Chapter 2: Cycles

In this chapter, we explore important concepts related to cycles and their fundamental role in graph theory. We begin by studying the decomposition of cycles and cocycles, the Arc Colouring Lemma proposed by Minty (1960), and the notions of cycle and cocycle bases. We then introduce the cyclomatic and cocyclomatic numbers, which measure the structure and connectivity of a graph. Next, we address the concept of planarity, where we learn how to determine whether a graph can be drawn on a plane without edge intersections. This section includes Euler's formula, Kuratowski's theorem (1930), and the idea of a dual graph. Finally, we study trees, forests, and arborescences, focusing on their definitions, main properties, and the construction of maximum (or spanning) trees, which are key structures in many applications of graph theory.

1. Cyclomatic and cocyclomatic number

1.1. Decomposition of cycles and cocycles into elementary sums

Decomposition of cycle into elementary cycles:

A cycle is said to be *elementary* if and only if it is minimal, meaning that it contains no other cycle as a proper subset. A cycle can be decomposed into a sum of pairwise arc-disjoint elementary cycles. This decomposition is achieved by traversing the cycle and extracting an elementary cycle each time a previously visited vertex is encountered. The procedure is formalized and presented as an explicit algorithm.

Algorithm:

Input: A graph and a cycle represented as a sequence of vertices, e.g., $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

Output: A list of elementary cycles that are pairwise arc-disjoint.

Steps:

1. Initialize an empty list **ElementaryCycles = []**.
   Initialize an empty set **Visited = {}**.
2. Traverse the cycle from the first vertex to the last:
   - If $\mathbf{v}_i$ is not in **Visited**, add it to **Visited**.
   - If $\mathbf{v}_i$ is already in **Visited**:
     - An elementary cycle is detected. It starts from the previous occurrence of $\mathbf{v}_i$ to the current position.
     - Extract this subsequence as a new cycle and add it to **ElementaryCycles**.
     - Remove the vertices of this cycle from **Visited**.
3. Return the list **ElementaryCycles**.

Example:

Consider the cycle **(f,h,f,c,e,f)** in the graph below. This cycle is *not elementary*, as it contains repeated vertices.

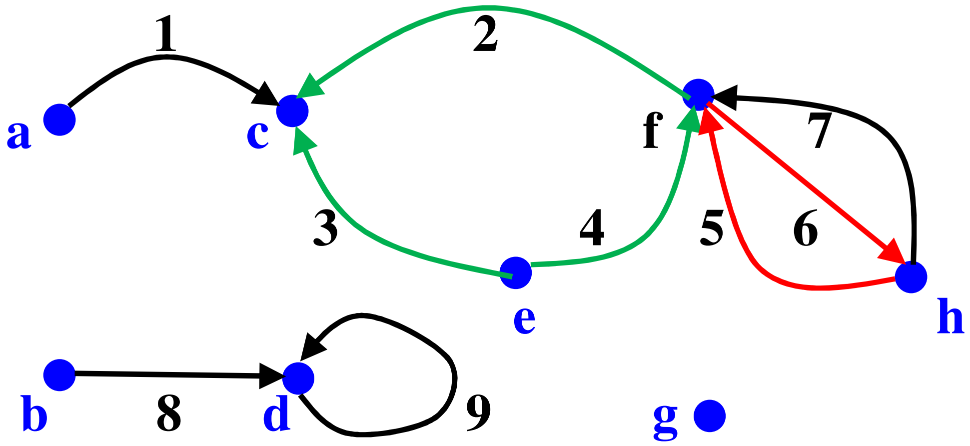


Figure 1: Decomposition of cycle into elementary cycles.

Traverse the cycle from the first occurrence of **f** to its second appearance: **(f,h,f)**, this is an *elementary cycle*.

The remaining chain is **(f,c,e,f)**, which we traverse again. From the next occurrence of **f** to the final one: **(f,c,e,f)**, this is another *elementary cycle*.

The cycle **(f,h,f,c,e,f)** is decomposed into the following elementary cycles: **(f,h,f)** and **(f,c,e,f)**.

Let μ be the vector associated with the cycle **(f,h,f,c,e,f)**, which corresponds to the sequence of arcs **(6,5,2,3,4)**. We have $\mu = (0,1,-1,1,1,1,0,0,0)$.

The vectors associated with the elementary cycles are: $\mu_1 = (0,0,0,0,1,1,0,0,0)$ corresponding to **(6,5)** or **(f,h,f)**.

$\mu_2 = (0,1,-1,1,0,0,0,0,0)$ corresponding to **(2,3,4)** or **(f,c,e,f)**.

Thus, we have the decomposition: $\mu = \mu_1 + \mu_2$.

