Chapter III: Flow problems

1. <u>Definitions</u>

1.1. <u>Transportation network:</u>

A transportation network is a graph without loops in which each arc **u** is assigned a positive number $\mathbf{C(u)}$, called the *capacity* of arc **u**. This network contains one vertex without predecessors, called the *source* or *entry* of the network, and another vertex without successors, called the *sink* or *exit* of the network. The network is denoted by $\mathbf{R} = (\mathbf{X}, \mathbf{U}, \mathbf{C})$.

<u>Example:</u>

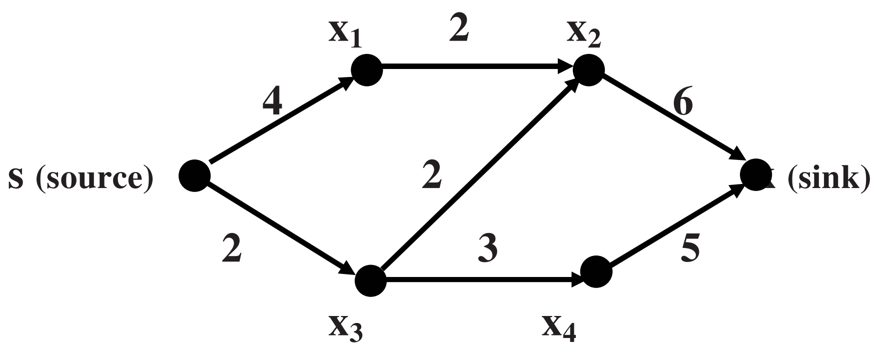


Figure 1: Transportation network.

<u>Flow in a transportation network:</u>

A flow **F** in a transportation network assigns to each arc **u** a value $\mathbf{F(u)}$, which represents the amount of flow that moves through this arc from the source to the sink.

A flow is *conservative* if it follows Kirchhoff's law at every vertex: "The total flow entering a vertex is equal to the total flow leaving that vertex."

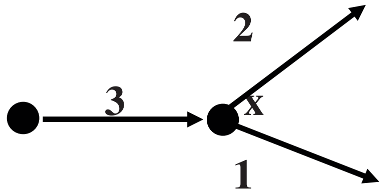


Figure 2: Kirchhoff's law**.**

In the graph shown in Figure 2, the amount of flow entering vertex **x** is equal to the sum of the flow amounts leaving **x**. The value of the flow is **2**. Kirchhoff's law is therefore satisfied at vertex **x**.

Compatible flow:

A flow is *compatible* in a network if, for every arc $\mathbf{u}=(\mathbf{x},\mathbf{y})$, we have $0 \le \mathbf{F}(\mathbf{u}) \le \mathbf{C}(\mathbf{u})$. In other words, for each arc $\mathbf{u}$, the flow passing through it does not exceed its capacity.

In what follows, we will use the following representation:

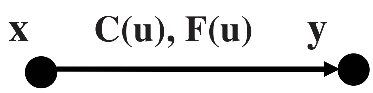


Figure 2: Representation "Capacity, Flow".

Example:

Consider the following network:

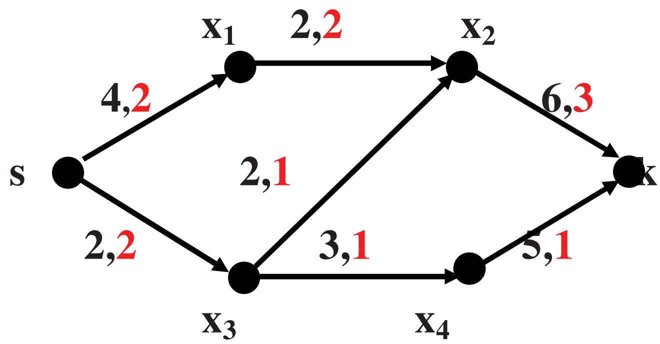


Figure 3: Compatible flow.

In the network shown in Figure 3, the flow passing through each arc does not exceed its capacity. Therefore, this flow is compatible.

Complete flow:

A flow is *complete* if, for every path from the source to the sink, there is at least one *saturated* arc, that is, an arc for which the flow equals its capacity $\mathbf{F(u) = C(u)}$.

Example:

In the network shown in Figure 3, there are three paths leading from $\mathbf{s}$ to $\mathbf{k}$, and in each of them, at least one arc is saturated. Therefore, the flow is *complete*.

