Chapter 1
Electrostatic field

1- Introduction

Electrostatics is the branch of physics which studies the phenomena (electrostatic field and potential) created by static electric charges for the observer. Electrostatic forces are described by Coulomb's law which has a certain analogy with gravitational interaction.

2- Structure of matter

The electric charge of a particle is a scalar (algebraic) quantity which characterizes the electromagnetic actions undergone or exerted by the particle. Despite the fact that matter is macroscopically neutral, it is composed of charged and neutral particles with discrete values (0, ±e, ±2e, ±3e, etc.) which are integer multiples of the elementary charge. This is the absolute value of the electron charge e = 1.60219x10-19 C.

The elementary particles, constituents of matter, have the following charges:

- electron (e-):

Charge: qe = -e = - l.60 x 10-19 C , Mass: me=9.1 x 10-31 kg

- Proton (H+): Charge: qp = + e = l.60 x 10-19 C, Mass: mp=1.67x10-24kg

- neutron: the charge is zero.

- The unit of charge is the coulomb C in the IS.

This quantification was first established in 1913 by Millikan's oil drop experiment. Electrification by friction is simply carried out by a transfer of electrons from a body of low electron affinity to another of higher affinity. On the other hand, the electric charge of particles does not depend on their speed or on physical conditions, such as temperature, pressure, etc., even in extreme conditions, such as in the hearts of stars or at the beginning of the formation of the Universe. The electron and the proton are absolutely stable. It is not possible to eliminate any of them individually but an electron and a proton can interact and produce a neutron and a neutrino.

3- Coulomb law

Let in a vacuum, two point charges q1 and q2, fixed at M1 and M2 over a distance of r. The two charges are stationary q1 and q2 exert on each other a force proportional to each of the charges and inversely proportional to the square of the distance which separates them. The electrostatic force is directed along the line which joins the charges (figure 1). It is attractive if the charges are of opposite signs (figure 1-a), repulsive when the charges are of the same sign (figure 1-b).

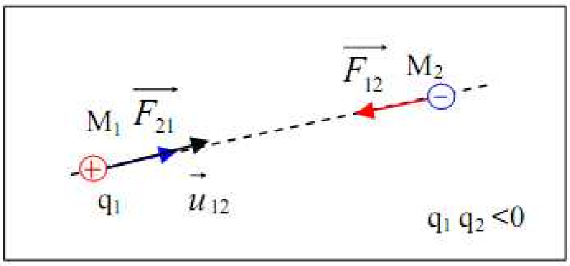

Figure 1-a

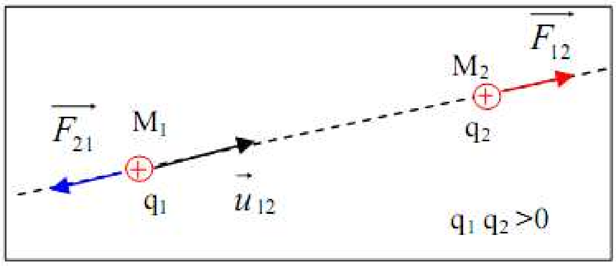

Figure 1-b

The force $\overrightarrow{F_{12}}$ exerted by q1 on the charge q2 is written:

$$\overrightarrow{F_{12}} = K\frac{q_1 q_2}{r^2}\overrightarrow{u_{12}} \qquad (1.1)$$

Where the distance between q1, q2 and $\overrightarrow{u_{12}}$ the unit vector carried by the support of M1M2 and which is oriented from M1 towards M2 is defined by:

$$\overrightarrow{u_{12}} = \frac{\overrightarrow{M_1M_2}}{\|\overrightarrow{M_1M_2}\|} = \frac{\overrightarrow{M_1M_2}}{r} \qquad (1.2)$$

The constant of proportionality is linked to the units chosen to express force, length and charge. In the international system of units (I.S), in its rationalized form, K is written:

$$K = \frac{1}{4\pi\varepsilon_0} \cong 9\ 10^9 (Vm/C)(IS) \qquad (1.3)$$

Where ε0 is the vacuum permittivity and has the value: $\varepsilon_0 = 8{,}85410^{-12}\ Fm^{-1}$