<u>Decomposition of cocycle into elementary cocycles:</u>

A cocycle can be decomposed into a sum of elementary cocycles that are pairwise arc-disjoint.

The following algorithm decompose a cocycle into disjoint elementary cocycles.

<u>Algorithm:</u>

Input:

A graph **G=(X,U)**, and a cocycle $\omega_G(A)$ associated with a vertex subset $A \subset X$.

Output:

A decomposition of $\omega_G(A)$ into pairwise arc-disjoint elementary cocycles.

Steps:

1. Identify the connected components $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ of the subgraph generated by the subset $\mathbf{A}$.
2. For each **i=1,…,k**, determine the connected component $\mathbf{C}_i$ of the graph $\mathbf{G}$ that contains $\mathbf{A}_i$.
3. For each **i=1,…,k**, determine the connected components $\mathbf{C}_{i1}, \mathbf{C}_{i2}, \ldots, \mathbf{C}_{im_i}$ of the subgraph generated by $\mathbf{C}_i - \mathbf{A}_i$.
4. For each **i=1,…,k**, and for each **j=1,…,m$_i$**, construct the cocycle $\boldsymbol{\omega}_G^{(i,j)} = -\boldsymbol{\omega}_G\left(\mathbf{C}_{ij}\right)$.

[$\boldsymbol{\omega}_G^{(i,j)}$ is an elementary cocycle since $\boldsymbol{\omega}_G\left(\mathbf{C}_{ij}\right)$ joins the connected subgraphs $\mathbf{C}_{ij}$ and $\mathbf{A}_i \cup \mathbf{C}_{i1} \cup \mathbf{C}_{i2} \cup \ldots \mathbf{C}_{im_i}$. Furthermore, $\boldsymbol{\omega}_G^{(i,j)}$ **(for i=1,…,k,** and **j=1,…,m$_i$**) are pairwise arc-disjoint.]

5. Return the list of all elementary cocycles $\boldsymbol{\omega}_G^{(i,j)}$ as a decomposition of $\boldsymbol{\omega}_G\left(\mathbf{A}\right)$ into pairwise arc-disjoint elementary cocycles, i.e., $\boldsymbol{\omega}_G\left(\mathbf{A}\right) = \sum_{\substack{1\le i\le k \\ 1\le j\le i_m}} \boldsymbol{\omega}_G^{(i,j)} = -\sum_{\substack{1\le i\le k \\ 1\le j\le i_m}} \boldsymbol{\omega}_G\left(\mathbf{C}_{ij}\right)$.

__Example:__

In the graph **G = (X , U)** shown below, let us consider a cocycle $\omega = (1,2,3,4,5)$ associated with the subset $\mathbf{A} = \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\} \subset \mathbf{X}$.

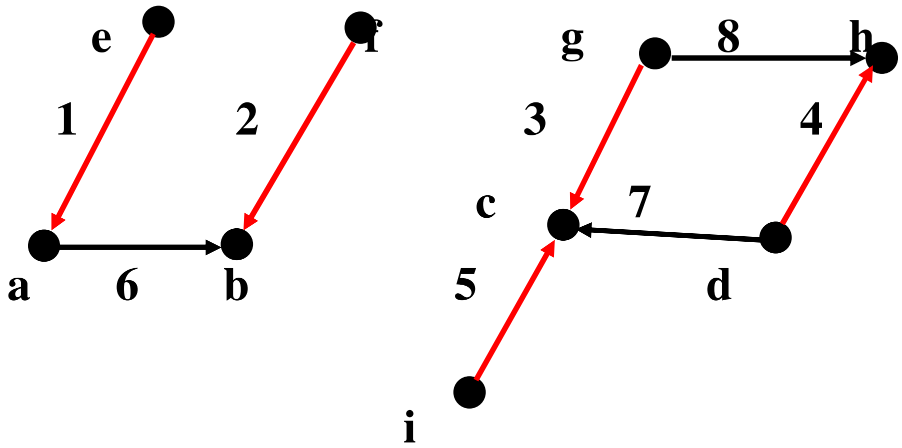


Figure 2. Example of a cocycle

To decompose the cocycle $\boldsymbol{\omega}$ into elementary cocycles, we proceed with the following steps:

The subgraph generated by $\mathbf{A}$ has two connected components, $\mathbf{A}_1 = \{\mathbf{a}, \mathbf{b}\}$ and $\mathbf{A}_2 = \{\mathbf{c}, \mathbf{d}\}$, which are respectively included in $\mathbf{C}_1 = \{\mathbf{a}, \mathbf{b}, \mathbf{e}, \mathbf{f}\}$ and $\mathbf{C}_2 = \{\mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{i}\}$, the two connected components of the graph.

