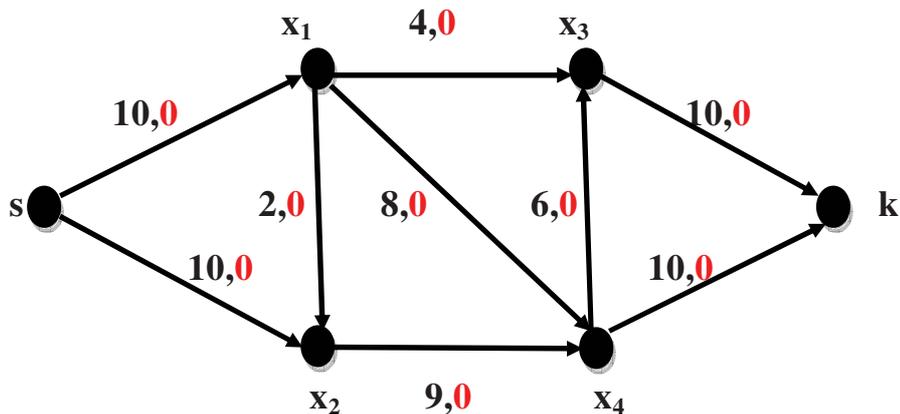


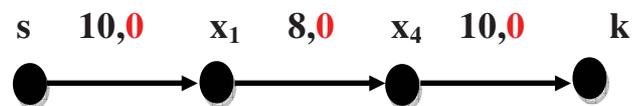
Tutorial Session 6 (Solution)

Exercise 1:

Initialize all arc flows to zero:

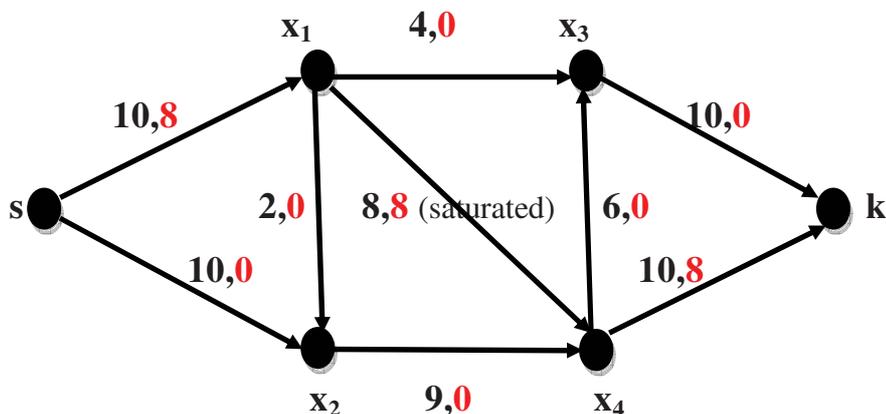


Find an **augmenting chain** between the source s and the sink k along which the flow can be increased. For example, we select the following chain:

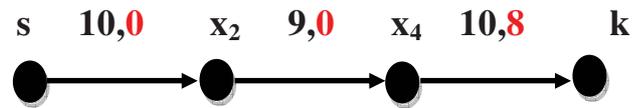


$$\text{Calculate: } y = \min \{ C(\mathbf{u}) - F(\mathbf{u}) \mid \mathbf{u} \in \mu^+ \} = \min \{ 10 - 0, 8 - 0, 10 - 0 \} = 8.$$

We improve the flow along this chain by adding the quantity **8** to the flow of the arcs in μ^+ . The flow of the arcs that do not belong to the chain remains unchanged.

