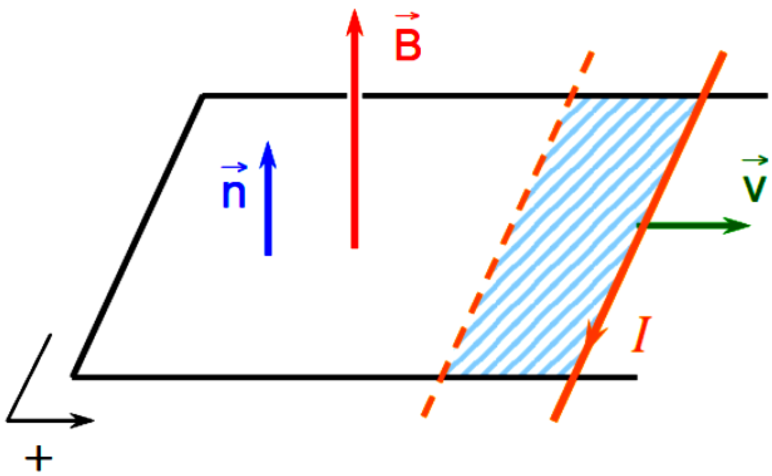
Chapter 3

Time-dependent phenomenon (quasi-stationary regime)

1. Experimental Observations

1.1. Circuit déformable dans un champ d'induction magnétique uniforme et constant

On a horizontal plane we define a positive direction of movement and the associated normal.

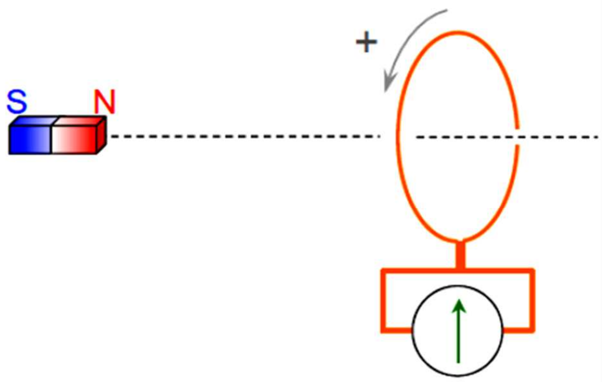


Moving the bar causes the appearance of a current with a sign opposite to the chosen direction (+).

1.2.Non-deformable circuit moving in a non-uniform magnetic induction field.

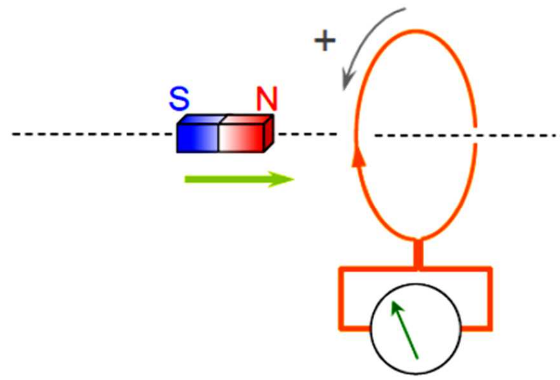
We consider a magnet moving with a translation movement towards a conductive circuit loop:

L'aimant est immobile à coté de la spire



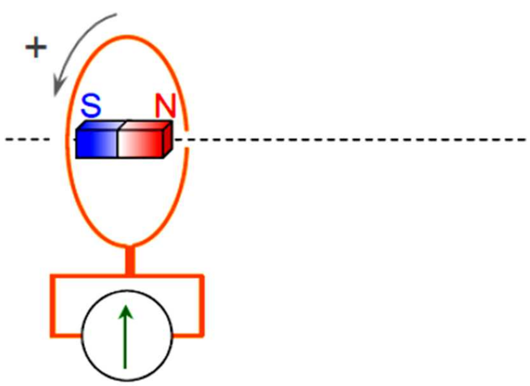
Le flux du champ $\vec{B}$ créé par l'aimant à travers la spire est constant

On approche l'aimant de la spire :



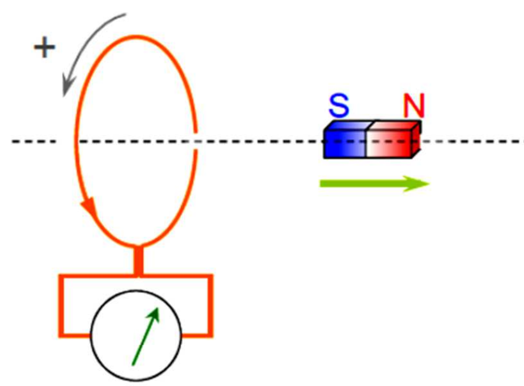
Le flux du champ $\vec{B}$ créé par l'aimant à travers la spire augmente.
Il apparaît un courant négatif dans la spire

L'aimant est immobile au centre de la spire



Le flux du champ $\vec{B}$ créé par l'aimant à travers la spire est constant et maximal
Il ne circule aucun courant dans la spire