The subgraph generated by $\mathbf{C}_1 - \mathbf{A}_1 = \{\mathbf{e}, \mathbf{f}\}$ has two connected components, $\mathbf{C}_{1,1} = \{\mathbf{e}\}$ and $\mathbf{C}_{1,2} = \{\mathbf{f}\}$. Similarly, the subgraph generated by $\mathbf{C}_2 - \mathbf{A}_2 = \{\mathbf{g}, \mathbf{h}, \mathbf{i}\}$ has two connected components, $\mathbf{C}_{2,1} = \{\mathbf{g}, \mathbf{h}\}$ and $\mathbf{C}_{2,2} = \{\mathbf{i}\}$.

For the vectors associated with the elementary cocycles, we obtain:

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{G}}(\mathbf{A}) &= \boldsymbol{\omega}_{\mathbf{G}}(\mathbf{A}_1) + \boldsymbol{\omega}_{\mathbf{G}}(\mathbf{A}_2) \\ &= -\boldsymbol{\omega}_{\mathbf{G}}(\mathbf{C}_{11}) - \boldsymbol{\omega}_{\mathbf{G}}(\mathbf{C}_{12}) - \boldsymbol{\omega}_{\mathbf{G}}(\mathbf{C}_{21}) - \boldsymbol{\omega}_{\mathbf{G}}(\mathbf{C}_{22}).\end{aligned}$$

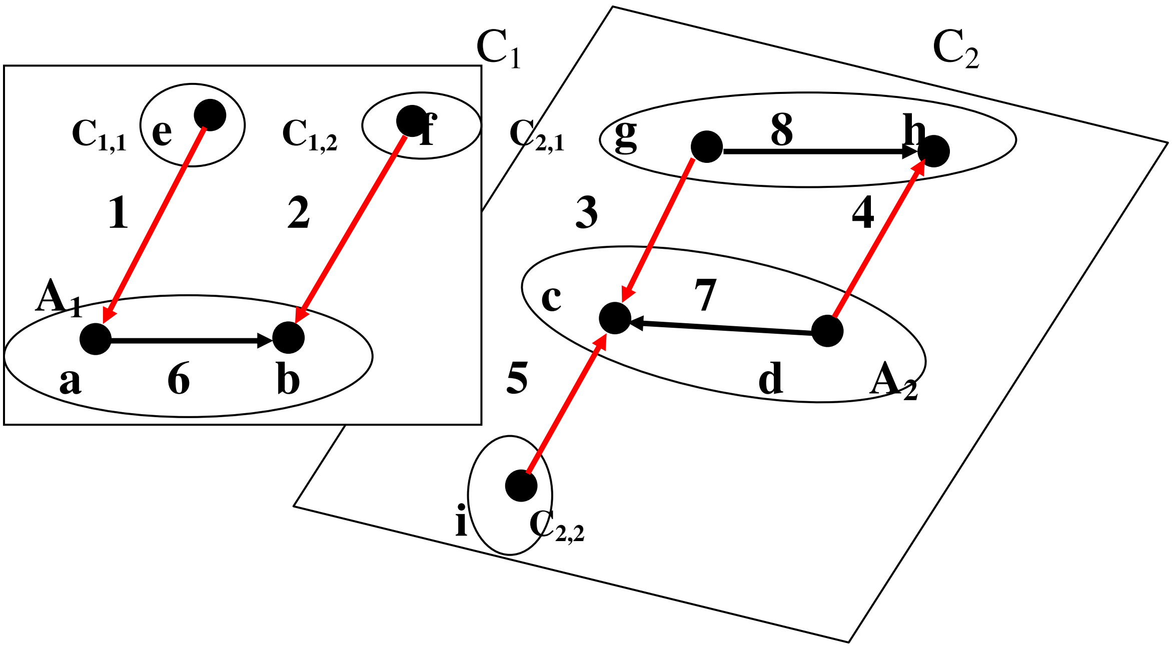


Figure 3. Illustration of the decomposition of a cocycle into elementary cocycles

1.2. Arc colouring lemma (Minty 1960):

Consider a graph with arcs numbered **1**, **2**, …, **m**, and colored either red, green, or black. Arc **1** is assumed to be colored black. Exactly one of the following conditions holds:

(1) there is an elementary cycle containing arc **1** and only red and black arcs with the property that all black arcs in the cycle have the same direction.

(2) there is an elementary cocycle containing arc **1** and only green and black arcs, with the property that all black arcs in the cocycle have the same direction.

***Consequence*:** Each arc belongs either to an elementary circuit or to an elementary cocircuit, but no arc belongs to both.

The following algorithm decides whether arc **1** belongs to an elementary cycle or cocycle:

Algorithm:

Input:

- A graph with arcs colored black, red, or green.
- Arc $1=(\mathbf{b},\mathbf{a})$, and it is colored black.

