Chapter 3: Fluid kinematics

Definitions:

- Fluid kinematics is the study of the movement of fluids without taking into account the forces that give rise to it.
- We only consider the relationships between the positions of fluid particles and time.
- In fluid kinematics, it is considered that the fluid is composed of fluid particles and that the study of the motion of these fluid particles constitutes the study of the motion of the fluid.
- To the fluid particle, we attach quantities:

1. *Position :* $\mathrm{M}(x, y, z, t)$,
2. *Velocity:* $\vec{V} = \vec{V}(x, y, z, t) \Longrightarrow$ it's variable vector field

   In Cartesian coordinates, we write: $\vec{V} = u(x, y, z, t)\vec{\imath} + v(x, y, z, t)\vec{\jmath} + w(x, y, z, t)\vec{k}$
3. *Acceleration*: $\vec{a} = \vec{a}(x, y, z, t)$

   In Cartesian coordinates, we write: $\vec{a} = a_x(x, y, z, t)\vec{\imath} + a_y(x, y, z, t)\vec{\jmath} + a_z(x, y, z, t)\vec{k}$
4. *Thermodynamic quantities* (density, temperature, pressure).

Notions of fluid flow:

Fluid flow is a fundamental concept in fluid mechanics, which is “the study of fluids and the forces acting on them”. The notions of fluid flow encompass various principles, theories, and models used to describe and analyze the behavior of fluids in motion.

Steady Flow (or Permanent Flow)

A flow is said to be steady (or permanent) when the velocity of the fluid do not change with time at any given point in space.

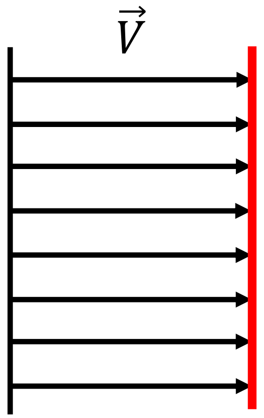
$$\frac{\partial V}{\partial t} = 0$$

Uniform flow:

If the velocity is invariant with respect to the position in a given section,

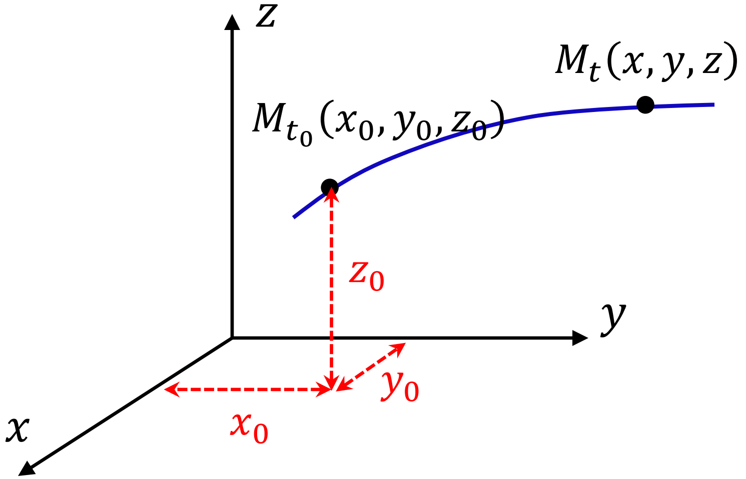
We speak of the uniform velocity profile:

$$\frac{\partial V}{\partial r} = 0$$



Description of the motion of a fluid particle:

To describe the motion of a fluid, two methods can be distinguished:

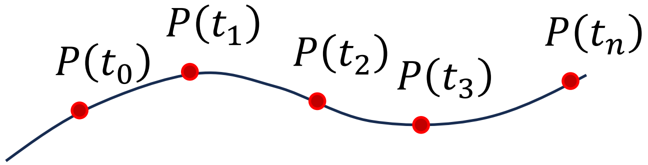


Lagrange method: This method consists of individualizing (isolate) a given particle of the fluid and following it in its motion.