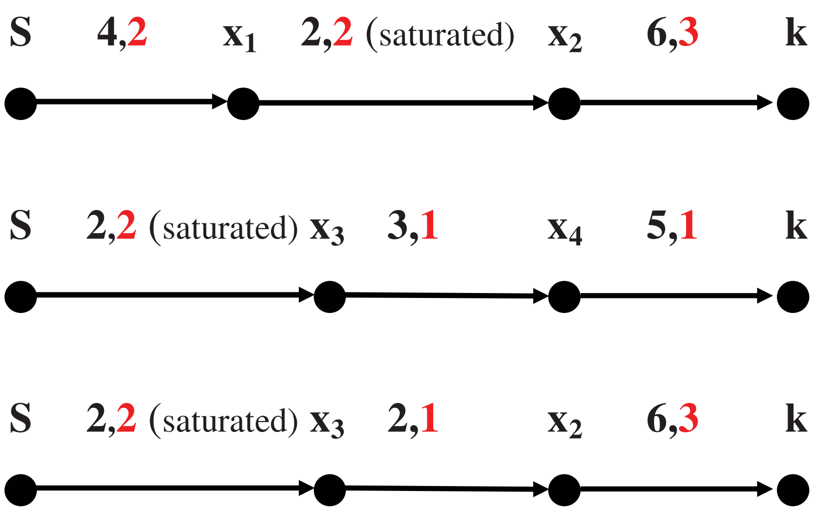


Figure 3: Complete flow**.**

2.2. Ford-Fulkerson theorem

Cut between two vertices and its capacity:

In a transportation network $\mathbf{R = (X, U, C)}$, a *cut* between **a** and **b** is defined to be a set of arcs of the form $\boldsymbol{\omega}^+(\mathbf{A})$ with $\mathbf{a} \in \mathbf{A}$ and $\mathbf{b} \notin \mathbf{A}$.

The *capacity of a cut* is defined to be the sum of the capacities of its arcs, i.e.,

$$\mathbf{C(\omega^+(A)) = \sum_{u \in \omega^+(A)} C(u)}.$$

Example:

Consider the following flow network:

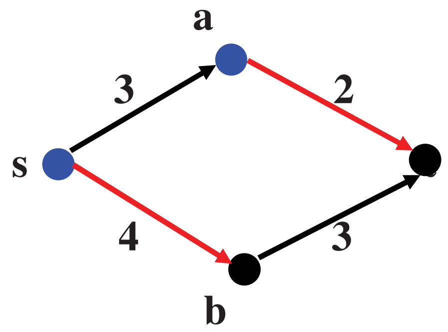


Figure 3: Cut and its capacity.

Let's create a cut between **s** and **t**.

Consider the set $\mathbf{A} = \{\mathbf{s}, \mathbf{a}\}$, in which $\mathbf{a} \in \mathbf{A}$ and $\mathbf{t} \notin \mathbf{A}$. The set of arcs $\omega^{+}(\mathbf{A}) = \{(\mathbf{s}, \mathbf{b}), (\mathbf{a}, \mathbf{t})\}$ forms a cut between **s** and **t**.

In this example, the arc **(s,b)** has a capacity of **4,** and the arc $(\mathbf{a}, \mathbf{t})$ has a capacity of **2**.

Therefore, the capacity of the cut is: $\mathbf{C}(\mathbf{s}, \mathbf{b}) + \mathbf{C}(\mathbf{a}, \mathbf{t}) = 4 + 2 = 6.$

Maximum flow theorem (Ford, Fulkerson [1957])

In a transportation network, the maximum flow that can go from the source to the sink is equal to the minimum capacity of a cut that separates the source from the sink.

2.3. Ford-Fulkerson algorithm:

The maximum flow problem consists in finding the largest amount of flow that can be sent from the source $\mathbf{s}$ to the sink $\mathbf{k}$, while taking into account the transport capacities of the arcs and the available quantity at the source $\mathbf{s}$. The most well-known algorithm for solving the maximum flow problem is the Ford–Fulkerson algorithm. The idea of the Ford–Fulkerson algorithm is to start with a compatible flow in the network (The simplest one being the zero flow) and then to improve it step by step until a complete flow is reached. A chain along which the flow can be increased is called an *augmenting chain.*

This is a chain where: oriented in the traversal direction and by μ^- the set of the other arcs of the cycle.

- the forward arcs (oriented in the traversal direction) have not yet reached their capacity, and
- the backward arcs (oriented in the opposite direction of the traversal) carry a positive flow that can be reduced.

In other words, a chain μ is said to be augmenting if:

- for every forward arc $\mathbf{u} \in \mu^+ : \mathbf{F}(\mathbf{u}) < \mathbf{C}(\mathbf{u})$,
- for every backward arc $\mathbf{u} \in \mu^- : \mathbf{F}(\mathbf{u}) > 0$.

The flow along this chain μ can be increased by the amount: $\mathbf{y} = \min\left\{\left\{\mathbf{C}(\mathbf{u}) - \mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^+\right\} , \left\{\mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^-\right\}\right\}$.