Steps:

1. Initialization:

- Let $\mathbf{M}$ be the set of marked vertices.
- Initialize $\mathbf{M}=\{\mathbf{a}\}$.

2. Marking Process:

Repeat until no new vertices can be added to $\mathbf{M}$:

For each vertex $\mathbf{x}\in\mathbf{M}$, do:

- For each unmarked vertex $\mathbf{y}$:
    - If there is a black arc $(\mathbf{x},\mathbf{y})$, then mark $\mathbf{y}$ (add $\mathbf{y}$ to $\mathbf{M}$).
    - If there is a red arc $(\mathbf{x},\mathbf{y})$ **or** $(\mathbf{y},\mathbf{x})$, then mark $\mathbf{y}$ (add $\mathbf{y}$ to $\mathbf{M}$).

3. Termination and Case Analysis:

- If $\mathbf{b} \in \mathbf{M}$, then:
  - The path used to mark **b** forms an elementary red-and-black cycle with all black arcs in the same direction.
  - Return: "Arc **1** belongs to an elementary cycle."
- Else:
  - Let $\mathbf{A} = \mathbf{M}$.
  - The arcs between **A** and its complement form a green-and-black cocycle with all black arcs oriented into **A**.
  - Return: "Arc **1** belongs to an elementary cocycle."

Example:

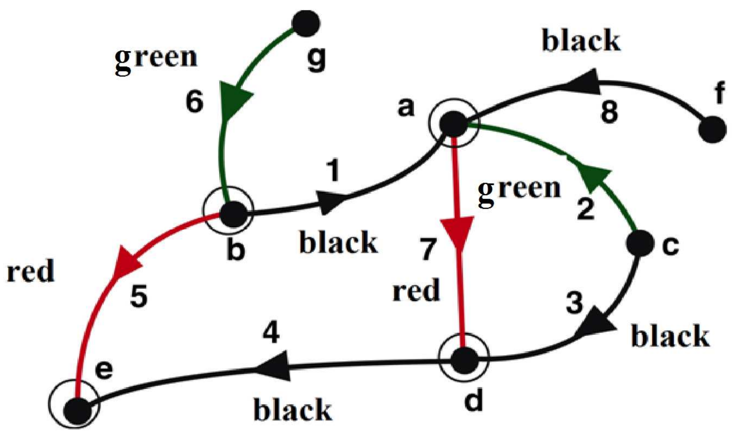


Figure 4: Example of a black and red cycle with all black arcs oriented in the same direction

Let the black arc be **1 = (b, a)**, and suppose that vertex **a** is initially marked.

Starting from vertex **a**, there is no black arc of the form **(a, x)**. However, we find a red arc **7 = (a, d)**. According to the marking rule, since there is a red arc from **a** to **d**, we mark vertex **d** and add it to the set **M** of marked vertices.

From vertex **d,** we find a black arc **4 = (d, e)**. Hence, we mark vertex **e** and add it to the set **M** of marked vertices.

From vertex **e,** there exists a red arc **5 = (b, e)** that connects back arc **1** to the marked vertex **e**. We marked vertex **b**. This completes a closed path **(1, 7, 4, 5)** that alternates between red and black arcs.

The path **(1, 7, 4, 5)** forms an elementary red-and-black cycle in which all black arcs are oriented in the same direction.

Therefore, arc **1** belongs to an elementary cycle.

Example:

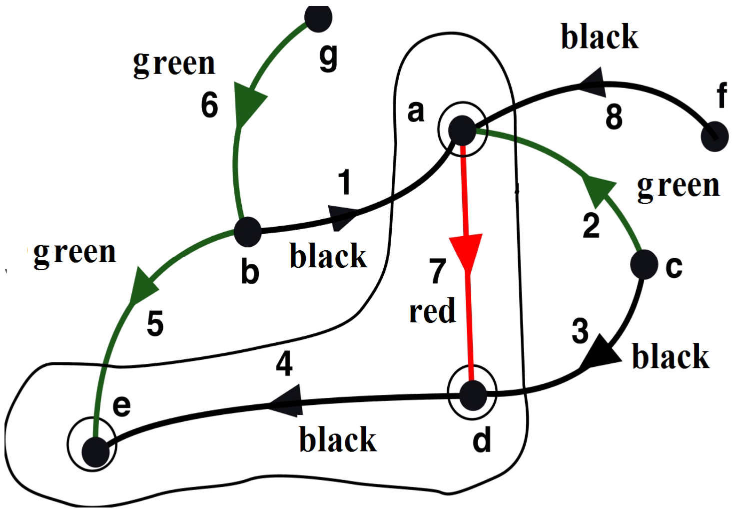


Figure 5: Example of a black and green cocycle with all black arcs oriented in the same direction

Contrary to the previous example, in this case, when the process reaches vertex **e**, there is no black arc of the form **(e, x)** and no red arc of the form **(e, x)** or **(x, e)**. Therefore, the marking process terminates at this point, and we conclude that there is no elementary red-and-black cycle to which arc **1** belongs. However, we can identify that arc

1 is part of an elementary black-and-green cocycle **(1, 2, 3, 5, 8)**, in which all black arcs are oriented in the same direction.