In accordance with the principle of action and reaction, the force $\overrightarrow{F_{21}}$ exerted by q2 on the charge q1 is equal and opposite to $\overrightarrow{F_{12}}$:

$$\overrightarrow{F_{21}} = -\overrightarrow{F_{12}} \qquad (1.4)$$

The forces $\overrightarrow{F_{12}}$ and $\overrightarrow{F_{21}}$ are carried by the line which joins the charges q1 and q2. This is a characteristic that can be explained by evoking the principle of isotropy: in an empty universe, no direction can be favored over another, all directions are equivalent. The presence of two point charges destroys this isotropy by introducing a single preferred direction, the line joining the charges.

4- Electric field

According to the principle of superposition, the total force that several charges qi located at points ri exert on a charge q placed at r is the vector sum of these forces exerted by each charge qi.

$F = \sum_i K_0\, q q_i / {R_i}^2$ (1.5)

Where**:** $R_i = r - r_i$ (1.6)

In the following, the charge q on which the force acts is considered as a test charge, while the charges qi which produce the force are considered as the source charges. If the source charges are distributed continuously in a volume V, on a surface S or a curve L, the source charge qi must be replaced by qv(r′) dV, qs(r′) dS or qL(r′) dL, where qv, qs and qL are the charge densities, respectively, per unit volume, per unit area, and per unit length, then integrate over the charge distribution of the source. According to the expression (1.1) of the force exerted by the point charges qi in ri on q in r, we deduce the electric field produced by these charges E = F/q and we can generalize it to distributions of continuous charges, and we will have:

$E(r) = \frac{\sum_i K_0 q_i}{{\mathrm{R_i}}^2}$ (1.7)

Where: $R_i = r - r_i$ or $R = r - r'$

Electric field in a uniformly charged volume:

$E(r) = K_0 \iiint_v \; dV'\, q_v(r')/R^2$ (1.8)

Electric field in a uniformly charged surface:

$E(r) = K_0 \iint_s \; dS'\, q_s(r')/R^2$ (1.9)

Electric field in a uniformly charged curve:

$$E(r) = K_0 \int_l dl' \, q_l(r')/R^2 \qquad (1.10)$$

5- Distribution of electrical charges

The distribution of loads is an assumption of a macroscopic load making it possible to define an infinitesimal load dq, to which we can apply the formulas established in the case of a point load defined as follows:

5-1- Linear distribution

Relating to the length of a wire, over which an electrical charge is distributed, is very large compared to the other dimensions, we can choose an extremely small length Δl, around the point of vector position $\vec{r}$, which contains a relatively small number of electrons of charge quantity ΔQ, and we define the distribution linearity of the electric charge as follows:

$$\lambda(\vec{r}) = \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l} = \frac{dQ}{dl} \quad (C/m) \qquad (1.11)$$

5-2- Surface distribution

It is sometimes found that the thickness of a volume containing electric charges is very small compared to the square root of its surface (d << $\sqrt{S}$). In this case, it is possible to consider with a good approximation that the charges are distributed on the surface S of the volume. We therefore delimit an infinitesimal surface ΔS, around the point of vector position $\vec{r}$, which contains a relatively small number of electrons with charge quantity ΔQ, and we define the surface distribution of the electric charge as:

$$\sigma(\vec{r}) = \lim_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS} \quad (C/m^2) \qquad (1.12)$$

5-3- Volume distribution

To do this, we delimit an infinitesimal volume ΔV, around the point of position vector $\vec{r}$, which contains a relatively small number of electrons with charge quantity ΔQ, and we define the distribution volume of the electric charge as:

$$\rho(\vec{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV} \quad (C/m^3) \qquad (1.13)$$

6- Electric dipole

An electric dipole is a distribution of charges that can be modeled by two charges −q and +q which we take at A− and A+ with coordinates A and B distant from A.

