

Series 1: Vector spaces

Exercise 1.

1. Is the subset $E = \{(x, y) \in \mathbb{R}^2, y = 2x\}$ of \mathbb{R}^2 , equipped with the usual laws of the vector space \mathbb{R}^2 , a vector space on \mathbb{R} ?
2. Is the subset $F = \{(x, y, z) \in \mathbb{R}^3, y^2 = 2x, z = 0\}$ of \mathbb{R}^3 , equipped with the usual laws of the vector space \mathbb{R}^3 , a subspace of \mathbb{R}^3 ?

Exercise 2.

Let $u_1 = (1, 2, 3, 4)$ and $u_2 = (1, -2, 3, -4)$ be two vectors of \mathbb{R}^4 .

Can one determine x and y so that $(x, 1, y, 1) \in \text{Span}(u_1, u_2)$? And so that $(x, 1, 1, y) \in \text{Span}(u_1, u_2)$?

Exercise 3.

In \mathbb{R}^4 , let's consider the sub-set E of vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$.

Is the subset E a subspace of \mathbb{R}^4 ? If it is, give a basis.

Exercise 4.

Let $a = (2, 3, -1)$, $b = (1, -1, -2)$, $c = (3, 7, 0)$ and $d = (5, 0, -7)$ be four vectors of \mathbb{R}^3 .

Let $E = \text{Span}(a, b)$ and $F = \text{Span}(c, d)$ be subspaces of \mathbb{R}^3 . Prove that $E = F$.

Exercise 5.

Let $E = \text{Span}(a, b, c, d)$ be a subspace of \mathbb{R}^3 with:

$$a = (2, -1, -1); \quad b = (-1, 2, 3); \quad c = (1, 4, 7) \text{ and } d = (1, 1, 2)$$

1. Is $\{a, b, c, d\}$ a basis of \mathbb{R}^3 ?
2. Prove that $\{a, b\}$ is a base of E .
3. Determine one or more equations characterizing E .
4. Complete a basis of E to a basis of \mathbb{R}^3 .

Exercise 6.

Let E and F be two subsets of \mathbb{R}^3 defined by:

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - 2z = 0 \text{ and } 2x - y - z = 0\},$$

and

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$$

We admit that F is a subspace of \mathbb{R}^3 . Let $a = (1, 1, 1)$, $b = (1, 0, 1)$ and $c = (0, 1, 1)$ be three vectors of \mathbb{R}^3 .

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1. Prove that E is a subspace of \mathbb{R}^3 .
 2. Determine a generating family (spanning set) of E and show that it is a basis.
 3. Prove that $\{b, c\}$ is a basis of F .
 4. Prove that $\{a, b, c\}$ is linearly independent in \mathbb{R}^3 .
 5. Do we have $E \oplus F = \mathbb{R}^3$?
 6. Let $u = (x, y, z)$, express u in the basis $\{a, b, c\}$.

Exercise 7.

Let $E = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 = 0 \text{ and } x_3 - x_4 = 0\}$

We admit that F is a subspace of \mathbb{R}^4 .

1. Determine a basis of E .
2. Complete this basis of E to a basis of \mathbb{R}^4 .

Exercise 8.

Let $E = \{P \in \mathbb{R}_2[X], P(1) = 0\}$.

1. Prove that E is a subspace of $\mathbb{R}_2[X]$.
2. Give a basis of E and deduce its dimension.

Exercise 9.

Let $E = \{P \in \mathbb{R}_3[X], P(-1) = 0 \text{ and } P(1) = 0\}$

1. Prove that E is a subspace of $\mathbb{R}_3[X]$.
2. Determine a basis and the dimension of E .

Exercise 10.

In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, are the following three functions $x \mapsto \sin(x)$, $x \mapsto \sin(2x)$ and $x \mapsto \sin(3x)$ linearly independent?

Exercise 11.

Let $f(x) = \cos(x)$, $g(x) = \cos(x) \cos(2x)$ and $h(x) = \sin(x) \sin(2x)$. Determine $\text{Span}(f, g, h)$.

Exercise 12.

Let's consider the vectors $v_1 = (1, 0, 0, 1)$, $v_2 = (0, 0, 1, 0)$, $v_3 = (0, 1, 0, 0)$, $v_4 = (0, 0, 0, 1)$ and $v_5 = (0, 1, 0, 1)$ of \mathbb{R}^4 .

1. Is $\text{Span}(v_1, v_2)$ supplementary to $\text{Span}(v_3)$ in \mathbb{R}^4 ?
2. Same question for $\text{Span}(v_1, v_3, v_4)$ and $\text{Span}(v_2, v_5)$.
3. Same question for $\text{Span}(v_1, v_2)$ and $\text{Span}(v_3, v_4, v_5)$.