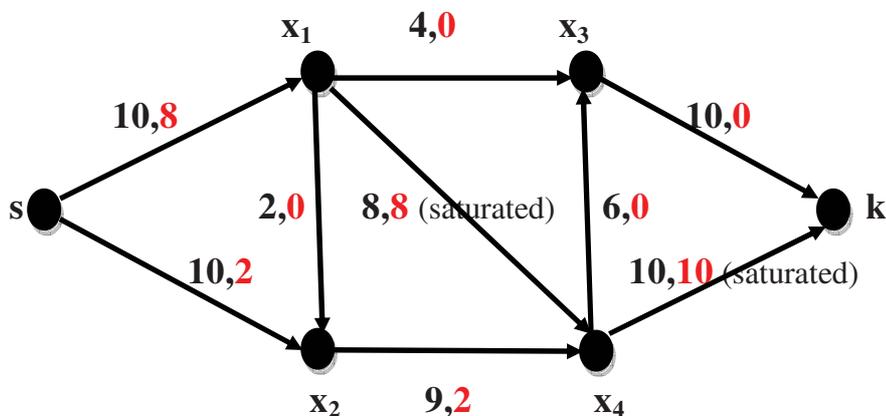


Now, find an augmenting chain in the residual network. We select:



Calculate: $y = \min \{C(\mathbf{u}) - F(\mathbf{u}) \mid \mathbf{u} \in \mu^+\} = \min \{10 - 0, 9 - 0, 10 - 8\} = 2.$

We increase the flow by 2 on the arcs of the chain, and all other arcs keep the same flow.

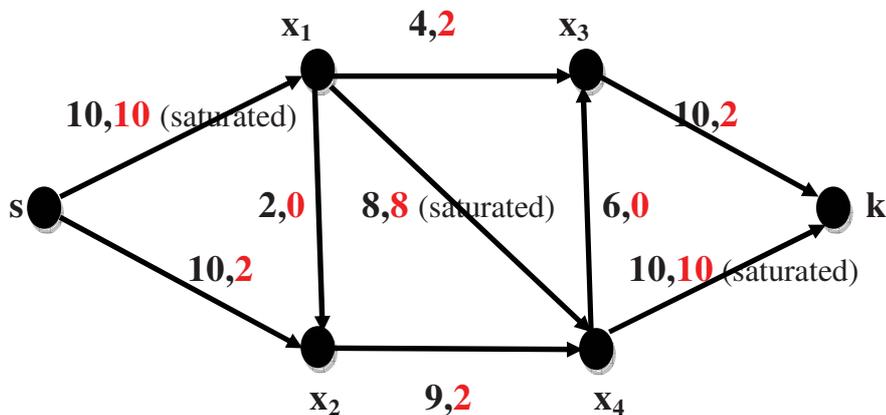


Select the augmenting chain:

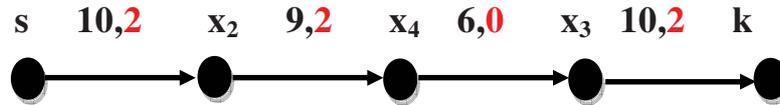


Calculate: $y = \min \{C(\mathbf{u}) - F(\mathbf{u}) \mid \mathbf{u} \in \mu^+\} = \min \{10 - 8, 4 - 0, 10 - 0\} = 2.$

We increase the flow by 2 on the arcs of the chain:

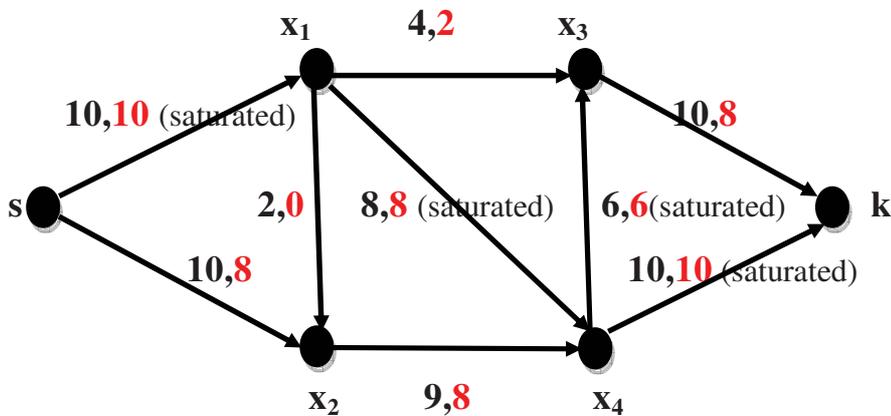


Select the augmenting chain:



Calculate: $y = \min \{C(\mathbf{u}) - F(\mathbf{u}) \mid \mathbf{u} \in \mu^+\} = \min \{10 - 2, 9 - 2, 6 - 0, 10 - 2\} = 6.$

We increase the flow by 6 on the arcs of the chain:



Select the augmenting chain:



Calculate:

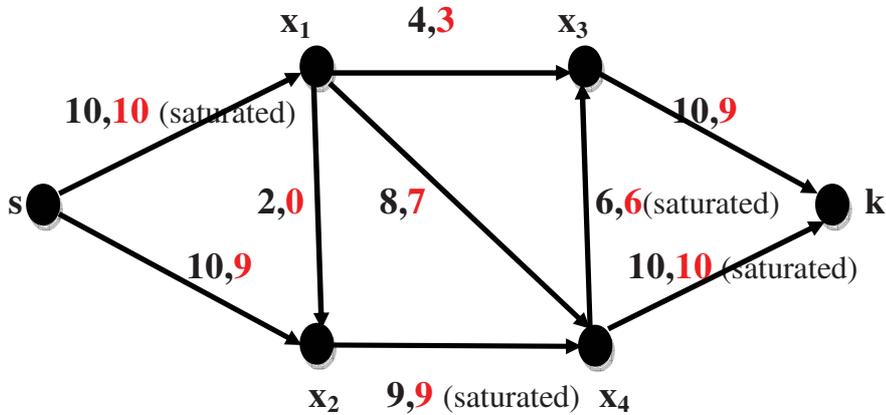
$y_1 = \min \{C(\mathbf{u}) - F(\mathbf{u}) \mid \mathbf{u} \in \mu^+\} = \min \{10 - 8, 9 - 8, 4 - 2, 10 - 8\} = 1$

and

$y_2 = \min \{F(\mathbf{u}) \mid \mathbf{u} \in \mu^-\} = 8.$

Then $y = \min \{y_1, y_2\} = \min \{1, 8\} = 1.$

We increase the flow by 1 on the forward arcs (μ^+) and decrease it by 1 on the backward arcs (μ^-).