- To do this, we consider at an initial moment t_0 a fluid particle M of coordinates $M_{t_0}(x_0, y_0, z_0)$and we follow it in its motion.

The Pathline of particle :

is the curve drawn by the successive positions occupied by the particle over time.



It is obtained experimentally by emerging in the fluid Colouring granules of the same density as him.

Each granule then draws the trajectory of the fluid particle that it occupies.

The position of this particle in time t, $M_t(x, y, z)$ is defined from the independent variables x_0, y_0, z_0 and t by :

$$M_t(x, y, z) = \begin{cases} x = f_1(x_0, y_0, z_0, t) \\ y = f_2(x_0, y_0, z_0, t) \\ z = f_3(x_0, y_0, z_0, t) \end{cases}$$

x, y *and* z are the Lagrange variables.

The components of velocity:

$$\vec{V}(v_x, v_y, v_z) = \begin{cases} v_x = \dfrac{\partial x}{\partial t} \\ v_y = \dfrac{\partial y}{\partial t} \\ v_z = \dfrac{\partial z}{\partial t} \end{cases}$$

The components of acceleration:

$$\vec{a}(a_x, a_y, a_z) = \begin{cases} a_x = \dfrac{\partial v_x}{\partial t} = \dfrac{\partial^2 x}{\partial t^2} \\ a_y = \dfrac{\partial v_y}{\partial t} = \dfrac{\partial^2 y}{\partial t^2} \\ a_z = \dfrac{\partial v_z}{\partial t} = \dfrac{\partial^2 z}{\partial t^2} \end{cases}$$

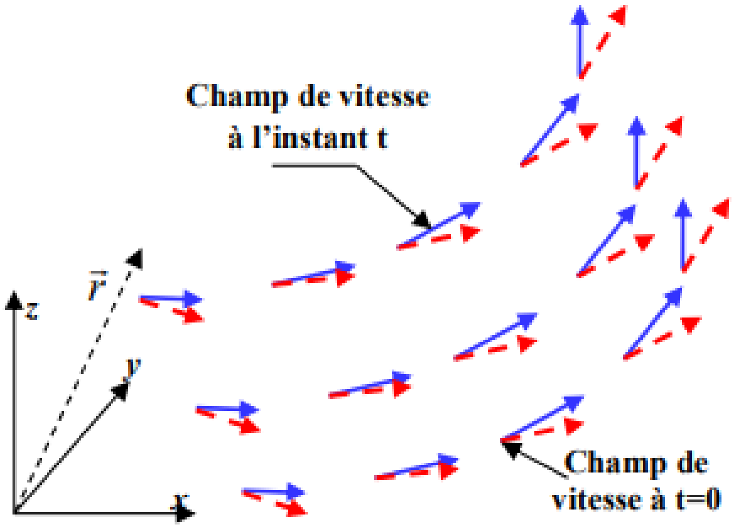
Euler's method:

In the Eulerian approach, instead of following the fluid particles in their motion, we place ourselves at a fixed point of the study frame of reference and we determine the velocity of the particle at a given moment t.

We will therefore determine; As a function of time, the velocity of the fluid particles that come successively passes through this point.

Euler's method is not concerned with the trajectory of particles, but with the field of velocity vectors, i.e. at each point in space, time (t) corresponds to a velocity ($\vec{V}$).