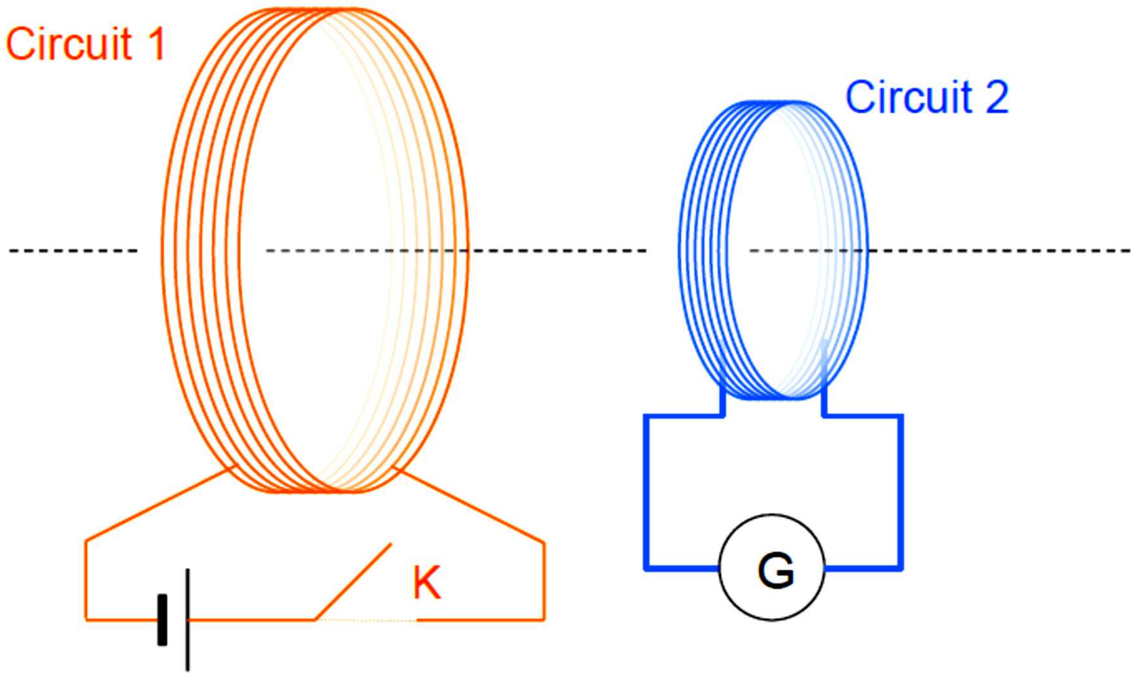
On éloigne l'aimant de la spire



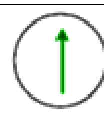
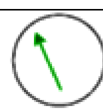
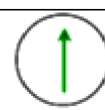
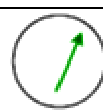
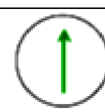
Le flux du champ $\vec{B}$ créé par l'aimant à travers la spire diminue
Il apparaît un courant négatif dans la spire

1.3.Undeformable circuit in a time-varying magnetic induction field

We consider the following experimental setup:



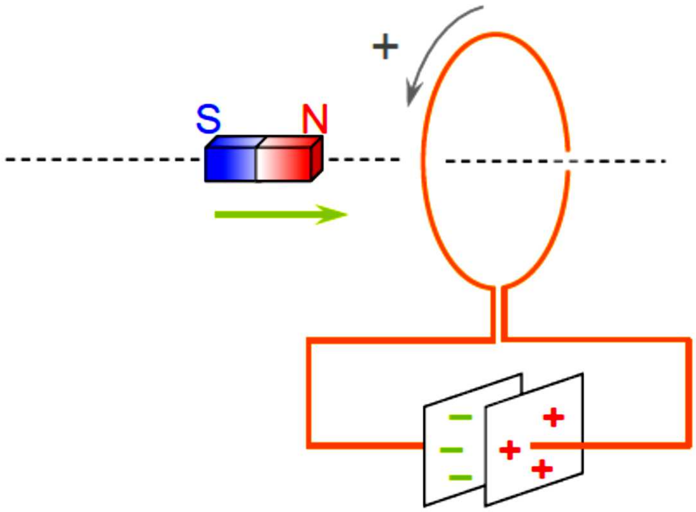
The following experiments are carried out successively:

Switch of circuit 1	Galvanometer of circuit 2
Open $\vec{B} = \vec{0}$	
Close $\vec{B} \neq \vec{0}$	 Then 
Open $\vec{B} = \vec{0}$	 Then 

1.4.Conclusions

- From these experiments, it appears that it is not the flow of $\vec{B}$ across the surface formed by the circuits that matters but the variation of this flow as a function of time.

- Experiments carried out with open circuits show that a potential difference appears between the terminals of the circuit subject to the flux variation of $\vec{B}$.
- The current which appears in closed circuits following the variation in flow is called induced current.



2. Interpretation of experiments: Faraday's law – Lenz's law

2.1.Faraday's law

Generally speaking, when a filiform circuit C is subjected to a variation in magnetic induction flux $\vec{B}$ over time and the origin of which is due to:

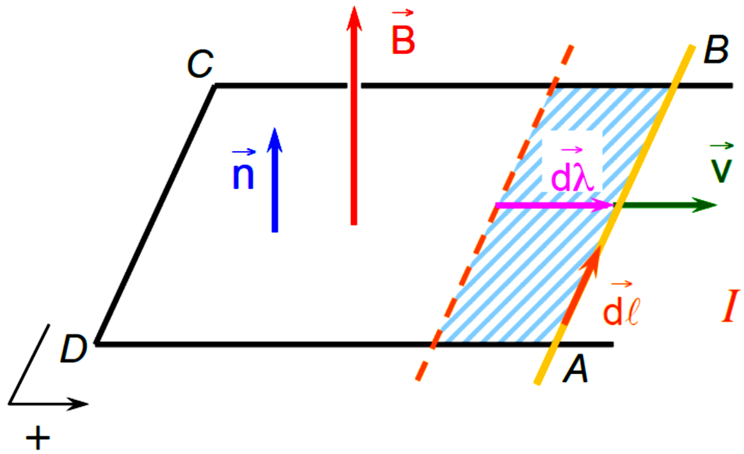
- a deformation of C, - a movement of C in a non-uniform magnetic field,

- a variation of $\vec{B}$ over time, we observe the appearance of an electromotive induction force:

$$e = - d\varphi/dt \qquad (3.1) \qquad \text{Faraday's law}$$

2.2.Results interpretation

2.2.1. Experience 1



The conductive bar is moved at speed $\vec{V}$ over the length dλ during the time dt.