6-1- Calculation of the potential at long distance

$$V_M = Kq\left(\frac{1}{MB} - \frac{1}{MA}\right)$$

$$= Kq\frac{MA - MB}{MB \times MA}$$

As $OM = r \gg a$ we have

$MA \cong r + \frac{a}{2}cos\theta,$

$MB \cong r - \frac{a}{2}cos\theta$

$MA \times MB \cong r^2$

$$V_M = K\frac{\vec{p}\,\overrightarrow{u_r}}{r^2} = \frac{Kqacos\theta}{r^2} \qquad (1.14)$$

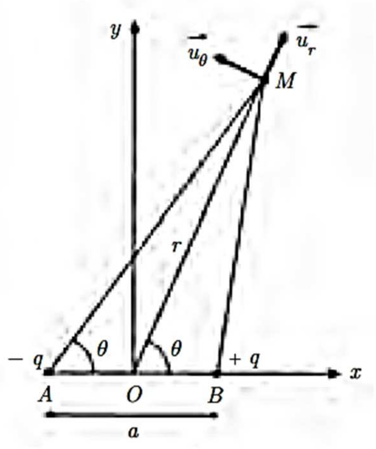

Figure 1-2

6-2- Calculation of the electric field at long distances

$\vec{E} = -\overrightarrow{grad}V$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2Kpcos\theta}{r^3} \qquad (1.15)$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{Kpsin\theta}{r^3} \qquad (1.16)$$

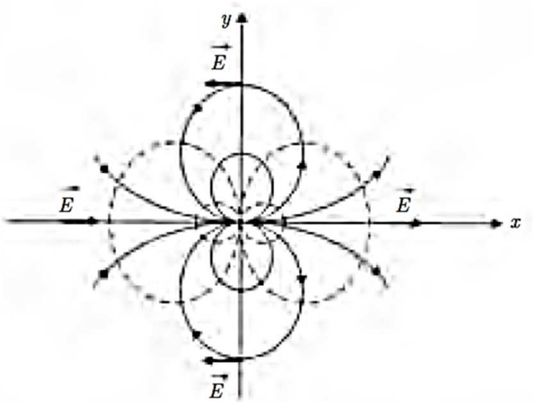

Figure 1-4

6-3- Force and couple exerted by an electric field on a dipole

6-3-1- Case of a uniform field

Let θ be the angle of AB with the axis $\overrightarrow{Ox}$taken in the direction of the applied field $\vec{E}$.

$\vec{F} = \overrightarrow{F_B} + \overrightarrow{F_A} = qE\overrightarrow{e_x} - qE\overrightarrow{e_x} = \vec{0}$ (1.17)

The resulting force is zero, but the resulting moment is not, $\overrightarrow{F_A}$ and $\overrightarrow{F_B}$ constitute a couple.

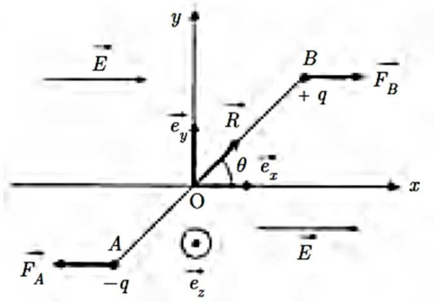

Figure 1-4

6-3-2- Case of a non-uniform field

In this case, the forces $\overrightarrow{F_B}$ and $\overrightarrow{F_A}$ are no longer equal and opposite. This results in a force that will move the dipole as a whole. We will therefore have a translational movement with center of mass O of the dipole, in addition to the rotational movement around O.

The resulting force is related to the potential energy by:

$$\vec{F} = -\overrightarrow{grad}\overrightarrow{E_p}$$

We will therefore have:

$\vec{F} = \overrightarrow{grad}(\vec{p}.\vec{E})$ (1.18)

7- Electric potential

When n point charges exist simultaneously at points M1, M2,. . . ,Mn, the principle of superposition allows us to write:

- For the resulting field at a point M (with ri = MiM ≠ 0):

$\overrightarrow{E_M} = K\sum_i \frac{q_i}{r_i^2}\vec{u}M_iM$ (1.19)

- And for the resulting potential:

$V_M = K\sum_i \frac{q_i}{r_i}$ (1.20)

