Chapter V: Hamiltonian and Eulerian Problems

In graph theory, two fundamental types of traversal problems are the *Hamiltonian* and *Eulerian* problems. Both deal with finding specific paths or cycles that visit the vertices or edges of a graph under certain conditions.

A Hamiltonian cycle is a closed path that visits each vertex of the graph exactly once before returning to the starting vertex. Determining whether such a cycle exists is a central and challenging question in graph theory, often associated with complex combinatorial structures. In this chapter, we present the main definitions and discuss both necessary and sufficient conditions for the existence of Hamiltonian paths and cycles.

On the other hand, an Eulerian cycle is a closed path that traverses every edge of the graph exactly once. The study of Eulerian graphs dates back to the 18th century with Euler's solution to the famous *Königsberg bridge problem.* We will recall the definitions, establish the necessary and sufficient conditions for the existence of an Eulerian cycle, and describe a simple local algorithm for its construction.

Finally, the chapter concludes with a discussion of the relations between Hamiltonian and Eulerian problems, highlighting the similarities and fundamental differences between these two classic graph-theoretic concepts.

1. Hamiltonian Problem

1.1. Definitions:

Let $\mathbf{G} = (\mathbf{X}, \mathbf{U})$ be a connected graph of order $\mathbf{n}$.

A *Hamiltonian path* (or *Hamiltonian chain*) is a path (or chain) that passes exactly once through each vertex of $\mathbf{G}$.

A Hamiltonian path is therefore an *elementary path* of length $\mathbf{n}-1$.

A *Hamiltonian circuit* (or *Hamiltonian cycle*) is a circuit (or cycle) that passes exactly once through each vertex of $\mathbf{G}$.

A Hamiltonian cycle is thus an *elementary cycle* of length $\mathbf{n}$.

An undirected graph is called *Hamiltonian* if it contains a Hamiltonian cycle. Similarly, a directed graph is called Hamiltonian if it contains a Hamiltonian circuit.

No polynomial-time solution is known for finding a Hamiltonian cycle. This problem is therefore very difficult. Moreover, there is no condition that is both necessary and sufficient for the existence of a Hamiltonian cycle. In addition, while many theorems concern the existence of a Hamiltonian cycle in a simple graph, only a few theorems address the existence of a Hamiltonian circuit in a directed graph.

Example:

The following graph has a Hamiltonian cycle: $(\mathbf{d}, \mathbf{b}, \mathbf{e}, \mathbf{a}, \mathbf{c}, \mathbf{d})$.

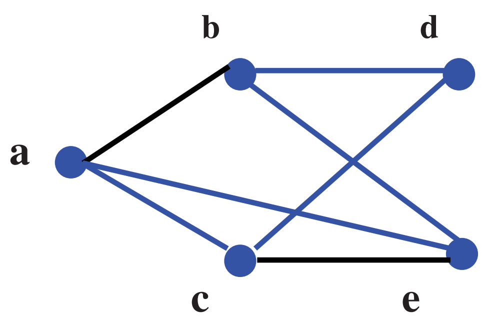


Figure 1: Hamiltonian cycle.

On the other hand, the following graph does not have a Hamiltonian cycle but has a Hamiltonian chain $(\mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{e}, \mathbf{d})$.

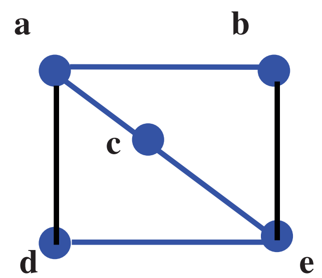


Figure 2: Hamiltonian chain.

1.2. Necessary condition for the existence of a Hamiltonian cycle:

Although no necessary and sufficient condition is known for determining whether a graph is Hamiltonian, several useful sufficient conditions provide guidance:

- If a multigraph $\mathbf{G}$ contains a vertex of degree 1, it cannot be Hamiltonian.
- For any simple multigraph $\mathbf{G}$ of order $\mathbf{n} \geq 3$, if every vertex has degree at least $\frac{\mathbf{n}}{2}$, then $\mathbf{G}$ is Hamiltonian (Dirac's Theorem).
- Every complete graph with at least three vertices is Hamiltonian.
- For an undirected graph to be Hamiltonian, it must be 2-connected; that is, the graph remains connected after the removal of any single vertex. The graph in the following figure is the smallest 2-connected non-Hamiltonian graph.