We have seen that the e– contained in the bar are subject to the magnetic component of the Lorentz force:

$$\overrightarrow{F_M} = q\vec{V} \wedge \vec{B} = q\overrightarrow{E_m} \qquad (3.2)$$

Where $\overrightarrow{E_m}$ is the electromotive field.

Let us calculate the circulation of the force on the closed contour (ABCD). Although the mobile charges move throughout the circuit, the magnetic force is only non-zero on the segment [AB], this circulation is:

$$\oint_{ABCD} \overrightarrow{F_M}\,\overrightarrow{dl} = q\int_A^B (\vec{V} \wedge \vec{B})\overrightarrow{dl} = -q\int_A^B \left(\vec{V} \wedge \overrightarrow{dl}\right).\vec{B}$$

With: $\vec{V} = \frac{\overrightarrow{d\lambda}}{dt}$

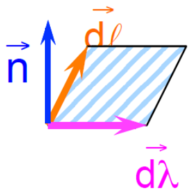
And: q=-1.602 10^{-19}C (électrons)

$$\oint_{ABCD} \overrightarrow{F_M}\,\overrightarrow{dl} = -q\int_A^B \left(\frac{\overrightarrow{d\lambda}}{dt} \wedge \overrightarrow{dl}\right).\vec{B} \qquad (3.4)$$

The integration over dl and the differentiation with respect to time being two independent operations, we can interchange them:

$$\oint_{ABCD} \overrightarrow{F_M}.\overrightarrow{dl} = -q\frac{1}{dt}\left[\int_A^B \vec{B}.\left(\overrightarrow{d\lambda} \wedge \overrightarrow{dl}\right)\right]$$

$(\overrightarrow{d\lambda} \wedge \overrightarrow{d\ell})$ is an oriented vector parallel to $\vec{n}$ and whose norm corresponds to the surface element d^2S:



The integration is only done on dℓ,

$$\int_A^B \vec{B}.\left(\overrightarrow{d\lambda} \wedge \overrightarrow{dl}\right) = d\emptyset$$

Hence, finally:

$$\oint_{ABCD} \overrightarrow{F_M}.\overrightarrow{dl} = -q\frac{d\emptyset}{dt}$$

The quantity:

$$\frac{1}{q}\oint_{ABCD} \overrightarrow{F_M}.\overrightarrow{dl}$$

is homogeneous at an electric potential and in fact corresponds to the electromotive force (e.m.f.) present in the circuit:

$$e = \frac{1}{q}\oint_{ABCD} \overrightarrow{F_M}.\overrightarrow{dl} \qquad (3.5)$$

We thus find Faraday's law:

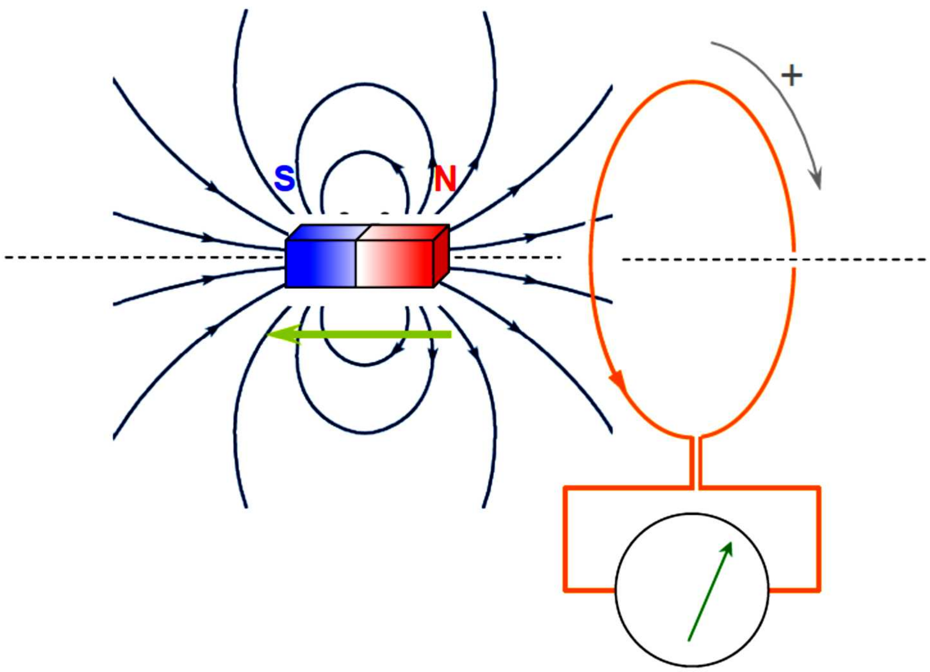
$$e = -\frac{d\emptyset}{dt} \qquad (3.6)$$

Noticed : From the previous demonstration, we can show that the circulation of this electromotive field on a closed circuit is not zero and is equal to the e.m.f.