1.3. Cycle basis and cocycle basis:

Definitions:

The cycles $\mu_1, \mu_2, \ldots, \mu_k$ are said to be *independent* if and only if a linear combination of their associated vectors is zero only when each coefficient is zero.

A *fundamental cycle basis* is a set of independent cycles such that any cycle can be expressed as a linear combination of these. The *cyclomatic number* of the graph is equal to the dimension of the cycle basis.

Similarly, cocycles $\omega_1, \omega_2, \ldots, \omega_k$ are independent if and only if a linear combination of their associated vectors is zero only when each coefficient is zero.

A *fundamental cocycle basis* is a set of independent cocycles such that any other cocycle can be expressed as a linear combination of these. The *cocyclomatic number* of the graph is equal to the dimension of the cocycle basis.

Notations:

The cyclomatic number of a graph G is denoted as $\upsilon(\mathbf{G})$, and the cocyclomatic number is denoted as $\lambda(\mathbf{G})$.

Example:

Consider the graph G in **Figure 6**:

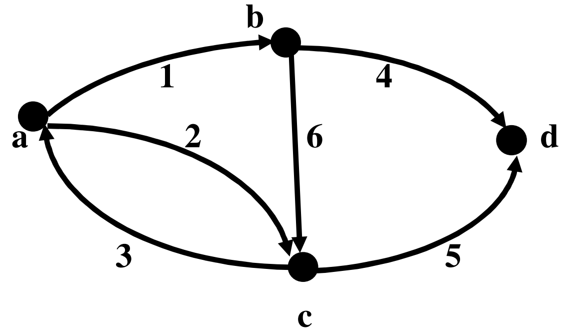


Figure 6: cyclomatic and cocyclomatic numbers.

The elementary cycles of the graph are :

$\mu_1 = (1,6,2) = (\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{a})$,

$\mu_2 = (1,6,3) = (\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{a})$,

$\mu_3 = (2,3) = (\mathbf{a},\mathbf{c},\mathbf{a})$,

$\mu_4 = (1,4,5,2) = (\mathbf{a},\mathbf{b},\mathbf{d},\mathbf{c},\mathbf{a})$,

$\mu_5 = (6,5,4) = (\mathbf{a},\mathbf{c},\mathbf{d},\mathbf{b})$,

$\mu_6 = (1,4,5,3) = (\mathbf{a},\mathbf{b},\mathbf{d},\mathbf{c},\mathbf{a})$.

These cycles are not independent, since we have, for example: $\mu_1 - \mu_2 + \mu_3 = 0$, but The cycles μ_1, μ_3, μ_4 form a cycle basis, and therefore, $\upsilon(\mathbf{G}) = 3$.

For the graph in Fig. 2.1, the elementary cocycles are:

$\omega(\{\mathbf{a}\}) = \{1,2,3\}$,

$\omega(\{\mathbf{a},\mathbf{b}\}) = \{6,2,3,4\}$,

$\omega(\{\mathbf{a},\mathbf{c}\}) = \{6,1,5\}$,

$\omega(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = \{4,5\}$,

$\omega(\{\mathbf{a},\mathbf{b},\mathbf{d}\}) = \{6,2,3,5\}$,

$\omega(\{\mathbf{a},\mathbf{c},\mathbf{d}\}) = \{6,1,4\}$,

Obviously, these cocycles are not independent. To form a basis, one could take, for example, $\omega(\{\mathbf{a}\})$, $\omega(\{\mathbf{a},\mathbf{b},\mathbf{c}\})$ and $\omega(\{\mathbf{a},\mathbf{b},\mathbf{d}\})$. Hence $\lambda(\mathbf{G}) = 3$.

We have the following result:

<u>Theorem:</u>

Let a graph **G** have **n** vertices, **m** arcs, and **p** connected components. Then: $\upsilon(\mathbf{G}) = \mathbf{m} - \mathbf{n} + \mathbf{p}$ and $\lambda(\mathbf{G}) = \mathbf{n} - \mathbf{p}$.

Example:

Taking **Figure 6,** since $\upsilon(\mathbf{G}) = 6 - 4 + 1 = 3$ and $\lambda(\mathbf{G}) = \mathbf{n} - \mathbf{p} = 4 - 1 = 3$, the outcome is consistent with the results obtained in previous Example**.**

2.1. Planar Graph

A graph **G** is said to be *planar* if it can be drawn on a plane such that its vertices are represented by distinct points and its arcs (or edges) are represented by simple curves that do not intersect except at their endpoints. A drawing of **G** that satisfies these conditions is called a *topological planar representation* of the graph.

