or energy per unit weight

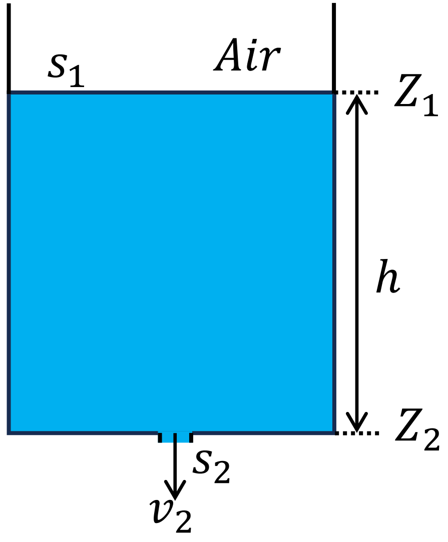
- Dividing the entire expression by ρ , we obtain: $$\frac{1}{2}v^2 + gZ + \frac{P}{\rho} = Cte \quad (\text{en } [J/K\ g]$$

All terms are energies per unit mass

Applications of Bernoulli's theorem:

1. Emptying a tank (reservoir) (Torricelli's theorem):

We consider a large tank open to the atmosphere containing a liquid of density ρ and pierced with a small orifice at its base at a height h of the free surface $(s_1 \gg s_2)$.



we apply Bernoulli's theorem between the two surfaces (1) and (2):

$$P_1 + \rho g Z_1 + \frac{1}{2}\rho {v_1}^2 = +P_2 + \rho g Z_2 + \frac{1}{2}\rho {v_2}^2$$

We have :

- $P_1 = P_2 = P_{atm}$
- $Z_2 = 0,\ Z_1 = h$ (Reference plane in 2)
- $s_1 \gg s_2 \Longrightarrow v_2 \gg v_1$ therefore $v_1 \cong 0$

We find: $v_2 = \sqrt{2gh}$ (Torricelli's formula)

Remark:

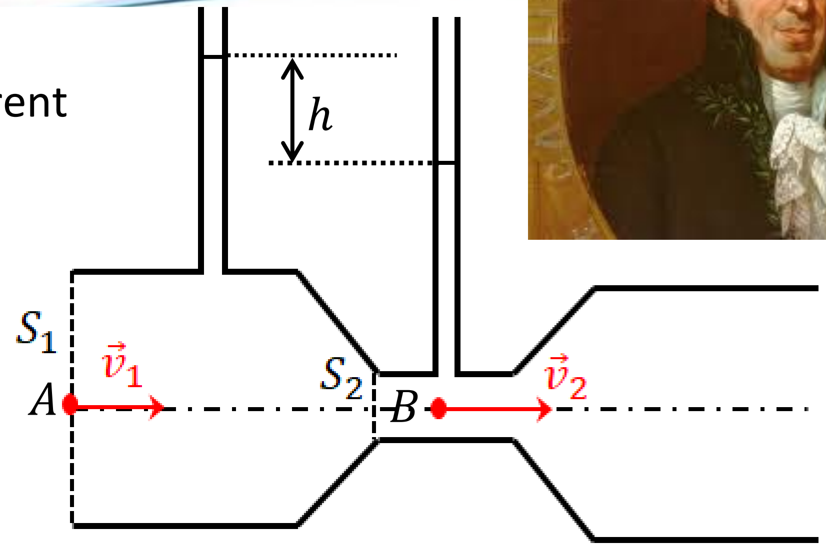
If s_2 is not very small compared to s_1, so we can't neglect v_1

We can then write, according to the continuity equation: $S_1 v_1 = S_2 v_2 \Longrightarrow v_1 = \frac{S_2}{S_1} v_2$

$$\cancel{P_1} + \cancel{\rho} g Z_1 + \frac{1}{2}\cancel{\rho} v_1^2 = \cancel{P_2} + \cancel{\rho} g \overset{0}{\cancel{Z_2}} + \frac{1}{2}\cancel{\rho} v_2^2 \Longrightarrow 2gh + v_1^2 = v_2^2 \Longrightarrow 2gh + \frac{s_2^2}{s_1^2} v_2^2 = v_2^2$$

$$\Longrightarrow 2gh = v_2^2 \left(\frac{s_1^2 - s_2^2}{s_1^2} \right) \qquad \boxed{\Longrightarrow v_2 = s_1 \left(\frac{2gh}{s_1^2 - s_2^2} \right)^{\frac{1}{2}}}$$

2. Venturi tube (Giovanni Battista Venturi (1746–1822)) :



A Venturi tube consists of a narrowing that separates two regions, of different cross-sections (S_1 and S_2 with $S_1 > S_2$), and a horizontal pipeline.