In the case of continuous load distributions, we will have the same:

- For a uniformly loaded wire:

$$\vec{E} = K_0 \int_l \frac{\lambda dl}{r^2} \vec{u}\, PM \qquad (1.20)'$$

$$V_l = K_0 \int_l \frac{\lambda dl}{r} \qquad (1.20)''$$

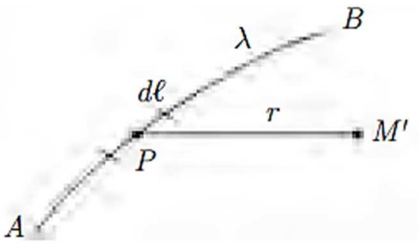

Figure 1-6

- For a uniformly loaded surface:

$$\vec{E} = K_0 \iint_s \frac{\sigma ds}{r^2} \vec{u}\, MM' \qquad (1.21)$$

$$V_s = K_0 \iint_s \frac{\sigma ds}{r} \qquad (1.21)'$$

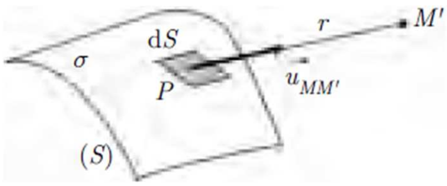

Figure 1-7

- And for a uniformly loaded volume:

$$\vec{E} = K_0 \iiint_i \frac{\rho dv}{r^2} \vec{u}\, MM' \qquad (1.22)$$

$$V_v = K_0 \iiint_v \frac{\rho dv}{r} \qquad (1.22)'$$

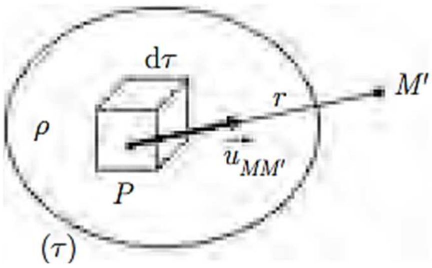

Figure 1-8

8- Relationship between the field E and the potential V

Here we are interested in the electric potential relative to the electric force such that $F = -\nabla U_E$. It can be shown that the electrostatic interaction of two charges q and qi corresponds to a potential energy $U_E = K_0 q q_i / R_i$, where $R_i = |r - r_i|$. In the case of a test charge q=1C, the force F becomes the electric field and the potential energy of the unit charge is the electrostatic potential V such that:

$$E_x = -\partial_x V \quad , \quad E_y = -\partial_y V, \quad E_z = -\partial_z V \qquad (1.23)$$

The electrostatic field $\vec{E}$derives from the scalar potential V. Through this local relation, which links the electrostatic field $\vec{E}$and the electrostatic potential V, the knowledge of V at a point in space is sufficient for the determination of $\overrightarrow{E(M)}$. This relation implies conditions of continuity and differentiability on the function V(M). According to the international system of units (IS) the unit of potential is the Joule per Coulomb (J/C) called volt (V), and the unit of the electric field is the Newton per Coulomb (N/C), which can also be called volt per meter (V/m).

9- Equipotential surface

The fundamental principle of an electric equipotential surface is the region where the value of the electric potential is the same at every point. Electrical equipotential have the following characteristics:

- The electric potential is in counterbalance at every point on the surface V=cte.
- The electric field is perpendicular to the equipotential surface$\vec{E} \perp \overrightarrow{dl}$.
- The direction of the electric field defines the direction in which there is a drop in potential.
- More the equipotential are closer, more the electric field $\vec{E}$ is higher.