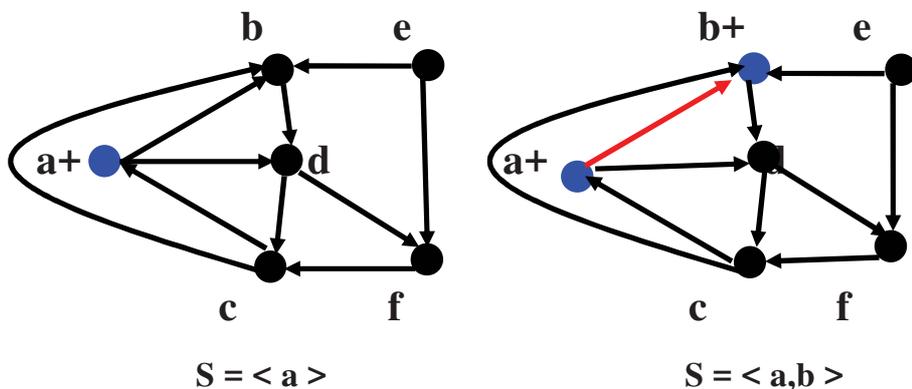


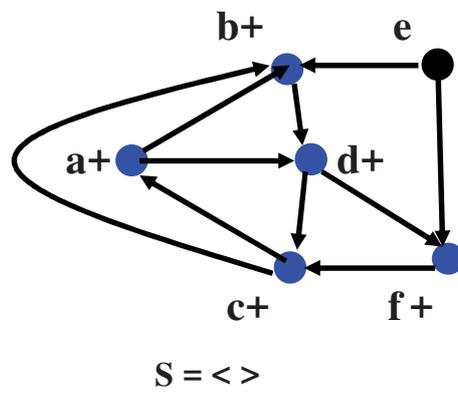
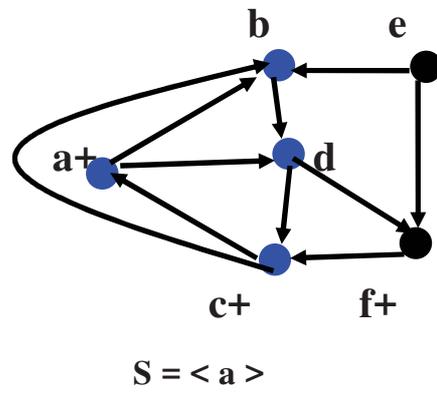
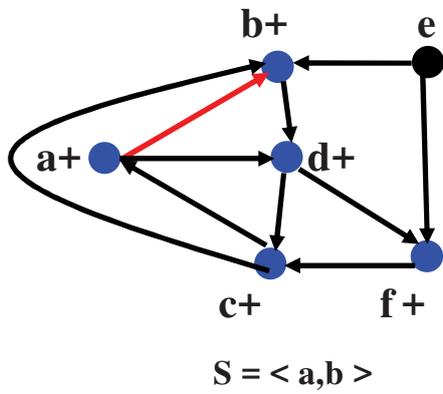
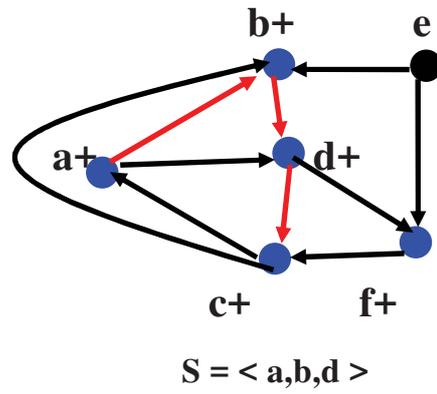
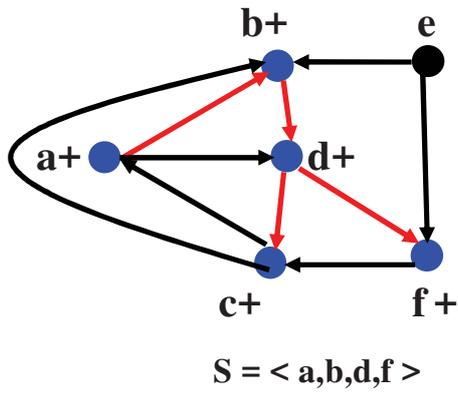
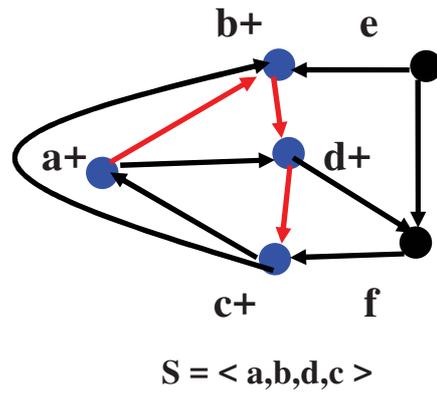
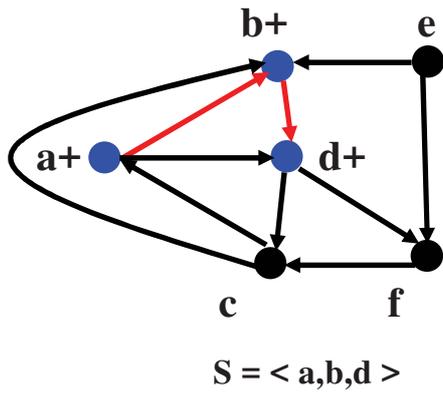
There is no augmenting chain in the residual graph. This means that the Ford–Fulkerson method has finished. Since the maximum flow is equal to the total flow leaving the source, in this example the maximum flow is $10+9 = 19$.