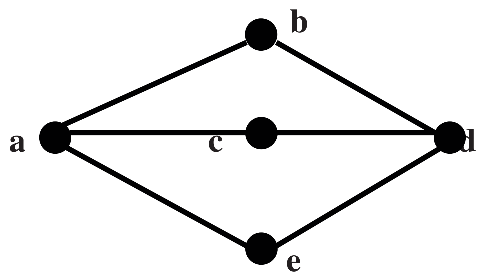


Figure 3: The smallest 2-connected non-hamiltonian graph.

- Every Hamiltonian graph contains a spanning subgraph of degree 2, meaning a partial graph that includes all the vertices of the original graph and in which each vertex is incident to exactly two edges (forming one or more cycles).

However, the existence of such a partial graph does not guarantee Hamiltonicity, as illustrated by the *Petersen graph*:

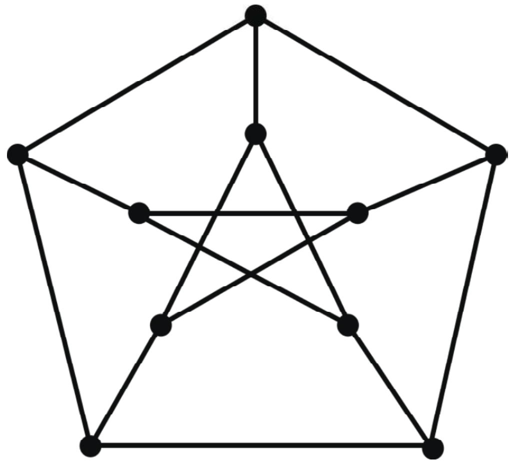


Figure 3: The Petersen graph is a well-known example of a non-Hamiltonian graph.

1.3. Sufficient condition for the existence of a Hamiltonian circuit:

Meyniel's Theorem (1973):

Let $\mathbf{G}$ be a loopless, strongly connected 1-graph of order $\mathbf{n}$. If, for every pair of non-adjacent vertices $\mathbf{x}$ and $\mathbf{y}$, the sum of their degrees satisfies $\mathbf{d}_G(\mathbf{x})+\mathbf{d}_G(\mathbf{y}) \geq 2\mathbf{n}-1$, then $\mathbf{G}$ contains a Hamiltonian circuit.

Corollary (Ghouila-Houri Theorem, 1960):

If a loopless, strongly connected 1-graph $\mathbf{G}$ of order $\mathbf{n}$ satisfies the condition $\mathbf{d}_G(\mathbf{x}) \geq 2\mathbf{n}$ for every vertex $\mathbf{x}$, then the graph contains a Hamiltonian circuit.

1.4. Sufficient condition for the existence of a Hamiltonian cycle:

Dirac's Theorem (1952):

A simple graph of order $\mathbf{n}$ that satisfies $\mathbf{d}_G(\mathbf{x}) \geq \frac{\mathbf{n}}{2}$ for every vertex $\mathbf{x}$, admits a Hamiltonian cycle.

This theorem is a restriction of Ore's theorem (1961).

Ore's Theorem (1961):

A simple graph of order $\mathbf{n}$ such that every pair of non-adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ satisfies $\mathbf{d}_G(\mathbf{x}) + \mathbf{d}_G(\mathbf{y}) \geq 2\mathbf{n}$ admits a Hamiltonian cycle.

Tutte's Theorem (1956):

If a planar graph is 4-connected, then it contains a Hamiltonian cycle.

2. Eulerian Problem:

The Eulerian problem is one of the oldest combinatorial problems, as demonstrated by Euler's solution to the Seven Bridges of Königsberg problem (now Kaliningrad).

2.1. Definitions:

Let $\mathbf{G}$ be a connected multigraph of order $\mathbf{n}$. An *Eulerian chain* (or *Eulerian cycle*) is a chain (or cycle) that uses each edge of $\mathbf{G}$ exactly once. An Eulerian chain is therefore simple. A multigraph is called *Eulerian* if it contains an Eulerian cycle.