The flow is then described by a set of velocity vectors called “***the velocity vector field***”



$$\vec{V} = \begin{cases} v_x = f_1(x, y, z, t) \\ v_y = f_2(x, y, z, t) \\ v_z = f_3(x, y, z, t) \end{cases}$$

v_x, v_y, v_z are Euler variables

Eulerian acceleration and particle derivative

The total variation of the velocity components is given by:

$$dv_x = \frac{\partial v_x}{\partial t} dt + \frac{\partial v_x}{\partial x} dx + \frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz$$

$$dv_z = \frac{\partial v_z}{\partial t} dt + \frac{\partial v_z}{\partial x} dx + \frac{\partial v_z}{\partial y} dy + \frac{\partial v_z}{\partial z} dz$$

$$dv_y = \frac{\partial v_y}{\partial t} dt + \frac{\partial v_y}{\partial x} dx + \frac{\partial v_y}{\partial y} dy + \frac{\partial v_y}{\partial z} dz$$

The acceleration is given by: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k}$

With:

$$a_x = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z$$

$$a_y = \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial x} v_x + \frac{\partial v_y}{\partial y} v_y + \frac{\partial v_y}{\partial z} v_z$$

$$a_z = \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial x} v_x + \frac{\partial v_z}{\partial y} v_y + \frac{\partial v_z}{\partial z} v_z$$

We have : $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$

$$= \left(\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z\right)\vec{i} + \left(\frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial x}v_x + \frac{\partial v_y}{\partial y}v_y + \frac{\partial v_y}{\partial z}v_z\right)\vec{j} + \left(\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial x}v_x + \frac{\partial v_z}{\partial y}v_y + \frac{\partial v_z}{\partial z}v_z\right)\vec{k}$$

$$= \frac{\partial v_x}{\partial t}\vec{i} + \frac{\partial v_y}{\partial t}\vec{j} + \frac{\partial v_z}{\partial t}\vec{k} + \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)v_x\vec{i} + \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)v_y\vec{j} + \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)v_z\vec{k}$$

$$= \frac{\partial\vec{V}}{dt} + \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}) \qquad = \frac{\partial\vec{V}}{dt} + (\vec{V}.\vec{\nabla})\vec{V}$$

Physical interpretation:

$\frac{\partial\vec{V}}{dt}$: is called local acceleration, it is zero for a permanent flow

$(\vec{V}.\vec{\nabla})\vec{V}$: is the convective acceleration, reflects the non-uniformity of the flow, it is zero for a uniform flow

To see if a flow is uniform, the velocity is measured at different points in the flow at the same time.

Généralisation

Any variation of a scalar quantity F estimated by following a fluid particle in its motion must (Pression, density…), in Eulerian formalism, be calculated using a particle derivative consisting of the sum of a local term and a connective term

$$\frac{dF}{dt} = \frac{\partial F}{dt} + (\vec{V}.\vec{\nabla})F$$

Particulate derivative of density - Incompressible flow

Since density is a scalar, its particle derivative is expressed simply:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{dt} + \left(v_x\frac{\partial\rho}{\partial x} + v_y\frac{\partial\rho}{\partial y} + v_z\frac{\partial\rho}{\partial z}\right) = \frac{\partial\rho}{dt} + (\vec{V}.\vec{\nabla})\rho$$

- A *flow is said to be incompressible* if the volume of all fluid particles is independent of time.
- Since the mass of a fluid particle is constant by definition, a flow is incompressible if the density of a given fluid particle does not vary, i.e: $\frac{d\rho}{dt} = 0$

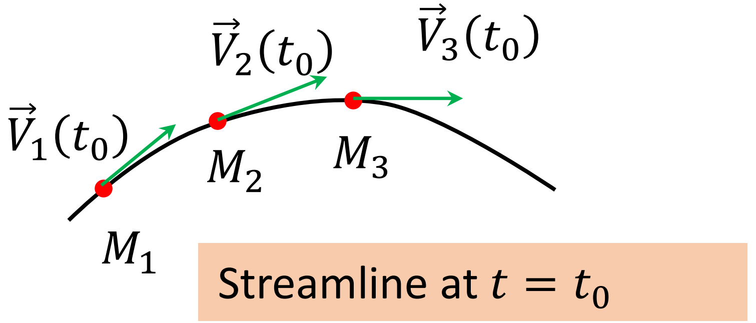
Remark:

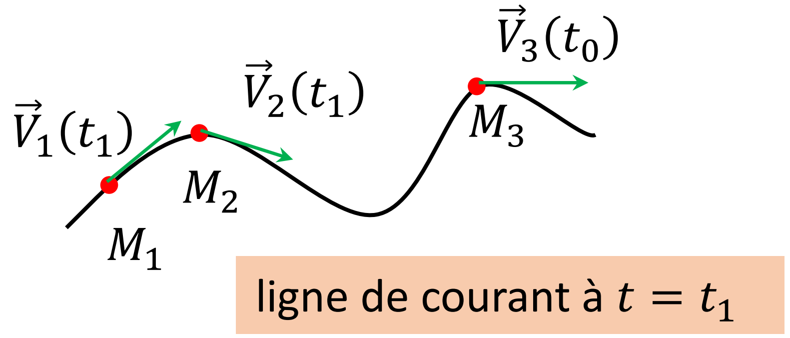
- For fluid mechanics, the most appropriate method is the ***Euleria*n method** which is concerned with the properties of flow at a fixed point in space as a function of time

Eulerian description of streamlines: The streamline is a line tangent at each point to the velocity.

Streamline equation

Mathematically, we can determine the equation of the streamlines at a given time from the velocity field data $\vec{V}(x,y,z,t)$, by setting the proportionality of the elementary displacement vector $d\overrightarrow{OM}(x,y,z,t)$ and the velocity vector $\vec{V}(x,y,z,t)$:





$$d\overrightarrow{OM} \parallel \vec{V} \Rightarrow d\overrightarrow{OM} \wedge \vec{V} = \vec{0} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$

$$\Rightarrow \begin{cases} wdy - vdz = 0 \\ wdx - udz = 0 \\ vdx - udy = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{v} = \dfrac{dz}{w} \\ \dfrac{dx}{u} = \dfrac{dz}{w} \\ \dfrac{dx}{u} = \dfrac{dy}{v} \end{cases} \Rightarrow \frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)}$$

Streamline equation

Exercise: Consider the velocity field given by: $u = y/(x^2 + y^2)$ and $v = -x/(x^2 + y^2)$.

Calculate the equation of the streamline passing through the point (0, 5).

Solution:

We ave: $$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)}$$

$$\Rightarrow \frac{dx}{u(x,y)} = \frac{dy}{v(x,y)} \Rightarrow dxv(x,y) = dyu(x,y) \quad \Rightarrow -dx\frac{x}{(x^2+y^2)} = \frac{dyy}{(x^2+y^2)} \quad \Rightarrow ydy = -xdx$$

By integration, we obtain: $y^2 = -x^2 + C$

Where C is a constant of integration.

For the streamline passed trough (0, 5), we have: $5^2 = -0^2 + C \Rightarrow c = 25$

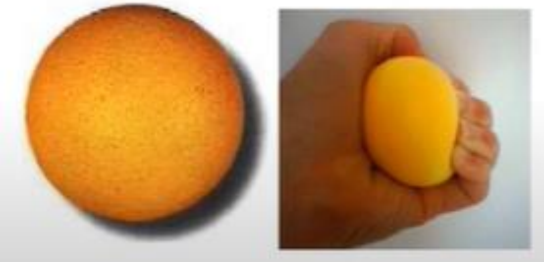
Thus, the equation of the streamline is: $y^2 = -x^2 + 25$

Note that the streamline is a circle with its center at the origin and a radius of 5 units.