Exercise 2:

The following figures present, in a stepwise manner, the marking procedure and the associated stack configurations observed during the forward and backward depth-first traversals:

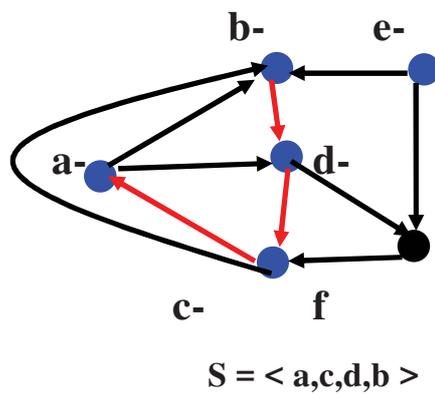
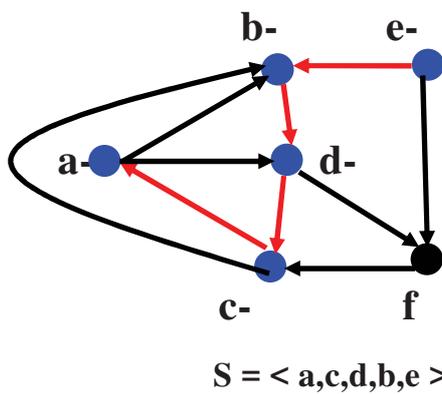
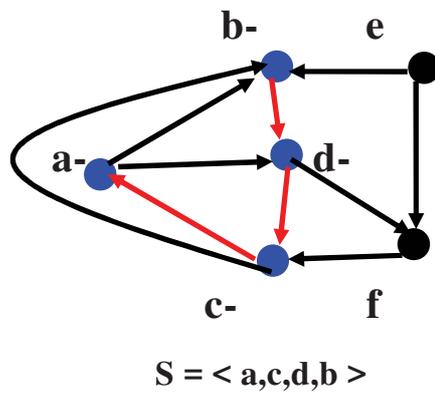
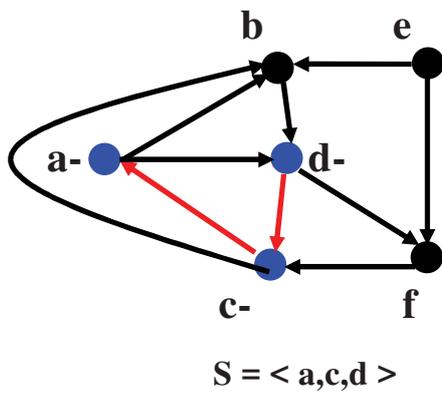
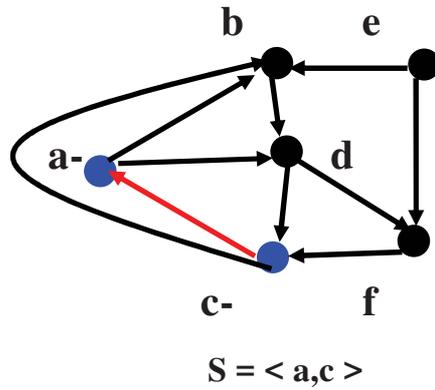
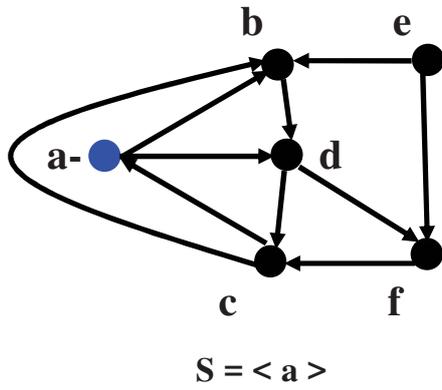
1) **Forward depth-first traversals:**