2.2. Necessary and sufficient condition for the existence of an Eulerian chain:

Euler's Theorem (1766):

A multigraph admits an Eulerian chain if and only if it is connected (except for isolated vertices) and if the number of vertices of odd degree is 0 or 2.

Corollary:

A multigraph admits an Eulerian cycle if and only if it is connected (except for isolated vertices) and has no vertices of odd degree.

2.3. Local algorithm to construct an Eulerian cycle:

Principle of the algorithm:

This algorithm aims to traverse **"in a single stroke"** all the edges of a connected multigraph in which all vertices have even degree.

Three traversal rules:

- Start from any vertex $\mathbf{a}$, and follow a chain without ever using the same edge twice.
- Upon reaching a vertex $\mathbf{x} \neq \mathbf{a}$ at the ***k***-th step, never take an edge that is a *bridge* in the graph $\mathbf{G_k}$ formed by the edges not yet used (except if $\mathbf{x}$ is an isolated vertex in $\mathbf{G_k}$). A bridge (also called an isthmus or cut-edge) is an edge whose removal increases the number of connected components of the graph.
- If you return to the starting vertex $\mathbf{a}$, continue along any unused edge if one exists; otherwise, stop.

<u>**Example:**</u>

Consider the following graph:

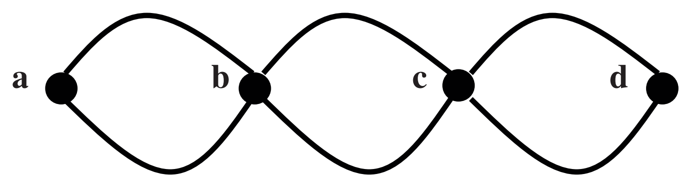


Figure 4: An Eulerian cycle

During traversal, some unused edges may become *bridges* in the partial graph formed by the remaining edges. When this happens, the algorithm avoids to use them by taking another way.

For example, we start at vertex **b**, so the current chain is $(\mathbf{b})$. We choose an unused edge from **b**, say **{b,c}**, obtaining the chain $(\mathbf{b},\mathbf{c})$. At this point, edge $\{\mathbf{c},\mathbf{b}\}$ becomes a bridge in the remaining graph, so we avoid using it for now. Instead, we select another edge, for example $\{\mathbf{c},\mathbf{d}\}$, and continue following the chain $(\mathbf{d},\mathbf{c},\mathbf{b},\mathbf{a},\mathbf{b})$ traversing unused edges without difficulty.

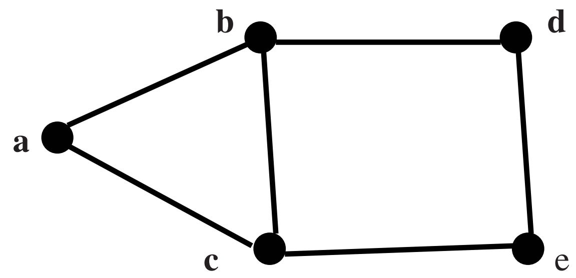
Finally, when we return to the starting vertex $\mathbf{b}$ and no unused edges remain, the algorithm stops. The resulting Eulerian cycle is $(\mathbf{b},\ \mathbf{c},\ \mathbf{d},\ \mathbf{c},\ \mathbf{b},\ \mathbf{a},\ \mathbf{b})$.

<u>2.4. Relation between Eulerian and Hamiltonian problem:</u>

An Eulerian graph without subdivisions (i.e., without edges that have been divided by inserting extra vertices) is Hamiltonian, but the converse is obviously false.

<u>Example:</u>

Consider the following graph:



There is a cycle visiting all vertices exactly once: $(\mathbf{a},\mathbf{b},\mathbf{d},\mathbf{e},\mathbf{c},\mathbf{a})$. Therefore the graph is Hamiltonian. However, vertex $\mathbf{b}$ has degree 3 (odd). Thus the graph is not Eulerian.

This clearly shows that **Eulerian ≠ Hamiltonian**, but some Eulerian graphs without subdivisions can also be Hamiltonian.