Example:

The following graph is planar:

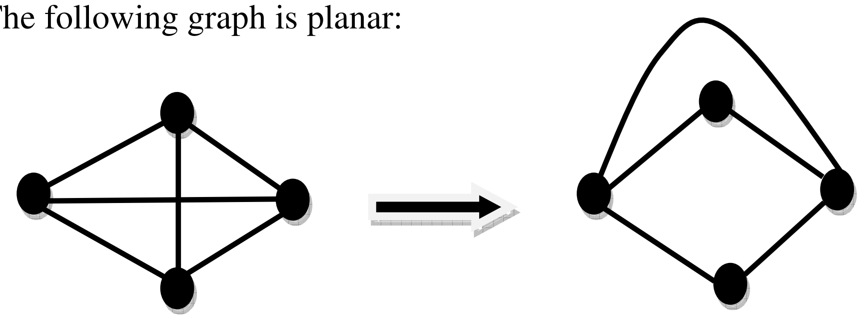


Figure 7: Topological representation of a planar graph

Example:

Three factories, labeled **a**, **b** and **c**, receive water from point **d**, gas from point **e**, and electricity from point **f** through underground supply lines. The question is whether it is possible to arrange the three factories and the three utility stations so that no two supply lines intersect except at their endpoints.

It can be shown that while eight supply lines can be drawn without crossings, the ninth line must necessarily intersect at least one of the others. Therefore, the graph $\mathbf{K}_{3,3}$ is not planar.

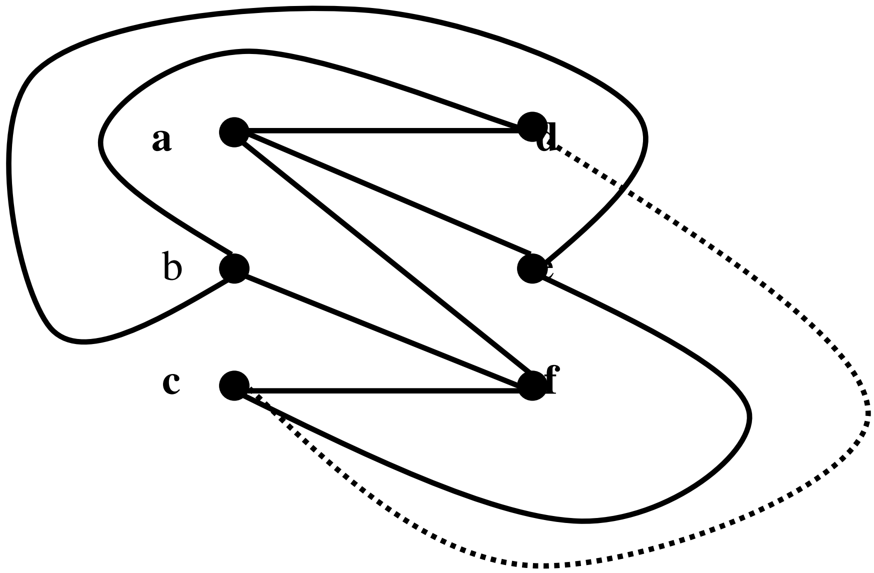


Figure 8: Problem of three factories and three utilities.

Let **G** be a topological representation of a planar graph.

A *face* of **G** is a region of the plane bounded by edges such that any two points within the region can be connected by a continuous curve that does not pass through any vertex or edge of **G**.

The *boundary* of a face is the set of edges that surround it.

The *contour* of a face is defined as an elementary cycle formed by the edges of its boundary that encloses the face **z** in its interior. Note that there is exactly one *unbounded face*, which has no contour, whereas all other faces are *bounded* and possess exactly one contour.

Two faces are said to be *adjacent* if their boundaries share a common edge (if they meet only at a vertex, they are not considered adjacent).

Example:

The following figure gives a geographic map corresponds to a topological planar multigraph whose edges are the borders between countries:

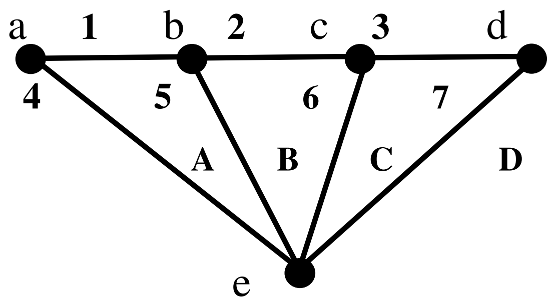


Figure 9: a topological planar graph

Faces A, B, and C are bounded, while D is the unbounded face. The edges 1, 4, and 5 form the boundary of face A. Faces A and B are adjacent.

