

Tutorial Session 5 (Solution)

Exercise 1:

1. The decomposition is achieved by traversing the cycle and extracting an elementary cycle each time a previously visited vertex is encountered. The first elementary cycle obtained is $(\mathbf{d}, \mathbf{c}, \mathbf{e}, \mathbf{f}, \mathbf{d})$. The remaining cycle, $(\mathbf{a}, \mathbf{d}, \mathbf{a})$, is also an elementary cycle.
2. To decompose the cocycle $\omega(\{\mathbf{a}, \mathbf{d}, \mathbf{f}\})$ into elementary cocycles, we proceed with the following steps:

The subgraph generated by $\mathbf{A} = \{\mathbf{a}, \mathbf{d}, \mathbf{f}\}$ has two connected components, $\mathbf{A}_1 = \{\mathbf{a}\}$ and $\mathbf{A}_2 = \{\mathbf{d}, \mathbf{f}\}$, both are included in $\mathbf{C} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$.

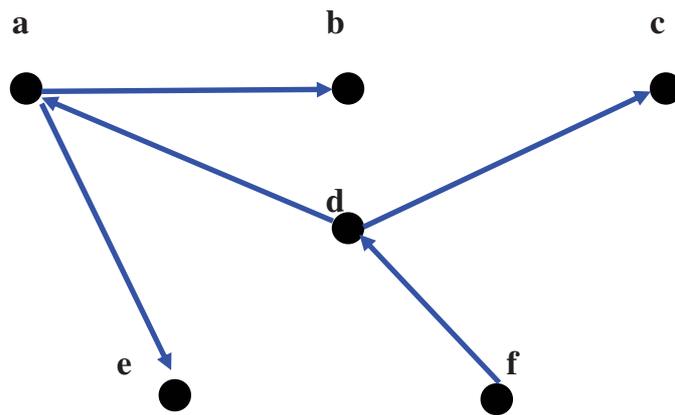
The subgraph generated by $\mathbf{C} - \mathbf{A}_1 = \{\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ has one connected components $\mathbf{C}_1 = \{\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. Similarly, the subgraph generated by $\mathbf{C} - \mathbf{A}_2 = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}\}$ has one connected component, $\mathbf{C}_2 = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}\}$.

For the vectors associated with the elementary cocycles, we obtain:

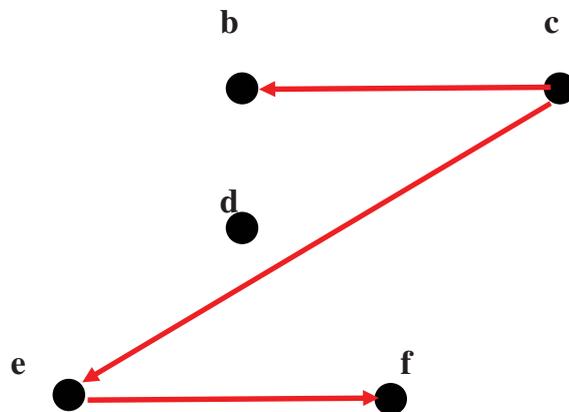
$$\begin{aligned}\omega_G(\mathbf{A}) &= \omega_G(\mathbf{A}_1) + \omega_G(\mathbf{A}_2) \\ &= -\omega_G(\mathbf{C}_1) - \omega_G(\mathbf{C}_2).\end{aligned}$$

3. When the marking process reaches vertex \mathbf{d} , there is no black arc of the form (\mathbf{d}, \mathbf{x}) and no red arc of the form (\mathbf{d}, \mathbf{x}) or (\mathbf{x}, \mathbf{d}) . Therefore, the marking process terminates at this point, and we conclude that there is no elementary red-and-black cycle to which arc (\mathbf{a}, \mathbf{b}) belongs. However, we can identify that arc (\mathbf{a}, \mathbf{b}) is part of an elementary black-and-green cocycle $\{(\mathbf{a}, \mathbf{b}), (\mathbf{d}, \mathbf{a}), (\mathbf{a}, \mathbf{e})\}$, in which all black arcs are oriented in the same direction.
4. The cyclomatic number : $\mathbf{v}(\mathbf{G}) = \mathbf{m} - \mathbf{n} + \mathbf{p} = 8 - 6 + 1 = 3$.
5. Let \mathbf{n} and \mathbf{m} be, respectively, the order and size of tree \mathbf{T} . Tree \mathbf{T} must have: $\mathbf{m} = \mathbf{n} - 1 = 6 - 1 = 5$ edges. But graph \mathbf{G} has 8 edges. So 3 edges must be removed.