To improve the flow, we:

- add $\mathbf{y}$ to the flow of all forward arcs in the chain, and
- subtract $\mathbf{y}$ from the flow of all backward arcs in the chain.

<u>Example:</u>

Consider the following transportation flow network:

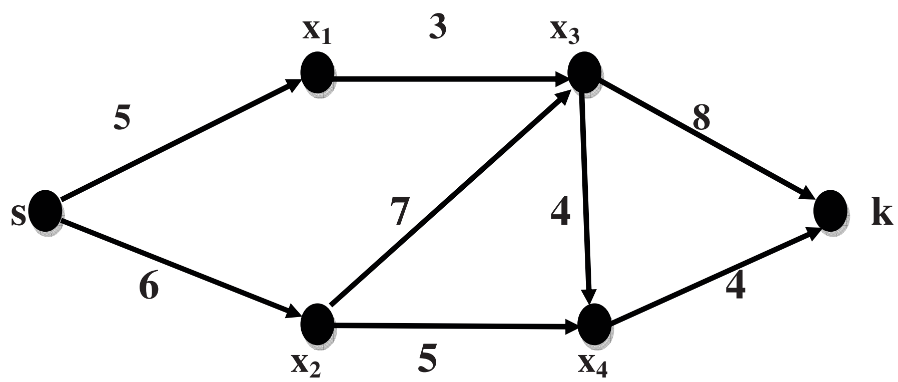
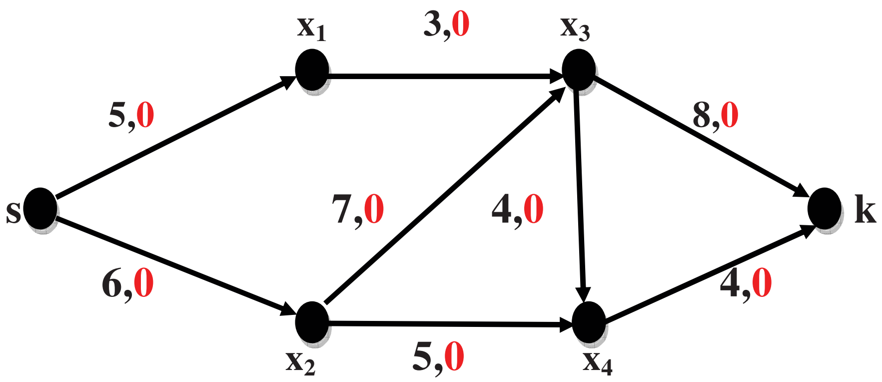


Figure 4: Transportation flow network.

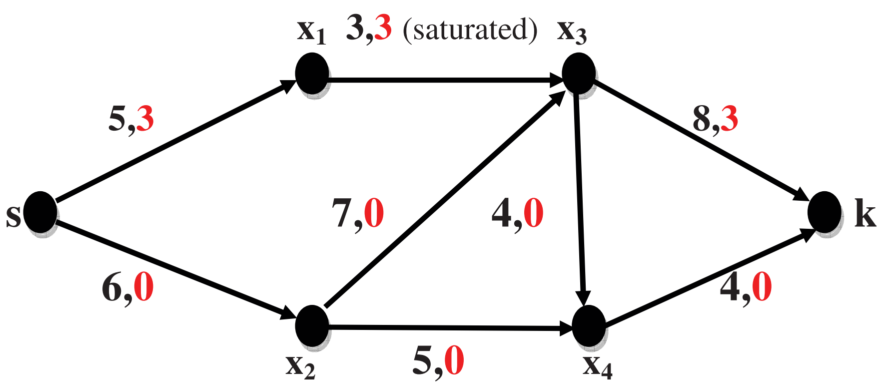
Initialize all arc flows to zero:



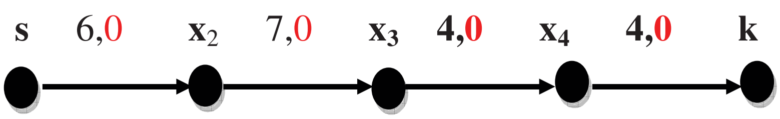
Find an augmenting chain between the source **s** and the sink **k** along which the flow can be increased. For example, we select the following chain:

Calculate: $\mathbf{y} = \min\{\mathbf{C}(\mathbf{u}) - \mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^+\} = \min\{5-0, 3-0, 8-0\} = 3$.

We improve the flow along this chain by adding the quantity 3 to the flow of the arcs in μ^+. The flow of the arcs that do not belong to the chain remains unchanged.