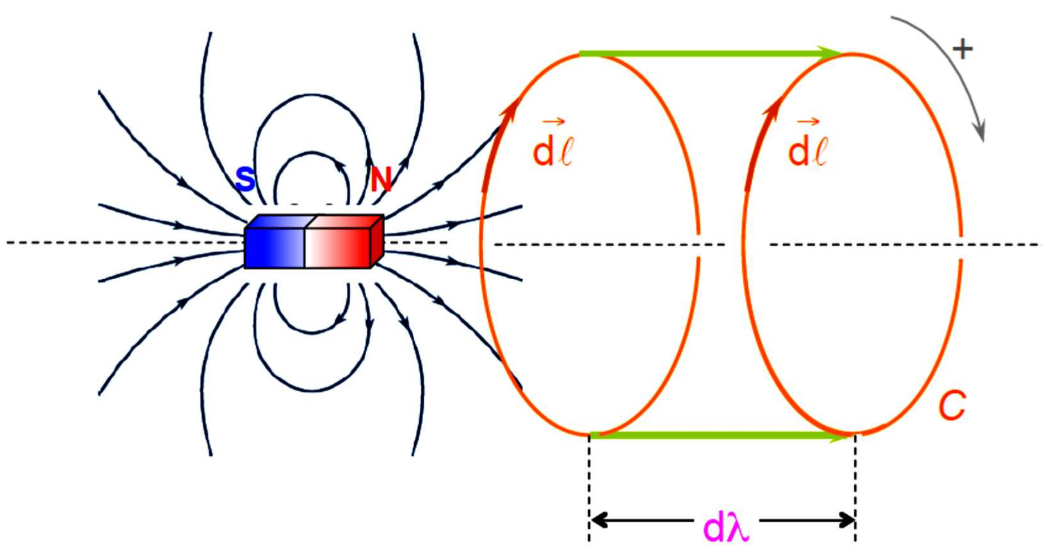
This induction electric field therefore does not derive from a scalar potential and its rotational is not zero.

2.2.2. Experience 2

We move the magnet away from turn C:



This amounts to spreading the turn of the magnet at speed $\vec{V}$:



Moving charges are subject to the magnetic component of the Lorentz force:

$\overrightarrow{F_M} = q\vec{V} \wedge \vec{B}$ (3.7)

Everything happens as if we had inserted an e.m.f. e generator. in the circuit.

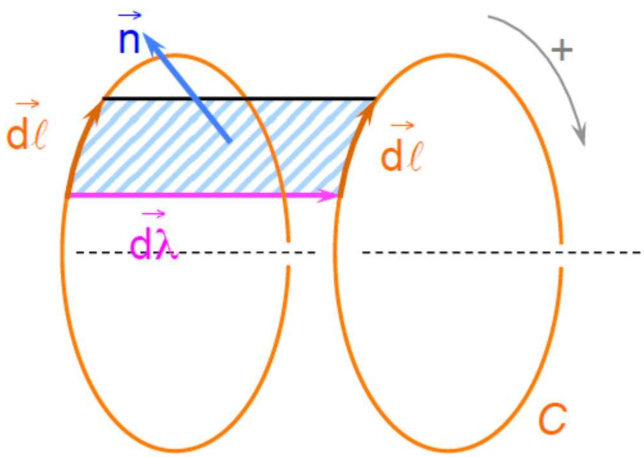
The work of the magnetic force is equal to the energy acquired by the charges:

$$q.e = \oint_c \overrightarrow{F_M}.\overrightarrow{dl} = \oint_c q(\vec{V} \wedge \vec{B}).\overrightarrow{dl}$$

therefore, the e.m.f.e in the circuit is equal to:

$e = \oint_c \left(\frac{\overrightarrow{d\lambda}}{dt} \wedge \vec{B}\right).\overrightarrow{dl} = -\oint_c \left(\frac{\overrightarrow{d\lambda}}{dt} \wedge \overrightarrow{dl}\right).\vec{B} = -\oint_c \frac{\left(\overrightarrow{d\lambda} \wedge \overrightarrow{dl}\right)}{dt}.\vec{B}$ (3.8)

With: $\left(\overrightarrow{d\lambda} \wedge \overrightarrow{dl}\right) = d^2 S\vec{n}$



$e = -\oint_c \frac{\vec{B}.\vec{n}}{dt} d^2 S = -\oint_c \frac{d^2\emptyset}{dt}$ (3.9)

where d2φ is the flow cut by $\overrightarrow{d\ell}$ during time dt.

now we know that the flow cut by a circuit during an elementary displacement dλ is equal to the variation of flux through this circuit during this displacement:

$$d\emptyset_c = d\emptyset$$

So finally:

$$e = -\frac{d\emptyset}{dt}$$

2.2.3. Experience 3

The previous experiments are interpreted coherently: there is a link between the magnetic force exerted on the mobile charges and the appearance of an induced electromotive force.

This explanation does not allow us to explain the third experiment since no circuit element is in motion, only the field $\vec{B}$ varies.

