

Series 2: Linear maps

Exercise 1. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the map defined for any $u = (x, y, z, t) \in \mathbb{R}^4$ by :

$$f((x, y, z, t)) = (x + y, z + t, x + y + z + t)$$

1. Prove that f is a linear map.
2. Determine a basis of $\ker(f)$.
3. Determine a basis of $\text{Im}(f)$.

Exercise 2. Let $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the map defined by:

$$u((x_1, x_2, x_3)) = (-2x_1 + 4x_2 + 4x_3, -x_1 + x_3, -2x_1 + 4x_2 + 4x_3)$$

1. Prove that u is linear.
2. Determine a basis of $\ker(u)$ and a basis of $\text{Im}(u)$.
3. Do we have $\ker(u) \oplus \text{Im}(u) = \mathbb{R}^3$?

Exercise 3. Let $\beta = \{e_1, e_2, e_3\}$ be the canonical basis of \mathbb{R}^3 . Let u be an endomorphism of \mathbb{R}^3 defined by :

$$u(e_1) = 2e_1 + e_2 + 3e_3; \quad u(e_2) = e_2 - 3e_3; \quad u(e_3) = -2e_2 + 2e_3.$$

1. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a vector. Determine the image by u of the vector x . (Calculate $u(x)$).
2. Let $E = \{x \in \mathbb{R}^3, u(x) = 2x\}$ and $F = \{x \in \mathbb{R}^3, u(x) = -x\}$. Prove that E and F are subspaces of \mathbb{R}^3 .
3. Determine a basis of E and a basis of F .
4. Do we have $E \oplus F = \mathbb{R}^3$?

Exercise 4. Let $\beta = \{e_1, e_2, e_3\}$ be the canonical basis of \mathbb{R}^3 . Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map such that :

$$\begin{aligned} f(e_1) &= -\frac{1}{3}e_1 + \frac{2}{3}e_2 + \frac{2}{3}e_3 = \frac{1}{3}(-e_1 + 2e_2 + 2e_3), \\ f(e_2) &= \frac{2}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3 = \frac{1}{3}(2e_1 - e_2 + 2e_3) \text{ and} \\ f(e_3) &= \frac{2}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3 = \frac{1}{3}(2e_1 + 2e_2 - e_3). \end{aligned}$$

Let $E_{-1} = \{u \in \mathbb{R}^3 \mid f(u) = -u\}$ and $E_1 = \{u \in \mathbb{R}^3 \mid f(u) = u\}$.

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1. Prove that E_{-1} and E_1 are subspaces of \mathbb{R}^3 .
 2. Demonstrate that $e_1 - e_2$ and $e_1 - e_3$ are belong to E_{-1} and that $e_1 + e_2 + e_3$ is belong to E_1 .
 3. What can we deduce about the dimensions of E_{-1} and E_1 ?
 4. Determine $E_{-1} \cap E_1$.
 5. Do we have $E_{-1} \oplus E_1 = \mathbb{R}^3$?
 6. Calculate $f^2 = f \circ f$ and deduce that f is bijective and determine f^{-1} .

Exercise 5. Let $\beta = \{e_1, e_2\}$ be the canonical basis of \mathbb{R}^2 . Let u an endomorphism of \mathbb{R}^2 so that $u(e_1) = e_1 + e_2$ with $\dim(\ker(u)) = 1$

1. Determine $u(e_2)$ depending on a parameter $a \in \mathbb{R}$.
2. Determine the image of a vector $x = (x_1, x_2) \in \mathbb{R}^2$ depending on a .
3. Determine a basis of the kernel $\ker(u)$.

Exercise 6. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ be the map defined for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by:

$$f(x) = x_1 + x_2 + x_3 + x_4$$

Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the canonical basis of \mathbb{R}^4 .

1. Calculate the images of vectors of the canonical basis by f . Deduce the dimension of $\text{Im}(f)$.
2. Determine the dimension of $\ker(f)$ and give a basis for it.

Exercise 7. Let $f : E \rightarrow F$ be a linear map. Prove that:

$$\ker(f) \cap \text{im}(f) = f(\ker(f^2)).$$

Exercise 8. Let u be an endomorphism of a vector space E on a field \mathbb{k} .

1. Prove that $\ker(u) \subset \ker(u^2)$.
2. Prove that $\text{Im}(u^2) \subset \text{Im}(u)$.

Exercise 9. Let u be an endomorphism of a vector space E on a field \mathbb{k} .

Show that the following assertions are equivalent

- (i) $\ker(u) \cap \text{im}(u) = \{0_E\}$.
- (ii) $\ker(u) = \ker(u \circ u)$.