Chapter 4

Variable speed - Maxwell's equations

4-1- Introduction

In 1865, Maxwell unified electricity and magnetism into a single theory, called electromagnetism. Fields E and B cannot be considered independent, the variation of one over time requires the presence of the other. In addition, they constitute a unique physical entity, called an electromagnetic field. This theory is verified by all these consequences in particular, the existence of electromagnetic waves which propagate in a vacuum at a speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$ equal to the speed of light. Electromagnetic theory is represented under very short electromagnetic wavelengths.

4-2- Principle of conservation of charge

The principle of conservation of electric charge is considered as a physical principle that has always been tested by experiments, including experiments on collisions between relativistic particles carried out in large accelerators. In a conducting medium comprising different types of carrier identified by the index (i), having a volume density of charges ρi and an overall speed vi, we define the volume current density vector j and the intensity of the current I(s) through a surface (S) oriented :

$$j = \sum_i \rho_i v_i \, ; \qquad I_{(s)} = \iint_{(S)} j.dS$$

Then the intensity I(s) is equal to the charge which passes through (S) per unit of time:

$\delta q = I_{(S)} dt$

Let us consider a fixed volume (V), limited by the closed surface (Σ) and make a charge balance between time t and t + dt. The charge contained in (V) is written:

$q = \iiint_{(V)} \rho(M, t) d\tau$ (4.3)

Where ρ(M,t) designates the total volume density of charges. Between times t and t + dt, this load varies from

$$dq = \iiint_{(V)} \rho(M, t + dt)d\tau - \iiint_{(V)} \rho(M, t)d\tau$$

$$= \iiint_{(V)} [\rho(M, t + dt) - \rho(M, t + dt)]d\tau$$

Either with a Taylor expansion of order 1

$$dq = \iiint_{(V)} \frac{\partial \rho}{\partial t} dt d\tau = dt \iiint_{(V)} \frac{\partial \rho}{\partial t} d\tau$$

Between the same instants, the charge δq(Σ) which leaves (V) by crossing (Σ) is by definition worth the intensity:

$$\delta q_{(\Sigma)} = I_{(\Sigma)} dt = dt \oiint_{(\Sigma)} j. dS$$

The postulate of the conservation of charge states that the variation of the charge q contained in the volume (V) is only due to the transfer of charges δq(Σ) through (Σ). Conventionally orienting the surface (Σ) outwards, δq(Σ) counts positively when it contributes to decreasing q, therefore:

$$dq = -\delta q_{(\Sigma)} \text{ soit } dt \iiint_{(V)} \frac{\partial \rho}{\partial t} d\tau = -dt \oiint_{(\Sigma)} j. dS$$

Using the definition of divergence, we obtain:

$$\iiint_{(V)} \left(div\, j + \frac{\partial \rho}{\partial t} \right) d\tau = 0$$

This integral is valid whatever the volume (V), so we obtain the local equation for conservation of charge

$$div\, j + \frac{\partial \rho}{\partial t} = 0$$

This local equation presents structural analogies with numerous local equations reflecting the conservation of an extensive scalar quantity.

4-3- Maxwell-Ampère Law

This law is identical to Maxwell's local equation inherited from Ampère's theorem. In local form, it is written in terms of current density vector $\vec{j}$:

$$\overrightarrow{rot}\vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

- **Introduction of displacement current**

The previous equation can be rewritten as follows:

$\overrightarrow{rot}\vec{B} = \mu_0(\vec{J} + \overrightarrow{J_D})$

By introducing the Maxwell displacement current of :

$\overrightarrow{J_D} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

The integral form links the circulation of the magnetic field on a closed contour C, and the currents which cross the surface relying on this contour:

$\oint_l \vec{B}.\overrightarrow{dl} = \mu_0 \int_S \vec{J}.\overrightarrow{dS} + \varepsilon_0\mu_0 \frac{\partial}{\partial t}\int_S \vec{E}.\overrightarrow{dS}$

4-4- Maxwell Equations

4-4-1- Maxwell Equations in any medium

The electric field E and the magnetic induction field B are defined by their action on a particle of charge q and speed v, called the Lorentz force:

$F = q(E + v \times B)$

In the presence of matter, it is necessary to introduce the polarization density P and the magnetization density M or, equivalently, the electric displacement $D = \varepsilon o E + P$ and the magnetic field $H = B/\mu o - M$. The fields E, D,B and H obey Maxwell's law equations in local form:

$\nabla. D = q_v$ (Gauss law)

$\nabla \times E + \partial_t B = 0$ (Faraday law)

$\nabla. B = 0$ (Absence of magnetics charges)

$\nabla \times H = j + \partial_t D$ (Ampère law)

These are the basic equations of electromagnetic phenomena in vacuum and in matter. They are valid in insulators $(j = 0)$ and in conductors. In the particular case of an ohmic conductor, $j = \sigma E$, where σ is the conductivity. The first three equations were established from experimental observations. The fourth equation is a generalization of Ampère's law. qv is the charge density of free charges and j is the free current density (i.e. conduction, convection and beam currents). The displacement current $\partial_t D$ was introduced by Maxwell in order to conserve the charge conservation equation $\partial_t q_v + \nabla. j = 0$. We can reverse the argument and derive this charge conservation equation as a consequence of the equations.

Sometimes it is useful to write Maxwell's equations in integral form instead of local differential form. For this, we integrate the two sides of [9.12] and [9.14] on a volume V and we use the Gauss-Ostrogradsky theorem to transform the integral volume of the divergence in the flow leaving through the surface S, which limits V. As with equations [9.13] and [9.15], we calculate the flow on either side through a surface S and we use Stokes' theorem to transform the flow of the loop into a circulation on the curve C, which limit S. Thus, we obtain the four equations

$\iint_s dS.n.D = Q^{(in)}$ où $Q^{(in)} = \iiint_v dV q_v$ (Gauss law)

$\int_l dr.E + \partial_t \iint_s dS.n.B = 0$ (Faraday law)

$\iint_s dS.n.B = 0$ (Absence of magnetic charges)

$\int_l dr.H - \partial_t \iint_s dS.n.D = I^{(in)}$

Where $I^{(in)} = \iint_s dS.n.j$ (Ampère law)

These equations hold even if the fields have discontinuities. On the other hand, the derivatives with respect to time of the flows receive contributions from both variations of the fields over time and displacement of the surfaces. For these reasons, Maxwell's equations in integral form have more general validity than the differential form. In these integral forms, fields, charges and currents are taken at the same time. For example, if a charge enters a closed surface S, the flux of its electric field leaving S is only non-zero when it is inside S. However, some effects (like propagation) depend on the local field properties; they are best analyzed using the local form of Maxwell's equations.

4-4-2- Maxwell's equations in a homogeneous, isotropic and linear medium

If the medium is homogeneous, linear and isotropic, D is proportional to E and B is proportional to H:

$D = \varepsilon E$ and $B = \mu H$

Where ε and μ are characteristics of the medium. We can then only use the fields E and B, and Maxwell's equations can be written as:

$\nabla . E = q_v / \varepsilon$ (Gauss law)

$\nabla \times E + \partial_t B = 0$ (Faraday law)

$\nabla . B = 0$ (Absence of magnetics charges)

$\nabla \times B = \mu j + \varepsilon\mu\partial_t D$ (Ampère law)

In particular, in vacuum ε becomes ε_o and μ becomes μ_o. To write the integral forms of the equations in a linear but inhomogeneous manner, it is necessary to assign the constants ε and μ corresponding to each element of area dS or element of volume dV; we find:

$\iint_s \ S.dS.\varepsilon.n.E = Q^{(in)}$ (Gauss Law)

$\int_l \ dr.E + \partial_t \iint_s \ dS.n.B = 0$ (Faraday law) (4.28)

$\iint_s \ dS.n.B = 0$ (Absence magnetics charges) (4.29)

$\int_l \ dr.{}^{B}/_{\mu} = I^{(in)} + \left({}^{d}/_{dt}\right) \iint_s \ dS.\varepsilon.n.D$ (Ampère law) (4.30)

4-4-2- propagation fields Equations

Maxwell's equations form a system of coupled partial differential equations of the first order. It is possible to write decoupled equations for each of the fields; but they are second order partial differential equations. Indeed, let us evaluate the loop of Faraday's equation [4.24], we find$\nabla \times (\nabla \times E) + \partial t(\nabla \times B) = 0$. Using the identity