We have the following results:

Theorem:

In a topological planar graph, the contours of the different bounded faces constitute a cycle basis.

2.2. Euler's formula

If a connected topological planar graph has **n** vertices, **m** arcs and **f** faces, then: $\mathbf{n}-\mathbf{m}+\mathbf{f}=2$.

Example:

In the graph shown by **Figure 9**, we have $\mathbf{n}-\mathbf{m}+\mathbf{f}=5-7+4=2$.

One consequence of Euler's Formula is:

Corollary:

A simple planar graph **G** has a vertex **x** of degree $\mathbf{d}_{\mathbf{G}}(\mathbf{x}) \leq 5$.

2.3. Kuratowski's theorem (1930)

Note first that $\mathbf{K}_5$ and $\mathbf{K}_{3,3}$ are not planar (see Tutorial Session 4).

A multigraph **G** is planar if, and only if, it contains no partial subgraph of type $\mathbf{K}_5$ or type $\mathbf{K}_{3,3}$.

2.4. Dual graph:

Let **G** be a planar graph that is connected and has no isolated vertices. The *dual graph* of **G**, denoted by **D**, is defined as follows:

Each vertex of **D** corresponds to a distinct face of **G**. An edge connecting two vertices in **D** represents an edge in **G** that is shared by the two corresponding faces. Graph **D** is also planar, connected and without isolated vertices.

See **Figure 10**. Graph **D** is called the *topological dual* of G. Note that:

(1) The topological dual of **D** is **G**.

(2) A loop in **G** corresponds to a pendant edge in D, and vice versa.

Example:

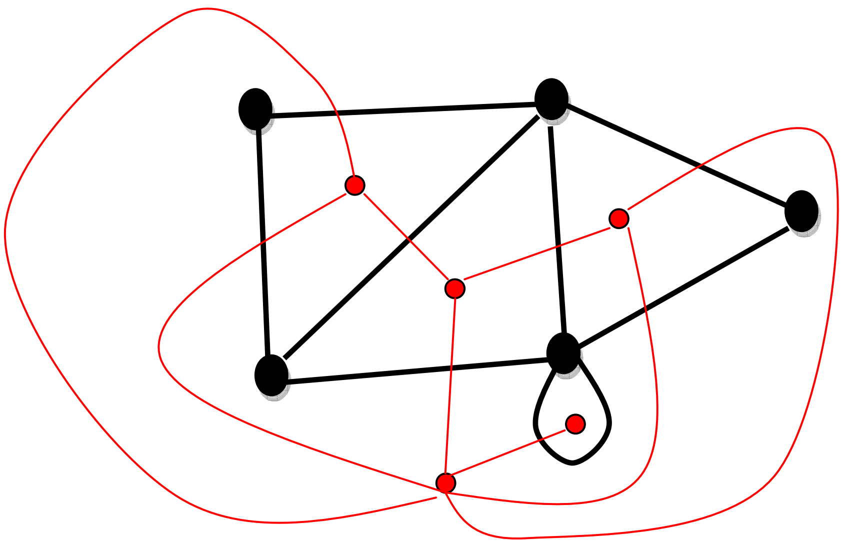


Figure 10: An example of a topological dual graph.

3.Tree, forest and arborescence

3.1. Tree and forest:

A *tree* is defined as a connected graph containing no cycles.

A graph that is acyclic but not necessarily connected is referred to as a *forest.*

Example:

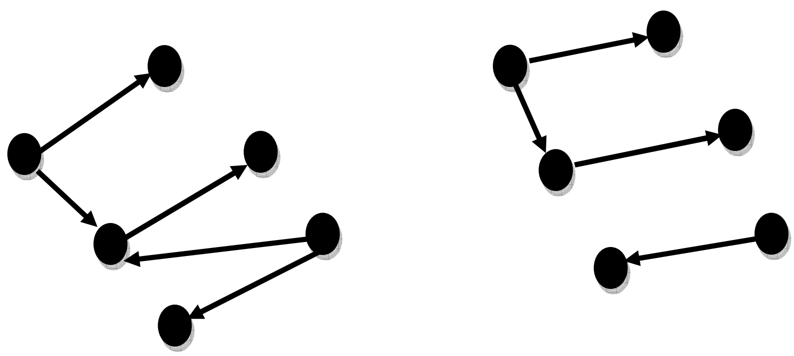


Figure 11: Example of a tree and forest.