6. A spanning tree T of G :



7. The cotree T^* associated with T :



8. A cycle basis for the graph G :

Cycles : $C_1 = (a, b, c, d, a)$, $C_2 = (d, c, e, a, d)$ and $C_3 = (a, e, f, d, a)$ form a cycle basis because there are three of them, and they are independent since each contains an arc of the cotree that the others do not.

9. Vectors associated with each cycle of the basis:

	(a,b)	(a,e)	(c,b)	(c,e)	(d,a)	(d,c)	(e,f)	(f,d)
C_1	1	0	-1	0	1	-1	0	0
C_2	0	-1	0	1	-1	1	0	0
C_3	0	1	0	0	1	0	1	1

10. The cocyclomatic number: $\lambda(G) = n - p = 6 - 1 = 5$.

11. A cocycle basis for the graph G:

Cocycles :

$$CO_1 = \omega(\{a, c, d, e, f\}) = \{(a, b), (c, b)\}$$

$$CO_2 = \omega(\{a, b, c, d, f\}) = \{(a, e), (c, e), (e, f)\}$$

$$CO_3 = \omega(\{a, b, d, e, f\}) = \{(d, c), (c, b), (c, e)\}$$

$$CO_4 = \omega(\{a, b, c, d, e\}) = \{(f, d), (e, f)\}$$

$$CO_5 = \omega(\{a, b, e\}) = \{(d, a), (c, e), (e, f), (c, b)\}$$

form a cocycle basis because there are five of them, and they are independent since each contains an arc of the tree **T** that the others do not.

12. The vectors associated with each cocycle of the basis:

	(a,b)	(a,e)	(c,b)	(c,e)	(d,a)	(d,c)	(e,f)	(f,d)
CO ₁	1	0	1	0	0	0	0	0
CO ₂	0	1	0	1	0	0	-1	0
CO ₃	0	0	-1	-1	0	1	0	0
CO ₄	0	0	0	0	0	0	1	-1
CO ₅	0	0	-1	-1	-1	0	1	0

Exercise 5:

a) Suppose that the multigraph $K_{3,3}$ (representing the 3 houses and 3 factories problem) is planar. Then Euler's formula must hold, so: $f = 2 - n + m = 2 - 6 + 9 = 5$. Each face must have at least 4 edges on its boundary; otherwise, two vertices from the same set would be connected. The number of arcs in the bipartite incidence graph between faces and edges therefore gives: $2m = 18 > 4f = 20$. The contradiction in this result proves that $K_{3,3}$ is not planar.

b) Now suppose that multigraph K_5 is planar. Then Euler's formula must hold, so: $f = 2 - n + m = 2 - 5 + 10 = 7$. Each face must have at least 3 edges on its boundary. The number of arcs in the bipartite incidence graph between faces and edges therefore gives: $2m = 20 > 3f = 21$. The contradiction in this result proves that K_5 is not planar.

Exercise 3:

The arithmetic expression $(e^x - y) * (c + d / z) + 4$ is evaluated according to operator precedence:

Parentheses \rightarrow Exponentiation \rightarrow Division \rightarrow Multiplication \rightarrow Addition/Subtraction.

The main operator is '+' (evaluated last). The left part $(e^x - y) * (c + d / z)$ has '*' as its root. Inside it: $(e^x - y)$ uses '-', with e^x (power) on the left and y on the right. $(c + d / z)$ uses '+', where d / z (division) is computed before addition. Finally, the result of multiplication is added to 4.