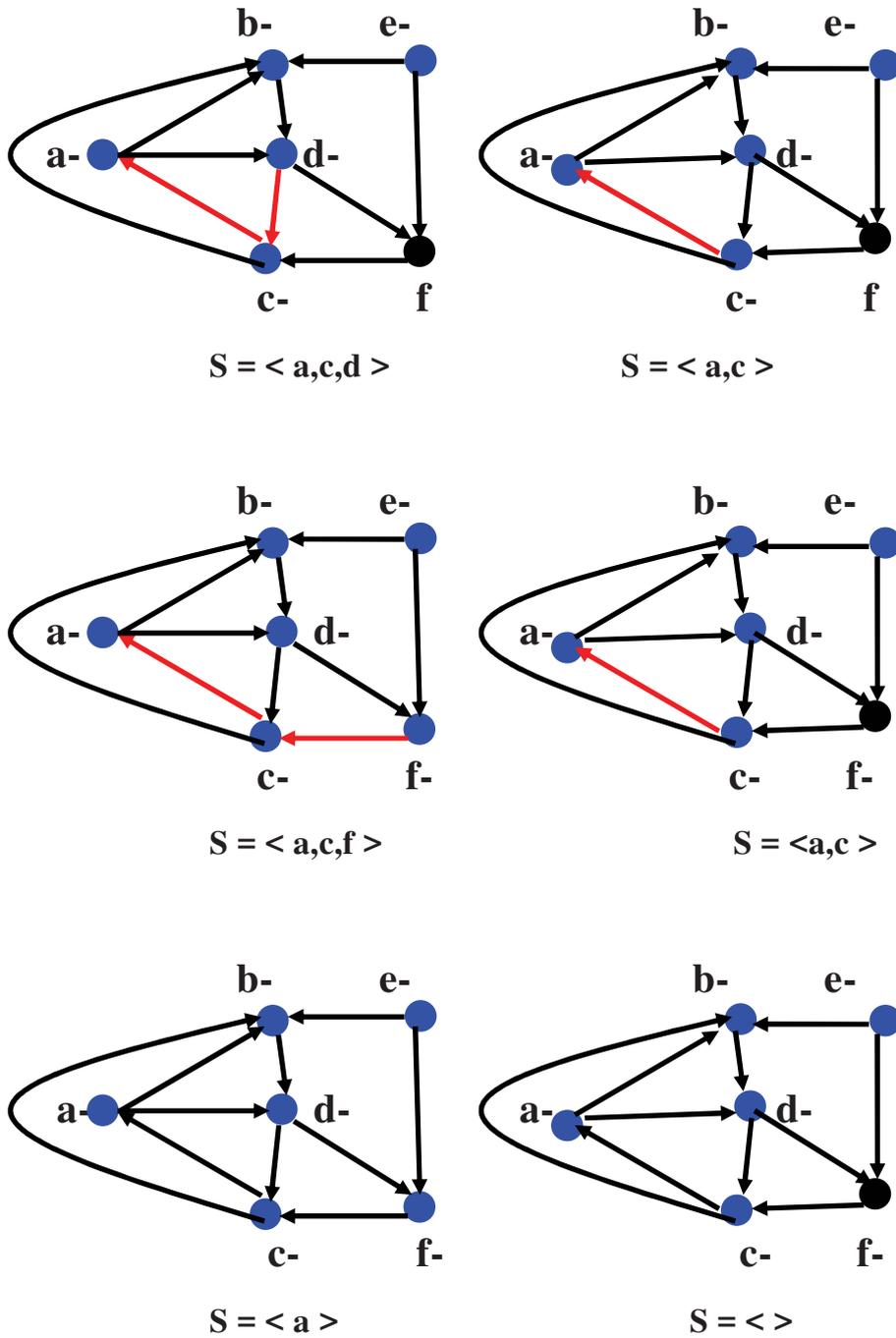




Thus, ForwardSet={ a, b, c, d, f }.

2) Backward depth-first traversals:





Thus, the **BackwardSet** = { a, b, c, d, e, f }.

Therefore, the first strongly connected component is:

$$\begin{aligned}
 C_1 &= \text{ForwardSet} \cap \text{BackwardSet} \\
 &= \{ a, b, c, d, f \} \cap \{ a, b, c, d, e, f \} \\
 &= \{ a, b, c, d, f \}.
 \end{aligned}$$

We remove from X the vertices belonging to C_1 , which gives $X=X \setminus C_1=\{e\}$.

All marks are then erased. We mark vertex e with the signs (+) and (-). Since there are no other vertices to be marked, we obtain $C_2=\{e\}$.

Next, we remove from X the vertices of C_2 , resulting in $X=X \setminus C_2=\emptyset$. The process is therefore complete.