Deformation of a fluid particle

- In a non-uniform velocity field, fluid particles deform. Deformations can be of several types: dilation, rotation in block, shear (cisaillement).
- It is the partial derivatives of the velocity field that determine how fluid particles deform.

rate of deformation tensor

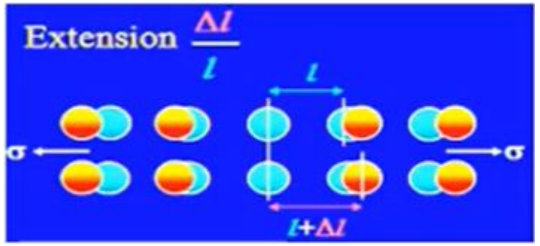


Extension (dilatatation) andGlissement (Shear):

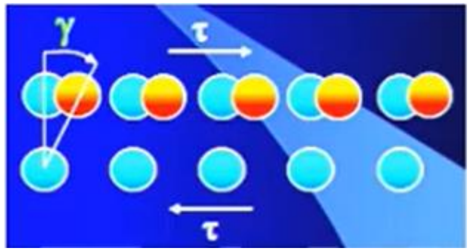
- To support a load, a material medium must deform
- We don't measure forces, stresses or strains (deformation),
- Only the amount of displacement that can be measured,

At the microscopic scale:

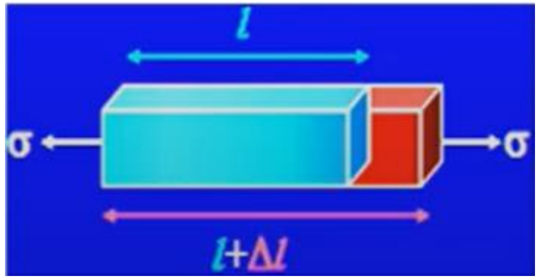
Extension $\frac{\Delta l}{l}$

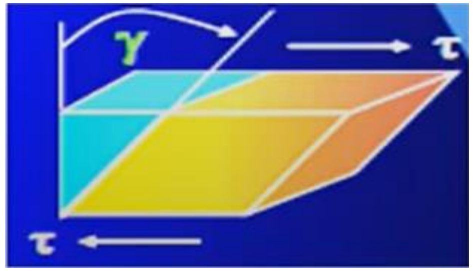


Glissement γ



On a macroscopic scale:

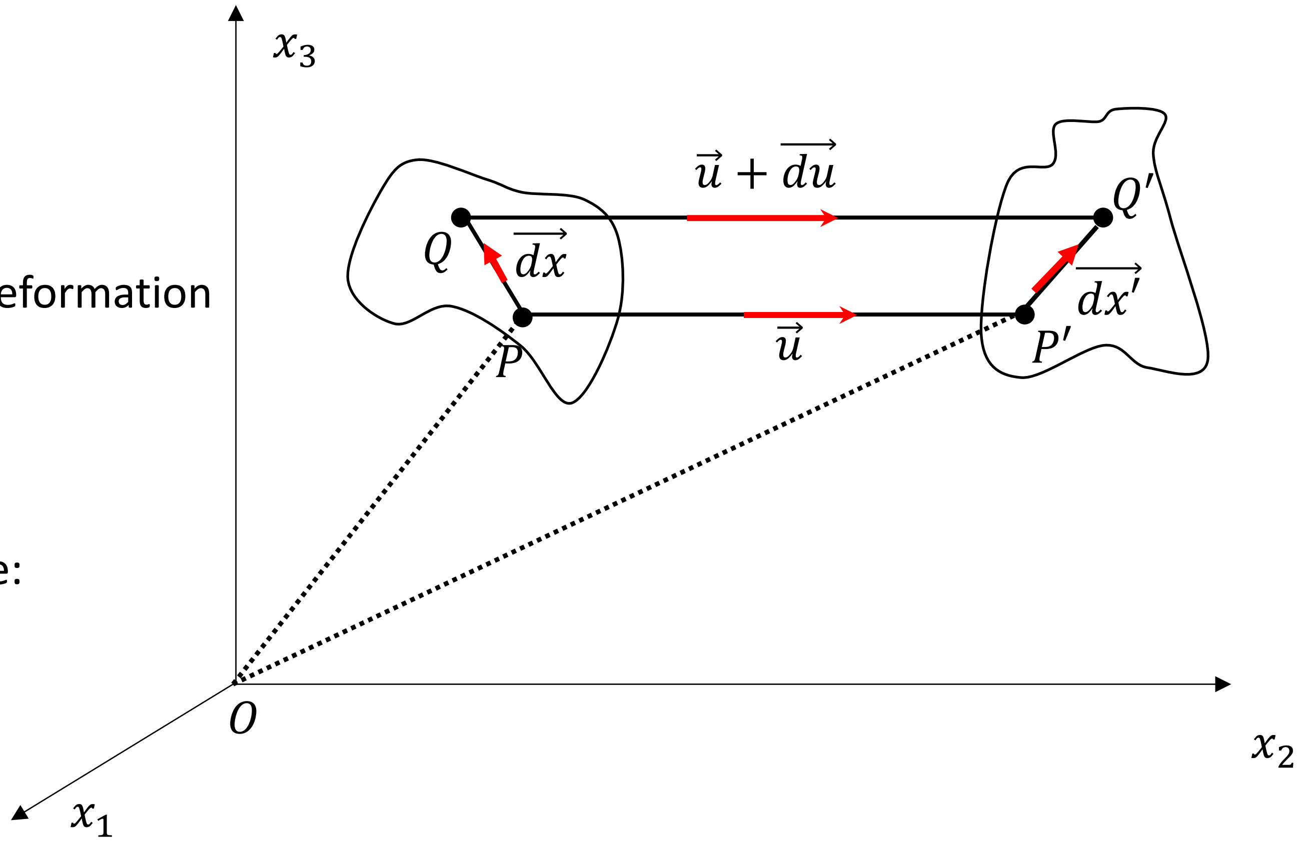




- The motions can be described either by Euler variable(u, v, w) or in Lagrange variable(x, y, z)
- Let us consider, in a Cartesian frame of reference, the position of the point P of a continuous body before and after the deformation
  - Before deformation: $P(x_1, x_2, x_3)$
  - After deformation: $P'(x'_1, x'_2, x'_3)$
- The displacement of the point of the body after the deformation is represented by the vector $\vec{u} = \overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}$
- For the components following the 3 axes, we also note:

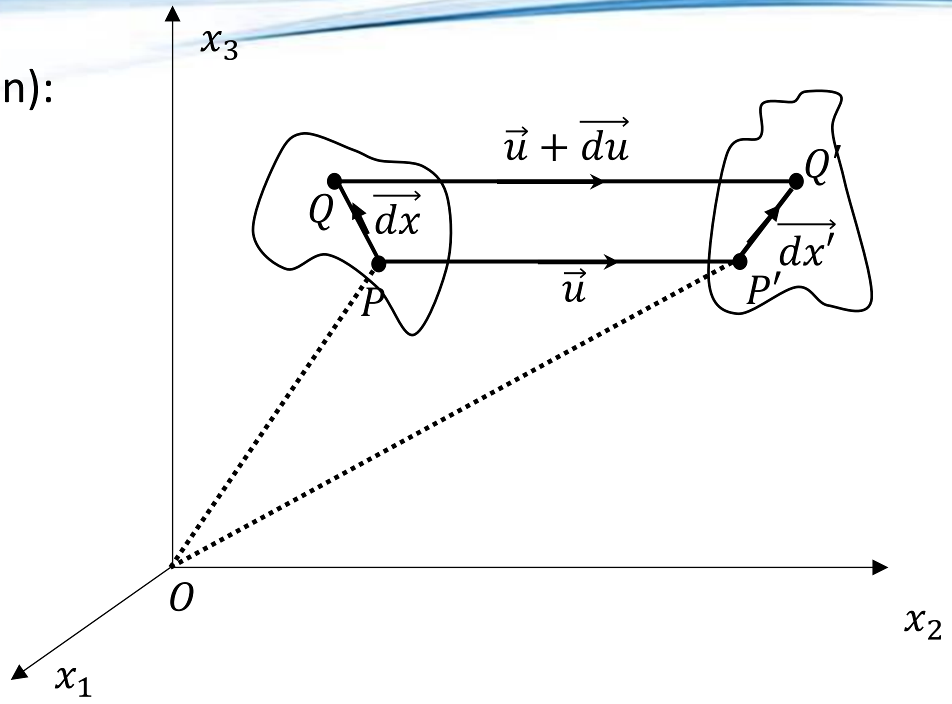
$$u_i = x'_i - x_i$$

- The given of $\vec{u}$ depending on the x_i completely determines the deformation of the body.



➢ Let be two infinitely neighboring points (before the deformation):

$P(x_1, x_2, x_3)$ et $Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$



➢ If dx_i is the vector radius between these two points before the deformation, it becomes after deformation $dx'_i = dx_i + du_i$

➢ The components of the vector $\overrightarrow{PQ}$ are: $\begin{cases} \overrightarrow{dx'}_1 = \overrightarrow{dx}_1 + \overrightarrow{du}_1 \\ \overrightarrow{dx'}_2 = \overrightarrow{dx}_2 + \overrightarrow{du}_2 \\ \overrightarrow{dx'}_3 = \overrightarrow{dx}_3 + \overrightarrow{du}_3 \end{cases}$

And as $\vec{u}_i$ depends on the $\overrightarrow{dx}_i$, consequently:

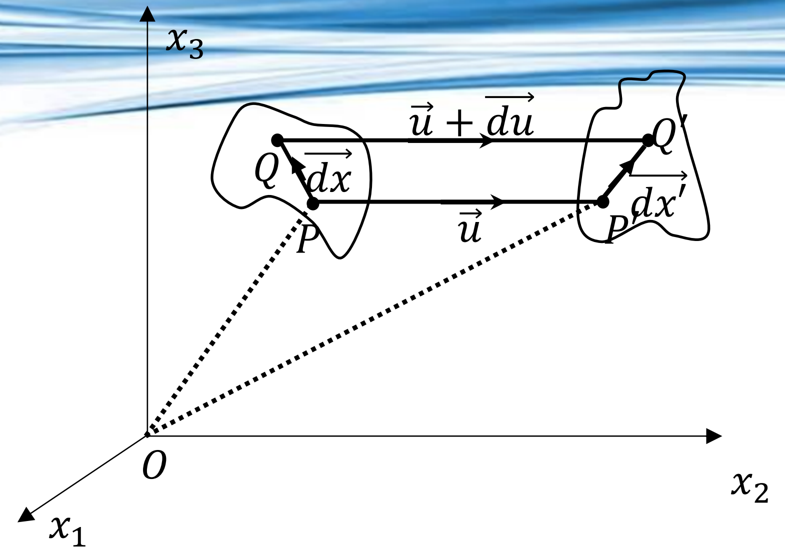
$$\overrightarrow{du}_1 = \frac{\partial u_1}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_1}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_1}{\partial x_3}\overrightarrow{dx}_3$$

$$\overrightarrow{du}_2 = \frac{\partial u_2}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_2}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_2}{\partial x_3}\overrightarrow{dx}_3$$

$$\overrightarrow{du}_3 = \frac{\partial u_3}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_3}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_3}{\partial x_3}\overrightarrow{dx}_3$$

$$\Rightarrow \begin{cases} \overrightarrow{dx'}_1 = \overrightarrow{dx}_1 + \frac{\partial u_1}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_1}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_1}{\partial x_3}\overrightarrow{dx}_3 \\ \overrightarrow{dx'}_2 = \overrightarrow{dx}_2 + \frac{\partial u_2}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_2}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_2}{\partial x_3}\overrightarrow{dx}_3 \\ \overrightarrow{dx'}_3 = \overrightarrow{dx}_3 + \frac{\partial u_3}{\partial x_1}\overrightarrow{dx}_1 + \frac{\partial u_3}{\partial x_2}\overrightarrow{dx}_2 + \frac{\partial u_3}{\partial x_3}\overrightarrow{dx}_3 \end{cases}$$