Two vertical tubes emerge from these regions and are open to the air.

- The velocity of the fluid flowing through the pipe increases in the throttle and its pressure decreases($v_2 > v_1 \Rightarrow P_2 < P_1$).
- Applying Bernoulli's theorem between the two points A and B:

$$P_1 + \rho g Z_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \rho g Z_2 + \frac{1}{2}\rho {v_2}^2 \ldots\ldots\ldots\ldots..(1)$$

- ✓ $Z_1 = Z_2$ (Same level)
- ✓ $S_1 v_1 = S_2 v_2$ (Continuity equation) $\Rightarrow v_2 = \frac{S_1}{S_2} v_1$
- ✓ The hydrostatic equation (Pascal's Law) between A and B : $P_1 - P_2 = \rho g h$

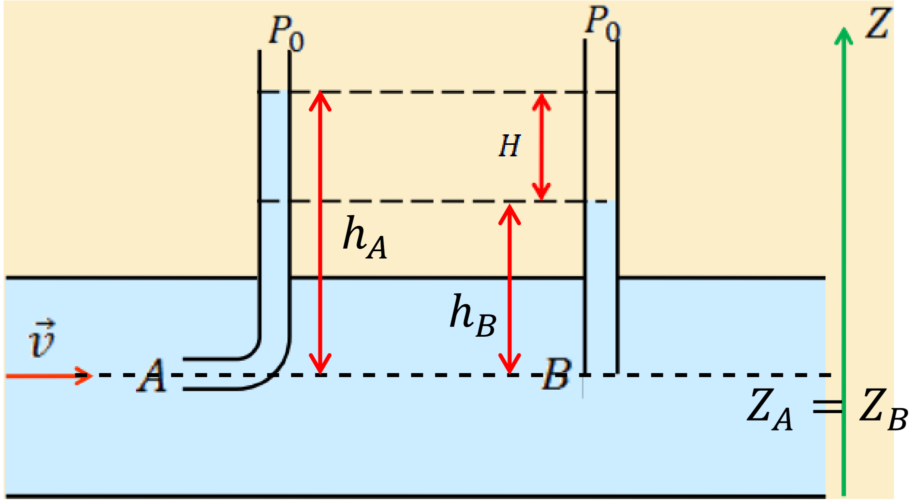
$$(1) \Leftrightarrow P_1 - P_2 = \frac{1}{2}\rho({v_2}^2 - {v_1}^2) = \frac{1}{2}\rho\left(\left(\frac{S_1}{S_2}\right)^2 - 1\right){v_1}^2 \Rightarrow v_1 = \sqrt{\frac{2(P_1 - P_2)}{\rho\left(\left(\frac{S_1}{S_2}\right)^2 - 1\right)}}$$

$$\boldsymbol{\Rightarrow v_1 = \sqrt{\frac{2gh}{\left(\left(\frac{S_1}{S_2}\right)^2 - 1\right)}}}$$

3. A Pitot tube (Henri Pitot (1695–1771))

- Consider a pipe with water flowing at a velocity of $\vec{v}$,
- The pipe was fitted with two tubes immersed in the liquid:

One opening at point A facing the flow, and the other at point B aligned with the streamlines. The liquid level difference H between the two tubes was measured.



✓ At point B: $\vec{v}_B = v$ (The liquid has the same speed as in the conduit)

✓ At point A : $\vec{v}_A = 0$ (stopping point; the flow velocity is zero)

✓ $Z_A = Z_B$: points A and B are on the same horizontal plane,

Applying Bernoulli's theorem between the two points A and B:

$$P_A + \rho g Z_A + \frac{1}{2}\rho {v_A}^2 = P_B + \rho g Z_B + \frac{1}{2}\rho {v_B}^2 \Rightarrow P_A - P_B = \frac{1}{2}\rho v^2 \;\ldots\ldots\ldots (2)$$

Applying Pascal's theorem (fundamental relation of hydrostatics):

At point A : $P_A = P_0 + \rho g h_A$

At point B : $P_B = P_0 + \rho g h_B$

$$\Rightarrow P_A - P_B = \rho g (h_A - h_B) = \rho g H \quad ; \qquad (2) \Leftrightarrow v = \sqrt{2gH}$$