Equipotential surfaces are spheres centered at a point where the charge is located. The direction of $\overrightarrow{E}$, that is to say of the gradient of V, is the direction of the normal to the equipotential surfaces, the one where V varies the most rapidly. The direction of the field lines depends on the polarity of the charge. If the source charge is positive, its field lines are directed radially outwards and vice versa with negative charges:

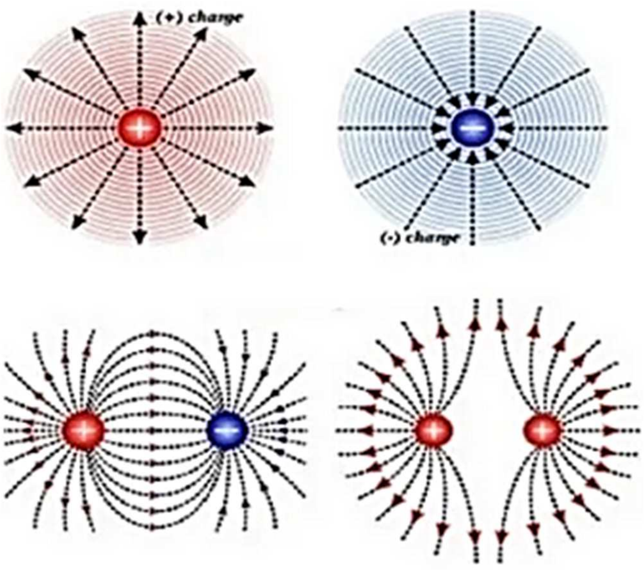

Figure 1-9

10- Gauss's theorem

Consider a charge q placed at point O. The field created by this charge at a point M, at a distance OM = r is given by:

$$\vec{E} = K\frac{q}{r^2}\overrightarrow{e_r} \qquad (1.24)$$

Considering a certain number of charges qi, some inside the volume τ, others outside.

If qi is inside:

We have: $$d\emptyset = \vec{E}\overrightarrow{ds} = \frac{Kq}{r^2}\overrightarrow{e_r}\vec{N}ds = KqdΩ \qquad (1.25)$$

$$d\emptyset' = \overrightarrow{E'}\overrightarrow{ds'} = \frac{Kq}{r'^2}\overrightarrow{e_r}\overrightarrow{N'}ds' = KqdΩ \qquad (1.26)$$

$$d\emptyset_i = d\emptyset + d\emptyset' \qquad (1.27)$$

$$\emptyset_i = \iint_s KqdΩ = 4\pi Kq \qquad (1.28)$$

So: $\emptyset_i = \frac{q_i}{\varepsilon_0}$ since: $K = \frac{1}{4\pi\varepsilon_0}$

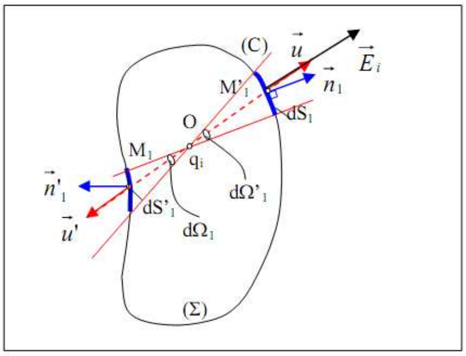

Figure 1-10

If qi is outside:

We have: $d\emptyset = \vec{E}\,\overrightarrow{ds} = K\frac{q}{r^2}\overrightarrow{e_r}\,\vec{N}ds = -Kqd\Omega$ (1.29)

$d\emptyset' = \overrightarrow{E'}\,\overrightarrow{ds'} = K\frac{q}{r^2}\overrightarrow{e_r}\,\overrightarrow{N'}ds' = -Kqd\Omega$ (1.30)

In total:

$d\emptyset_i = d\emptyset + d\emptyset' = 0 \Rightarrow \emptyset_i = 0$ (1.31)

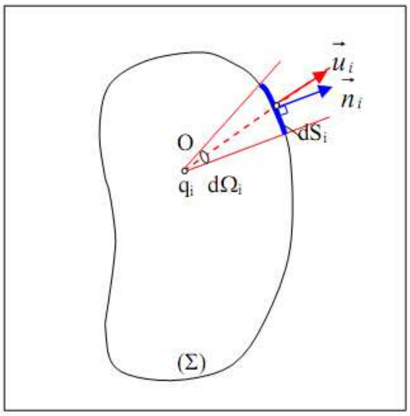

Figure 1-11

As a result, the flow of the resulting field through (S) is due only to charges internal to S:

$\emptyset = \iint_s \vec{E}\,\overrightarrow{ds} = \sum_i \frac{q_i}{\varepsilon_0}$ this is only for inside charge (1.32)

The case of Gauss' theorem where v is the volume limited by the closed surface Σ is given by the following equation:

$$\emptyset = \oiint_{\Sigma} \vec{E}\,\overrightarrow{ds} = \frac{1}{\varepsilon_0}\iiint_v \rho d\tau \qquad (1.33)$$

11- Capacitance and capacitor

A conductor carried at potential V appears on its surface, a charge q defined by:

$$q = \oiint_S \sigma ds \qquad (1.34)$$

So if the potential transforms into V1, then V2, then V3, the charge becomes q1, q2, q3. The potential charge relationships are linear and can be written as follows:

$$\frac{q}{V} = \frac{q_1}{V_1} = \frac{q_2}{V_2} = \frac{q_3}{V_3} = C \qquad (1.35)$$

C represents the coefficient of proportionality also called the capacitance of the conducting body. The latter is independent of q and V. If q is in coulomb (C) and V in volt (V), C will be measured in Farad (F).

Generally speaking, the relationship between charges and potentials in a system of n Conductors in equilibrium. In matrix form, it is written as follows:

$$[Q_i] = [C_{ij}] \times [V_j] \qquad (1.36)$$

Where the indices i and j vary between 1 and n. This writing means that, for each value of i, this expression must be summed over j. Properties of matrix C:

- It is symmetrical: Cij = Cji (Gaussian identity),

- The diagonal terms are positive: Cii > 0, they constitute the capacity coefficients,

- The non-diagonal terms are negative: Cij < 0, these are the influence coefficients.

Consider the following relation originating from (1.24):

$$C = \frac{Q}{V_1 - V_2} = \frac{-Q}{V_2 - V_1} \qquad (1.37)$$

From this equation, we see that the charge Q and the potential difference between (V1 and V2) determine the capacitance C of the capacitor.

There are three types of capacitors:

11-1- The Spherical capacitor

This type of capacitor contains two armatures in the shape of a concentric sphere of radius R1 and R2. For a point M, located between the two reinforcements and such that OM=r, we can write:

$$\vec{E} = E(r)\overrightarrow{e_r} = K\frac{Q_1}{r^2}\overrightarrow{e_r}$$

We have: $\int_1^2 \vec{E}\,\overrightarrow{dl} = V_1 - V_2$

So: $KQ_1\int_1^2 \frac{dr}{r^2} = V_1 - V_2$

Either: $KQ_1\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = V_1 - V_2$ So: $C = K\frac{R_1R_2}{R_2 - R_1}$ (1.38)

Note: if $e << R_2 - R_1$ we have $C = \varepsilon_0\frac{4\pi R^2}{e} = \varepsilon_0\frac{S}{e}$ which is a plane capacitor.

11-2- The cylindrical capacitor

In this case, the armatures are made up of two coaxial cylinders. According to Gauss' theorem:

$$\overrightarrow{E} = E(r)\overrightarrow{e_r} = 2K\frac{Q_1}{rh}\overrightarrow{e_r}$$

We have : $\int_1^2 \vec{E}\,\overrightarrow{dl} = V_1 - V_2$

So : $2K\frac{Q_1}{h}\int_1^2 \frac{dr}{r^2} = V_1 - V_2$ Either : $Q_1 = \frac{2\pi\varepsilon_0 h}{ln\frac{R_2}{R_1}}(V_1 - V_2)$

Where the capacitor :

$$C = \frac{Q_1}{V_1 - V_2} = \frac{2\pi\varepsilon_{0h}}{ln\frac{R_2}{R_1}} \qquad (1.39)$$

11-3- the plane capacitor

The armatures are made up of two parallel planes of surface S, spaced e apart. If the first is positively charged with a density +σ and the second negatively with a density −σ. between the two frames, we have:

For the first armature: $\overrightarrow{E_1} = \frac{\sigma}{2\varepsilon_0}\overrightarrow{e_r}$

For the second armature : $\overrightarrow{E_2} = \frac{-\sigma}{2\varepsilon_0}\overrightarrow{e'_r}$