$$\overrightarrow{du} = \begin{cases} \overrightarrow{du}_1 = \dfrac{\partial u_1}{\partial x_1}\overrightarrow{dx}_1 + \dfrac{\partial u_1}{\partial x_2}\overrightarrow{dx}_2 + \dfrac{\partial u_1}{\partial x_3}\overrightarrow{dx}_3 \\ \overrightarrow{du}_2 = \dfrac{\partial u_2}{\partial x_1}\overrightarrow{dx}_1 + \dfrac{\partial u_2}{\partial x_2}\overrightarrow{dx}_2 + \dfrac{\partial u_2}{\partial x_3}\overrightarrow{dx}_3 \\ \overrightarrow{du}_3 + \dfrac{\partial u_3}{\partial x_1}\overrightarrow{dx}_1 + \dfrac{\partial u_3}{\partial x_2}\overrightarrow{dx}_2 + \dfrac{\partial u_3}{\partial x_3}\overrightarrow{dx}_3 \end{cases} = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{vmatrix} \begin{pmatrix} \overrightarrow{dx}_1 \\ \overrightarrow{dx}_2 \\ \overrightarrow{dx}_3 \end{pmatrix}$$



$$\Longrightarrow \overrightarrow{du} = |G|\overrightarrow{PQ}$$

$$G = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{vmatrix}$$ Denotes the gradient tensor of the displacement vector

$\overrightarrow{P'Q'} = \overrightarrow{PQ} + \overrightarrow{du} \quad = \overrightarrow{PQ} + |G|\overrightarrow{PQ}$ Avec $\overrightarrow{QQ'} = \vec{u} + \overrightarrow{du} \quad = \overrightarrow{PP'} + \overrightarrow{du} \quad = \overrightarrow{PP'} + |G|\overrightarrow{PQ}$

The rate of deformation tensor $|G|$ can be decomposed into a sum of a symmetric tensor and an antisymmetric tensor:

Symmetrical part

Antisymmetric part

$$|G| = |\Delta| + |\Omega|$$

$$\Delta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{And} \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$|\Delta| = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

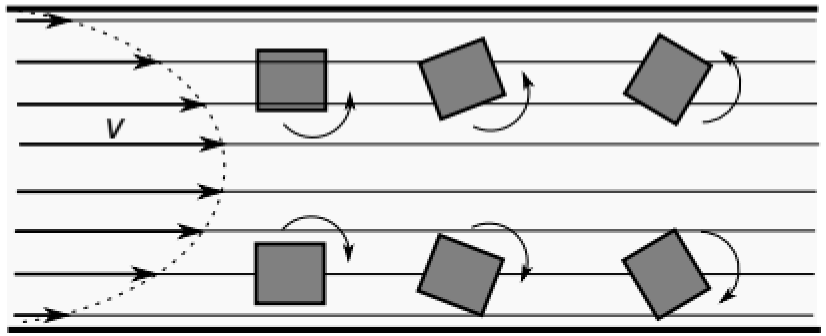
$$|\Omega| = \begin{vmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{vmatrix}$$

$$|\Delta| = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix}$$

$$|\Omega| = \begin{vmatrix} 0 & \xi_{12} & \xi_{13} \\ \xi_{21} & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{vmatrix}$$

Rotational and irrotational flow:

- Flow is ***rotational*** if the particles rotate around their center when they are moving.
- In this example, the velocity near the edge is lower than the velocity near the center, causing the fluid particles to rotate.
- En général, l’écoulement est rotationnel si les lignes de courant se referment sur elles-mêmes



- The flow is said to be ***irrotational*** if the particles do not rotate around their center when they are in motion and if at each instant the velocity field rotational is zero: $\overrightarrow{rot}\vec{V} = 0$