$\nabla \times (\nabla \times E) = \nabla(\nabla.E) - \Delta E$, Gauss's law [9.23] and Ampère's law [9.26], we find $\Delta E - \left({}^{1}/_{v^2}\right) \partial_{tt}{}^{2} E = \mu \partial_{tt}.j + \nabla q_v/\varepsilon$

Where : $v = {}^{1}/_{\sqrt{\mu\varepsilon}}$

Similarly, let's calculate the curvature of both sides of [9.26] and use equations [9.24] and [9.21], we find the equation

$\Delta B - \left({}^{1}/_{v^2}\right) \partial_{tt}{}^{2} B = -\mu \nabla \times \mathrm{j}$

The uncoupled equations, [9.31] and [9.32], for E and B determine properties of the fields E and B if we know the charge density and the current density at each point in space. If these densities are equal to zero everywhere, these equations become:

$\Delta E - \left({}^{1}/_{v^2}\right) \partial_{tt}{}^{2} E = 0,$ $\Delta B - \left({}^{1}/_{v^2}\right) \partial_{tt}{}^{2} B = 0$ (4.33)

These are the propagation equations (or wave equations) called d'Alembert equations. The speed of propagation is

$$v = {}^{1}/_{\sqrt{\mu\varepsilon}}$$

In a vacuum, this speed is the same as the speed of light in a vacuum.

$$c = {}^{1}\!\big/_{\sqrt{\mu_0 \varepsilon_0}} = 2.997\,924\,58 \times 10^8 m/s$$

4-5- Ohm's law localized:

Ohm's local law for a fixed conducting medium is defined by the following relation:

$$j = \gamma E \;\; (Am^{-2})$$

γ: Conductivity of electrical energy of the medium $(S.m^{-1})$.

Resistivity is its inverse $1/\,\gamma$ (Ωm)

Ohm's law for a filiform circuit AB

$$V_A - V_B = Ri$$

Where

i : intensity (A)

$R = {}^{L}\!/_{\gamma S}$

R : resistance of string (Ω) between A et B, with L its length (m) and S its section (m2)

The Joule power dissipated between A and B is given by the following equation:

$$P = (V_A - V_B)i = Ri^2 \qquad \text{(W)}$$

Local Ohm's law for a mobile conducting medium

$$j = \gamma(E + v \wedge B)$$

Where

v : driver's movement speed.

$v \wedge B$: Electromotor field, comes from the fact that the driver ‘sees’ an electric field in his own frame of reference$E' = E + v \wedge B$ differ from that (E) of the fixed frame of reference.

4-6-Conditions to the limits

Let us focus on the interface between two media 1 and 2. This interface is likely to carry a surface charge σ as well as a surface current $\vec{j}$. The integration of Maxwell's equations between two points very close to the interface, a point $M1$ in medium 1 and a point $M2$ in medium 2, leads to the following relationships:

$div\vec{E} = \frac{\rho}{\varepsilon_0}$ gives $\overrightarrow{E_{2n}} - \overrightarrow{E_{1n}} = \frac{\sigma}{\varepsilon_0}\overrightarrow{n_{1\to2}}$

$div\vec{B} = 0$ gives $\overrightarrow{B_{2n}} - \overrightarrow{B_{1n}} = \vec{0}$

$\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ gives $\overrightarrow{E_{2t}} - \overrightarrow{E_{1t}} = \vec{0}$

$\overrightarrow{rot}\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ gives $\overrightarrow{B_{2t}} - \overrightarrow{B_{1t}} = \mu_0 \overrightarrow{J_s} \wedge \overrightarrow{n_{1\to 2}}$

These equations can be summarized in two equations, which for a point M1 in medium 1 and a point M2 in medium 2, neighbors of the interface:

• Transition relation for boundary conditions

$\overrightarrow{E_2} - \overrightarrow{E_1} = \frac{\sigma}{\varepsilon_0} \overrightarrow{n_{1\to 2}}$

$\overrightarrow{B_2} - \overrightarrow{B_1} = \mu_0 \overrightarrow{J_s} \wedge \overrightarrow{n_{1\to 2}}$