The total field is equal to :

$$\vec{E} = \overrightarrow{E_1} + \overrightarrow{E_2} = \frac{\sigma}{\varepsilon_0}\overrightarrow{e_r}$$

We deduce: $V_1 - V_2 = Ee = \frac{\sigma e}{\varepsilon_0} = \frac{Qe}{S\varepsilon_0} = \frac{Qe}{S\varepsilon_0}$

where:

$$C = \frac{Q_1}{V_1 - V_2} = \frac{S\varepsilon_0}{e} \qquad (1.40)$$

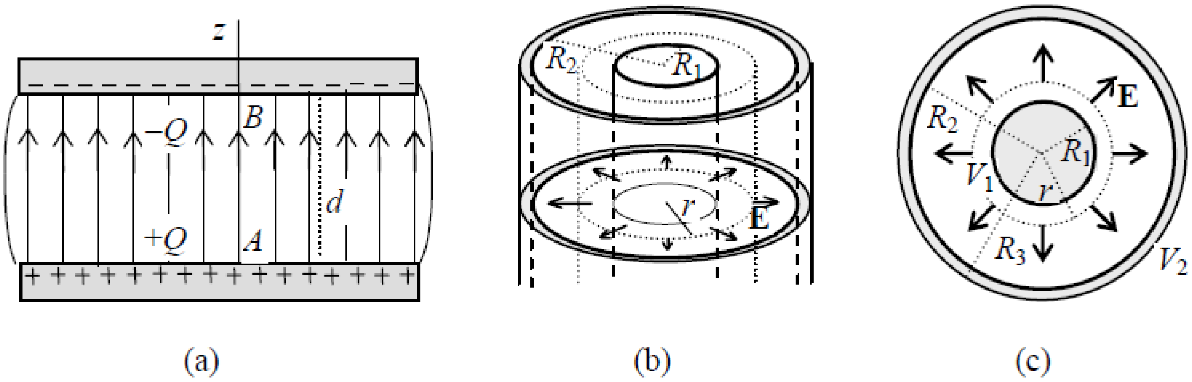

Figure 1-12

12- Electrostatic energy

The electrostatic energy U of a system of charges, initially assumed far from each other corresponds to the work that must be done to bring these charges to their final positions.

12-1- Energy of a point charge placed in a field E

The work of the electrostatic force for a charge q moving from A to B in the field is defined by the following equation:

$$U_{AB} = q(V_A - V_B) = qV \qquad (1.41)$$

12-2- Energy of a point charge system

In this case, Each of the charges is subject to the action of the electrostatic field created by the other charges:

To remember $W_{ij} = q_i V_{j \to i}$ which give $\sum_j W_{ij} = q_i \sum_j V_{j \to i} = q_i V_i$

So we obtain $\sum_i \sum_j W_{ij} = \sum_i q_i V_i$

The total energy will be : $W = \frac{1}{2} \sum_i q_i V_i$ (1.42)

The term ½ comes from the fact that in the interaction between qi and qj is counted twice.

12-3- Energy of a continuous distribution of charges

In a set of point charges dividing the total charge into dq the energy equations are given as follows: Volume distribution: $W = \frac{1}{2} \iiint \rho V d\tau$ (1.43)

Surface distribution :$W = \frac{1}{2} \iint \sigma V ds$ (1.44)

Distribution line : $W = \frac{1}{2} \int \lambda V ds$ (1.45)

12-4- Energy of a system of conductor charges in electrostatic equilibrium

In the case of only one conductor, the energy is as follows:

$W = \frac{1}{2} Q_i V_i$ (1.46)

For n conductors the total energy will be:

$W = \frac{1}{2} \sum_i q_i V_i$ (1.47)

12-5- Energy of a charged capacitor

The energy of a capacitor whose armature charges are +Q and –Q respectively and are at potentials V1 and V2, with expressions:

$$W = \frac{1}{2}Q(V_1 - V_2) = \frac{1}{2}C(V_1 - V_2)^2 = \frac{1}{2}\frac{Q^2}{C} \qquad (1.48)$$