To explain the result of experiment 3, we will establish the local expression of Faraday’s law:

$$e = -\frac{d\emptyset}{dt} = \oint_c \overrightarrow{E_m} . \overrightarrow{dl}$$

By applying Stokes' theorem:

$$-\frac{d\emptyset}{dt} = \oint_c \overrightarrow{E_m} . \overrightarrow{dl} = \iint_s \left(\overrightarrow{rot}\,\overrightarrow{E_m}\right)\overrightarrow{dS}$$

On the other hand, we know that:

$$\emptyset = \iint_s \vec{B} . \overrightarrow{dS} \qquad (3.10)$$

In experiment 3, only $\vec{B}$ depends on time (the circuits are non-deformable so S = Cte)

$$\frac{d\emptyset}{dt} = \iint_s \frac{d\vec{B}}{dt} . \overrightarrow{dS}$$

Hence, finally:

$$\iint_s \left(\overrightarrow{rot}\,\overrightarrow{E_m}\right)\overrightarrow{dS} = -\iint_s \frac{d\vec{B}}{dt} . \overrightarrow{dS} \qquad (3.11)$$

Or in local form:

$$\overrightarrow{rot}\overrightarrow{E_m} = -\frac{\partial \vec{B}}{\partial t} \qquad (3.12)$$ (Maxwell-Faraday’s Equation)

Any temporal variation of the magnetic induction field results in the appearance of an induced electric field of magnetic origin.

The explanation of the phenomenon observed in experiment 3 is therefore found: it is the variation in flow which leads to the appearance of an induced electromotive force. The electromotive field $\overrightarrow{E_m}$ or even induction electric field is the same as the electric field perceived in the frame of reference of the moving conductors described in the previous experiments:

$$\overrightarrow{E_m} = \vec{V} \wedge \vec{B} \qquad (3.13)$$

Noticed :

If a "classical" electrostatic field $\overrightarrow{E_s}$ is superimposed on this induced field, Maxwell's equation remains valid:

$$\overrightarrow{rot}\left(\overrightarrow{E_s} + \overrightarrow{E_m}\right) = \overrightarrow{rot}\overrightarrow{E_s} + \overrightarrow{rot}\overrightarrow{E_m} = \vec{0} - \frac{\partial \vec{B}}{\partial t} \qquad (3.14)$$

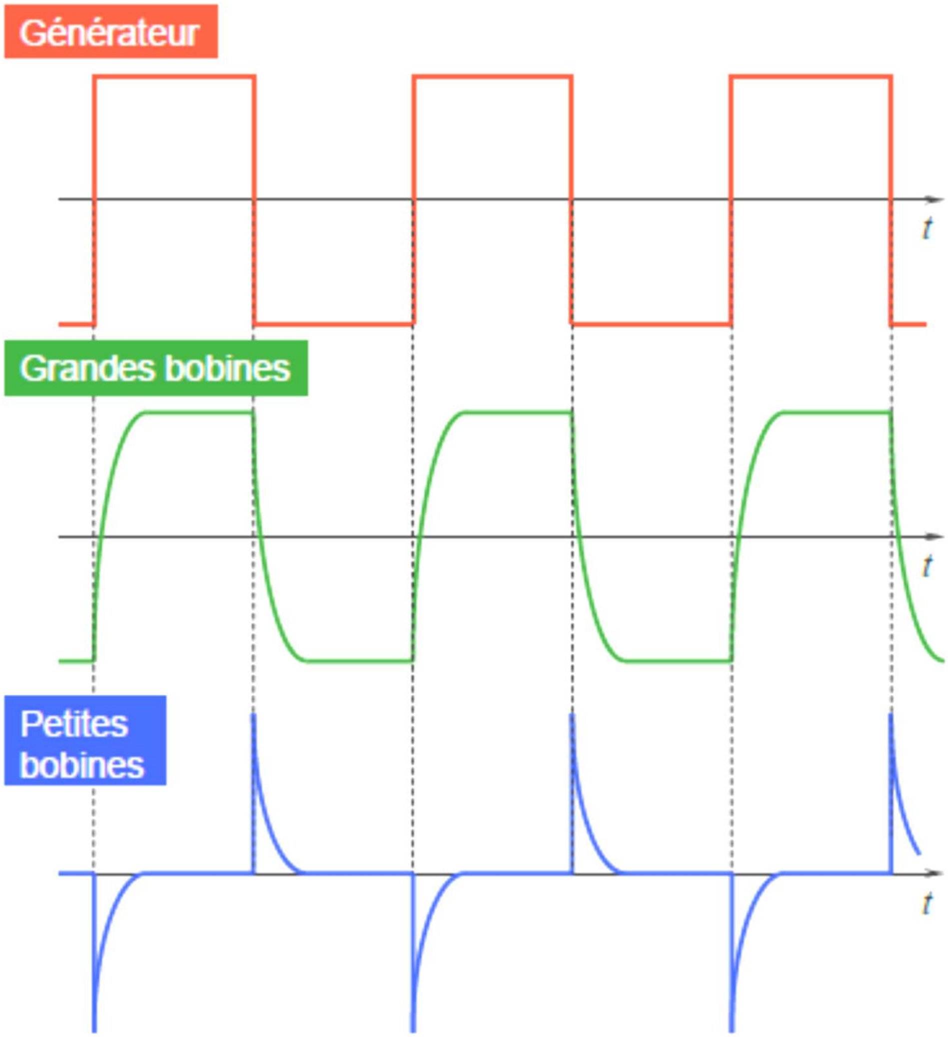
It is also appropriate to introduce the notion of vector potential $\vec{A}$ from which magnetic induction is derived $\vec{B} = \overrightarrow{rot}\,\vec{A}$.