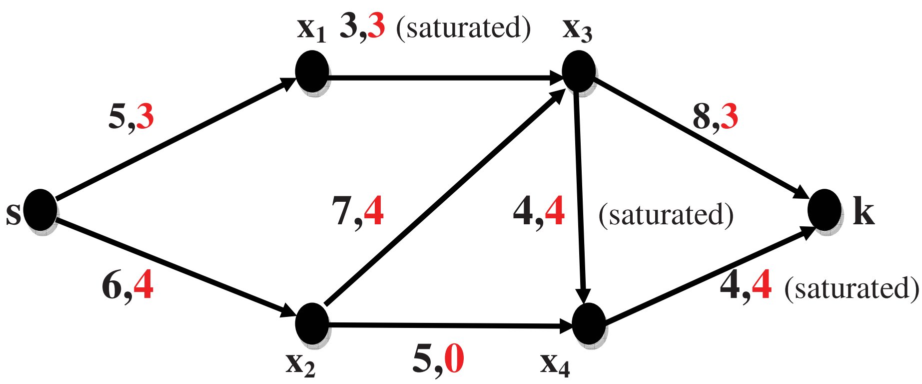


Now, find an augmenting chain in the residual network. We select:

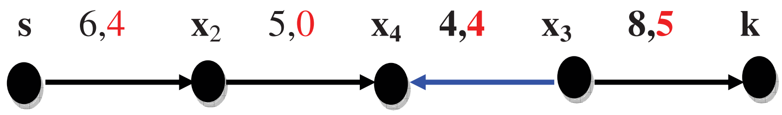


Calculate: $\mathbf{y} = \min\{\mathbf{C}(\mathbf{u}) - \mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^{+}\} = \min\{6-0, 7-0, 4-0, 4-0\} = 4$.

We increase the flow by 4 on the arcs of the chain, and all other arcs keep the same flow.



Select the augmenting chain:



Calculate:

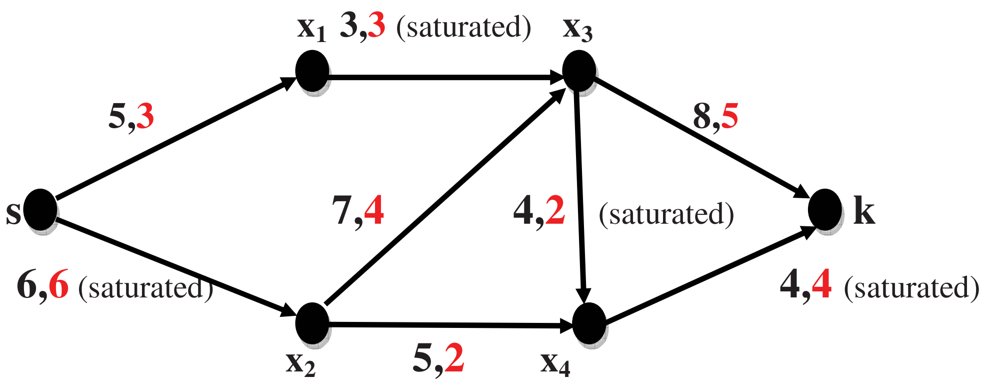
$$\mathbf{y}_1 = \min\{\mathbf{C}(\mathbf{u}) - \mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^{+}\} = \min\{6-4,\ 5-0,\ 8\text{-}5\} = 2$$

and

$$\mathbf{y}_2 = \min\{\mathbf{F}(\mathbf{u}) \mid \mathbf{u} \in \mu^{-}\} = 4.$$

Then $\mathbf{y} = \min\{\mathbf{y}_1\ ,\ \mathbf{y}_2\} = \min\{2\ ,\ 4\} = 2.$

We increase the flow by 2 on the forward arcs (μ^{+}) and decrease it by 2 on the backward arcs (μ^{-}).



There is no augmenting chain in the residual graph. This means that the Ford–Fulkerson method has finished. Since the maximum flow is equal to the total flow leaving the source, in this example the maximum flow is **6+3 = 9**.

3. **Search for a compatible flow**

If the graph **G** is not a transportation network, the existence of a compatible flow is not guaranteed. The following algorithm search for the existence of such flow:

Algorithm:

1) Start with the zero flow.

2) Search for an arc **u** such that $\mathbf{F(u) < C(u)}$. If no such arc exists END. Otherwise, set **s** as the terminal endpoint of **u**, and **k** as the initial endpoint of **u**.

3) Search for a chain from **s** to **k** such that:

the arcs in the positive direction are not saturated, and the arcs in the negative direction have a flow strictly greater than their lower bound.

If such a chain does not exist END.

4) Use this chain to improve the flow, by adjusting it to satisfy the bound constraint on arc **u**, then return to step (2).