The number of strongly connected components of G is two, namely:

$C_1=\{a,b,c,d,f\}$ and $C_2=\{e\}$.

Exercise 3:

Initialization:

Vertex x	A	B	C	D	E
$d(x)$	0	∞	∞	∞	∞

$S=\{A\}$, $T=\emptyset$, $\alpha = A$.

1) Update of distances

We examine the two arcs (A,B) and (A,C):

$$d(A) + C(A,B) = 2 < d(B) = \infty \Rightarrow d(B) = 2.$$

$$d(A) + C(A,C) = 6 < d(C) = \infty \Rightarrow d(C) = 6.$$

Vertex x	A	B	C	D	E
$d(x)$	0	2	6	∞	∞

Selection

$$d(B) = \min \{ d(B), d(C), d(D), d(E) \} = 2$$

Update

$S = \{A, B\}$, $T=\{(A,B)\}$, $\alpha = b$.

2) Update of distances

We examine the arc (B,E):

$$d(B) + C(B, E) = 4 < d(E) = \infty \Rightarrow d(E) = 4.$$

Vertex x	A	B	C	D	E
d(x)	0	2	6	∞	4

Selection

$$d(E) = \min\{d(C), d(D), d(E)\}.$$

Update

$$S=\{A, B, E\}, T=\{(A,B), (B,E)\}, \alpha = E, .$$

3) Update of distances

We examine the two arcs (E,C) and (E,D):

$$d(E) + C(E,C) = 5 < d(C) = 6 \Rightarrow d(C) = 5.$$

$$d(E) + C(E,D) = 11 < d(D) = \infty \Rightarrow d(D) = 11.$$

Vertex x	A	B	C	D	E
d(x)	0	2	5	11	4

Selection

$$d(C) = \min\{d(C), \pi(D)\}.$$

Update

$$A=\{A, B, E, C\}, T=\{(A,B), (B,E), (E,C)\}, \alpha = C.$$

4) Update of distances

We examine the arc (C,D):

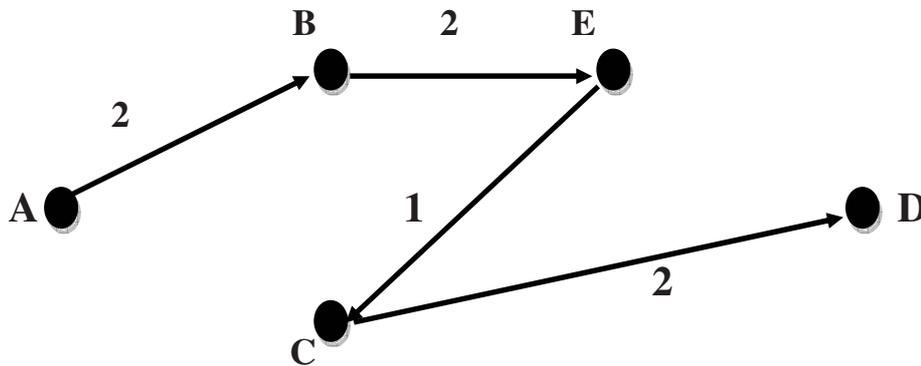
$$d(C)+C(C,D) = 7 < d(D) = 11 \Rightarrow d(D) = 7.$$

Vertex x	A	B	C	D	E
d(x)	0	2	5	7	4

Update

$$A=\{A, B, E, C, D\}, T=\{(A,B), (B,E), (E,C), (C,D)\}.$$

The shortest-path tree originating from vertex A is:



Exercise 3:

Sorted edges by increasing weight:

$$C(B,C) \leq C(D,E) \leq C(D,E) \leq C(C,D) \leq C(B,E) \leq C(A,B) \leq C(A,C)$$

Construction process:

$$W = \emptyset$$

$$W = \{(B,C)\}$$

$$W = \{(B,C), (C,E)\}$$

$$W = \{(B,C), (C,E), (E,D)\}$$

$$W = \{(B,C), (C,E), (E,D)\}$$

$$W = \{(B,C), (C,E), (E,D), (A,B)\}$$

The final set W represents the minimum spanning tree of (G) :