In this case:

$$\vec{E} = \overrightarrow{E_s} + \overrightarrow{E_m} = -\overrightarrow{grad}V - \frac{\partial \vec{A}}{\partial t} \qquad (3.15)$$

2.3.Lenz's law

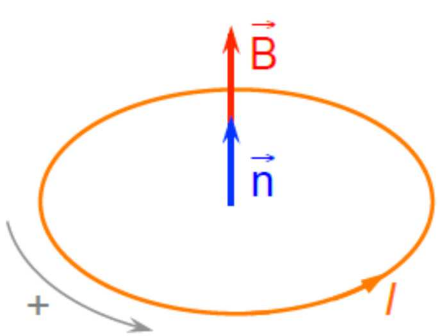
Induction produces effects that oppose the causes that gave rise to it. In this case the signals delivered by this oscilloscope are as follows:



3. Application of induction phenomena

3.1. Auto-induction phenomena

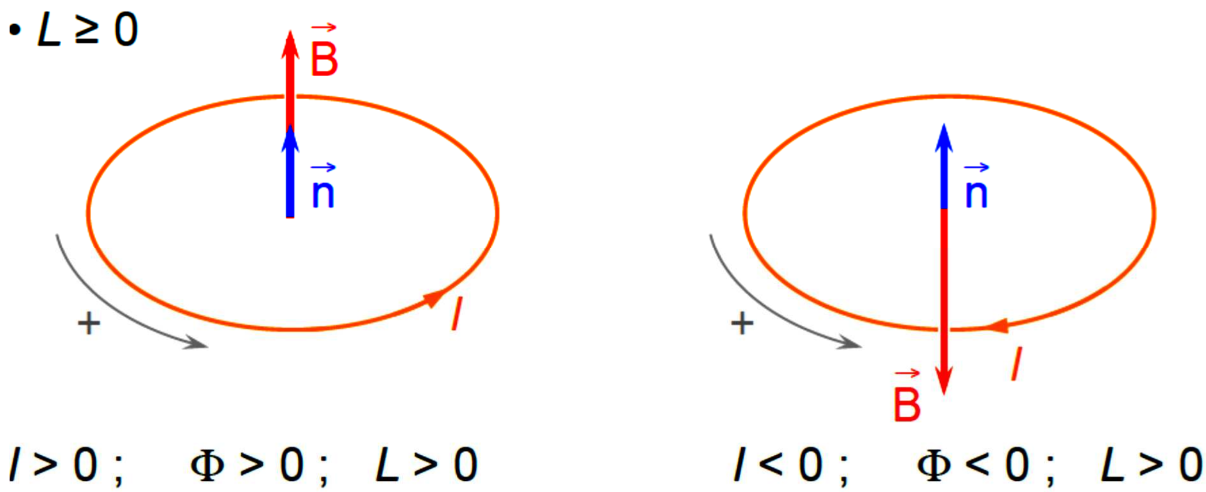
We consider an oriented turn traversed by a current of intensity I.



The proper flux of the field created by the turn through itself is proportional to I:

$$\emptyset = L.I \qquad (3.16)$$

L: self-induction coefficient.



Self-induction coefficient of a solenoid (N, ℓ, R, I)

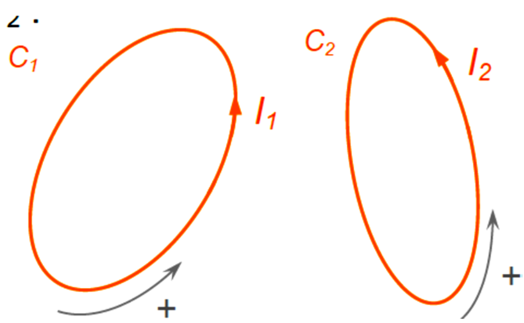
$$B = \mu_0 \frac{N}{l} I \qquad (3.17)$$

$$\emptyset = N.B.S = \mu_0 \frac{N^2}{l} I \pi R^2 \qquad (3.18)$$

$$\Rightarrow L = \frac{\mu_0 N^2 \pi R^2}{l} \qquad (3.19)$$

3.2.Mutual induction coefficient

We consider two circuits, C1 and C2, oriented through which currents I1 and I2 pass:



Circuit C1 creates, at C2, a magnetic induction field B1 proportional to I1.

The flow of $\overrightarrow{B_1}$ (created by C1) through circuit C2 is therefore proportional to I1:

$$\emptyset_{2,1} = kI_1 = M_{2,1}.I_1 \qquad (3.20)$$

Likewise, for the flow of the field $\overrightarrow{B_2}$ (created by C2) through the circuit C1:

$\emptyset_{1,2} = kI_2 = M_{1,2}.I_2$ (3.21)

M1.2 and M2.1 are algebraic quantities which depend only on the shape, relative positions and respective orientations of circuits C1 and C2.

We can show that:

$M_{2,1} = M_{1,2} = M$ (3.22)

M1.2 and M2.1 are the mutual induction coefficients.

Unit: the henry (H)

3.3.Mutual induction coupling

If I_1 varies overtime, then $\Phi_{2.1}$ varies.

An e.m.f. then appears. induced e_2 in circuit C_2:

$e_2 = -M\frac{dI_1}{dt}$ (3.23)

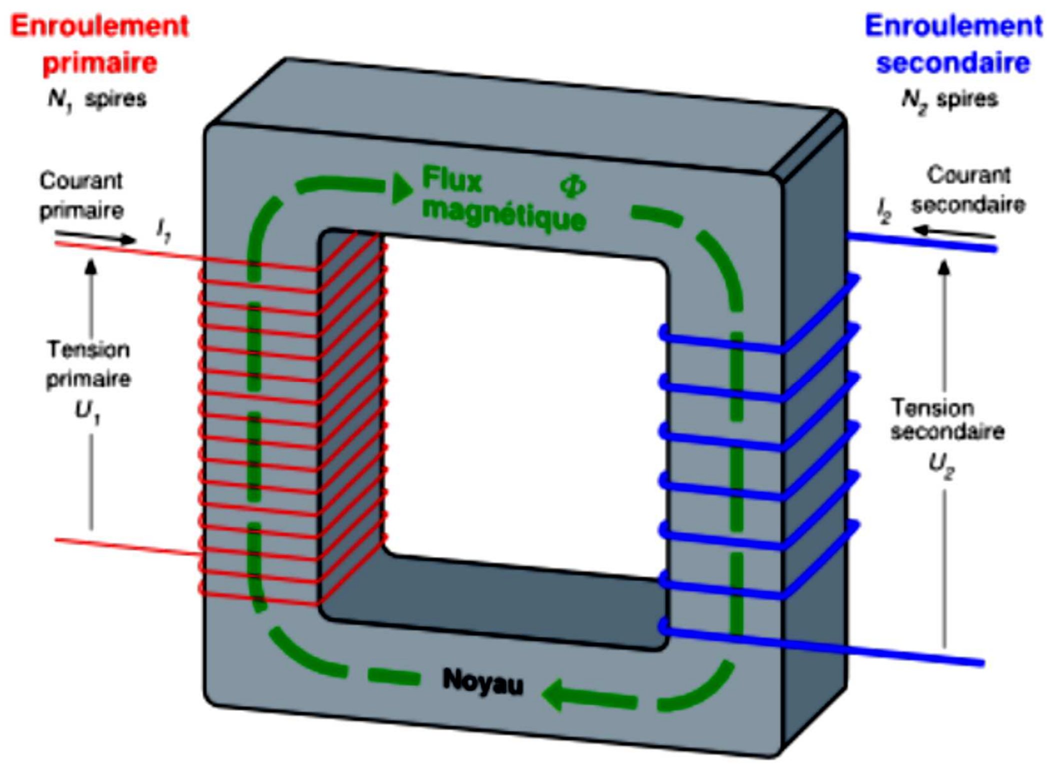
The current I_2 circulating in C_2 is therefore variable as well as the field $\overrightarrow{B_2}$ (created by C1) in C_1.

An e.m.f. then appears. induction in C1:

$e_1 = -M\frac{dI_2}{dt}$ (3.24)

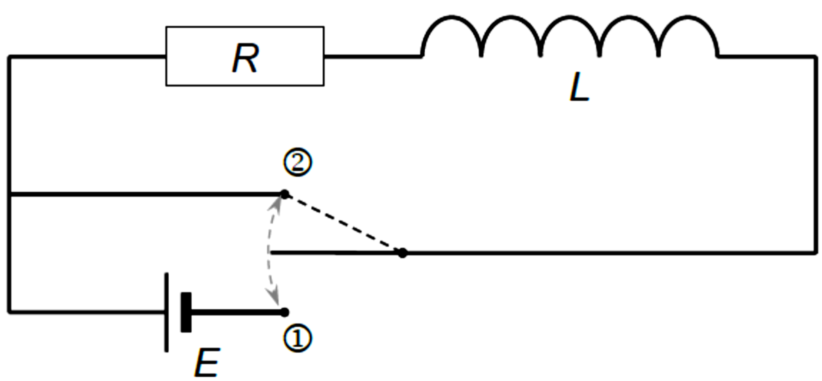
The circuits are said to be coupled by mutual inductance.

Application: transformers



3.4.Establishment of a current in a circuit R, L.

We consider the following circuit:



At time t = 0, we close the circuit in position 1.

- Current begins to flow through the coil.
- The flux of the $\vec{B}$ field created by the coil through itself begins to increase.
- Appearance of an e.m.f. induced whose effect is to oppose the causes which gave rise to it (Lenz's law):

$$e = -\frac{d\phi}{dt} = -L\frac{dI}{dt} \qquad (3.25)$$

- The circuit is conductive ⇒ appearance of a current I_{ind} whose direction is such that the magnetic induction field it creates opposes the increase in flux (Lenz's law).

This is a transient phenomenon which only disrupts the establishment of the permanent regime for short periods of time. We will try to determine the evolution of the current in the circuit. We express Ohm's law:

$E + e = R.I(t)$

$E - L\frac{dI}{dt} = R.I(t)$ e opposes E

$\frac{dI(t)}{dt} + \frac{R}{L}.I(t) = \frac{E}{L}$ 1st order differential equation with 2nd member

The general solution of this equation is the sum of a particular solution of the equation with 2nd member and a solution of the equation without 2nd member:

Special solution:

$$I(t) = \frac{E}{R} = I_0 \qquad (2.26)$$

Solution of the equation without 2nd member:

$$\frac{dI(t)}{dt} = -\frac{R}{L}.I(t) \Leftrightarrow \frac{\left(\frac{dI(t)}{dt}\right)}{I(t)} = -\frac{R}{L} \Leftrightarrow I(t) = k.e^{-\left(R/_L\right)t} \qquad (2.27)$$

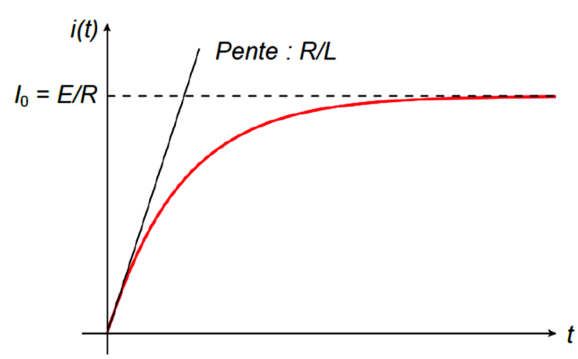
Hence the general solution of the equation:

$$I(t) = I_0 + ke^{-\left(R/_L\right)t} \qquad (2.28)$$

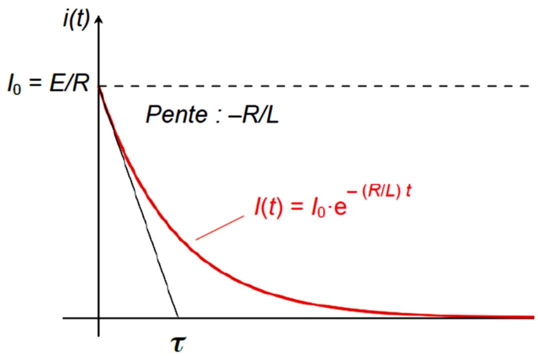
The constant k is determined by the initial conditions:

at t = 0, I(0) = 0. In this case, it comes : k = – I_0

$$I(t) = I_0(1 - e^{-(R/L)t}) \qquad (2.29)$$

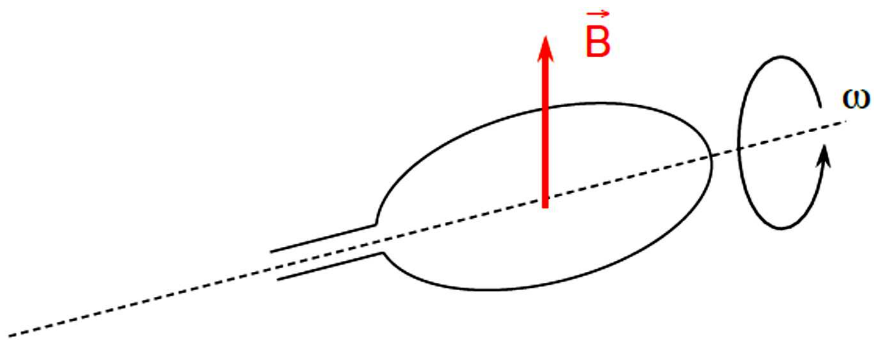


Likewise, if we flip the switch to position 2, the current will not drop instantly because the turn will react by establishing a current which will oppose the reduction in the flow of $\vec{B}$ through itself.



$\tau = L/R$ Circuit time constant

AC generator: We consider a turn which rotates around an axis Δ at the angular speed ω. The whole is immersed in a homogeneous and constant magnetic induction field $\vec{B}$:



As the orientation of the turn varies, the flow of $\vec{B}$ through it varies, hence the appearance of an e.m.f. e who opposes the causes which gave birth to it:

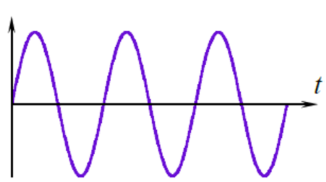
$$\emptyset = \iint_S \vec{B}.\overrightarrow{dS} = \vec{B}.\vec{S} = B.S\cos\omega t \qquad (3.30)$$

$$e = -\frac{d\emptyset}{dt} = +\omega.B.S.sin\omega t \qquad (3.31)$$

Assuming that the turn has a resistance R, we deduce the current:

$$i = {}^{e}/_{R} = \frac{\omega.B.S}{R} sin\omega t \qquad (3.32)$$

→ electric motors



3.5.Electro (magnetic) energy

We consider the circuit from the previous paragraph and we are interested in the energy involved in this circuit to go from the initial state ($I = 0, \Phi_{coil} = 0$) to the final state ($I = I_0, \Phi_{coil} \neq 0$): From Ohm's law:

$$E = R.I(t) + L.\frac{dI}{dt} \qquad (3.33)$$

We can determine the work provided by the generator between times t and t + dt:

$$E.I.dt = R.I^2(t)dt + L.I.\frac{dI}{dt}dt \qquad (3.34)$$

Between t = 0 and the instant t for which we have the current I0, the generator has provided the work:

$$W = \int_0^t E.I.dt = \int_0^t R.I^2\,(t)dt + \int_0^t L.I.\frac{dt}{dt}dt \qquad (3.35)$$

Where:

$$W = \int_0^t R.I^2(t)dt + \frac{1}{2}L.{I_0}^2 \qquad (3.36)$$

joule effect with :$\int_0^t R.I^2(t)dt$

electromagnetic Energy: $\frac{1}{2}L.{I_0}^2$

What happens when we close the circuit on itself?

i.e. when the current goes from I_0 to 0: It was shown that the current decreased exponentially:

$$I(t) = I_0.e^{-(R/L)t} \qquad (3.37)$$

The energy dissipated by the Joule effect by the circulation of this current of t = 0 (we close the circuit on itself) and the moment for which the current is zero (t = ∞) is:

$$W = \int_0^{+\infty} R.I^2(t)dt = \int_0^{+\infty} R{I_0}^2 e^{-(2R/L)t}dt = \frac{1}{2}L.{I_0}^2 \qquad (3.38)$$

The energy stored by the circuit when the permanent current is established is released when the circuit is closed on itself and dissipated by the Joule effect.

4. Comparison between the stationary regime and the quasi-stationary regime

SR	QSR
$\vec{E} = -\overrightarrow{grad}\, V$	$\vec{E} = -\overrightarrow{grad} V - \frac{\partial \vec{A}}{\partial t}$
$div\vec{B} = 0$	$div\vec{B} = 0$
$\overrightarrow{rot}\vec{E}=0$	$\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\overrightarrow{rot}\vec{B} = \mu\vec{j}$	$\overrightarrow{rot}\vec{B} = \mu\vec{j}$