The following properties are equivalent and each characterizes a tree:

Properties:

Let **H=(X,U)** be a graph of order **|X|=n>2**. The following statements are equivalent and characterize a tree:

1. **H** is connected and contains no cycles.
2. **H** contains **n-1** arcs and has no cycles.
3. **H** is connected and contains exactly **n-1** arcs.
4. **H** has no cycles, and adding any arc to **H** creates exactly one cycle.
5. **H** is connected, and removing any arc disconnects the graph.
6. Every pair of vertices of H is Connected by one and only one chain.

3.2. Arborescence:

In a graph **G,** a vertex **x** is called a *root* if all the vertices of **G** can be reached by paths starting from **x**.

An *arborescence* is defined as a tree that has a root.

Example:

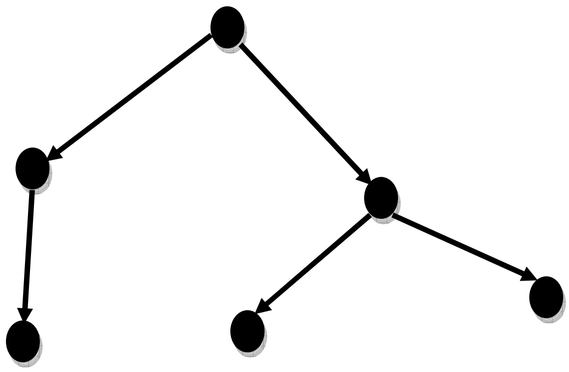
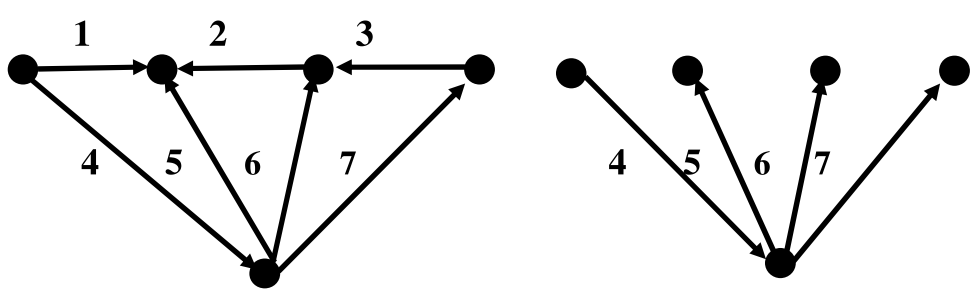


Figure 12: Example of an arborescence.

3.2.Maximum (or spanning) tree:

From any connected graph, one can obtain a partial graph that forms a tree. A *spanning tree* of a connected graph **G** is a partial graph that includes all the vertices of **G** and forms a tree. Equivalently, a spanning tree can be defined as a minimal connected partial graph of **G** or, equivalently, as a maximal acyclic partial graph of **G**. If **G** has **n** vertices, every spanning tree of **G** has exactly **n-1** arcs.

Example:



Graph **G** Spanning tree **H**

Figure 12: The partial graph of G generated by the set of arcs **{4,5,6,7}** form a spanning tree **H**.

A spanning tree can be constructed by the following algorithm:

Algorithm:

Input: A connected graph $\mathbf{G} = (\mathbf{X}, \mathbf{U})$.

Output: A spanning tree $\mathbf{T} = (\mathbf{X}, \mathbf{U_T})$.

Steps:

1. **Initialization:**

   Set $\mathbf{U_T} = \{\mathbf{u}_0\}$, where $\mathbf{u}_0$ is any arc (edge) of **G**.

2. **Iteration:**

   Repeat the following process:

- Find an arc $\mathbf{u_i} \in \mathbf{U} \setminus \mathbf{U_T}$ such that adding $\mathbf{u_i}$ to $\mathbf{U_T}$ does not create a cycle in the partial graph $(\mathbf{X}, \mathbf{U_T} \cup \{\mathbf{u_i}\})$.
- Add $\mathbf{u_i}$ to $\mathbf{U_T}$.

3. **Termination:**

   When no further arc can be added without forming a cycle, stop.

   The resulting partial graph $\mathbf{T} = (\mathbf{X}, \mathbf{U_T})$ is a spanning tree of **G**.

Example:

Consider the connected graph $\mathbf{G} = (\mathbf{X}, \mathbf{U})$:

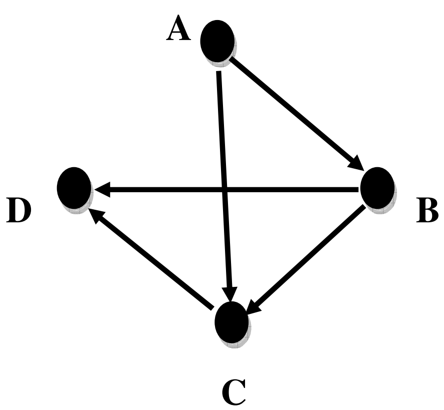


Figure 13: A connected graph.

Choose any arc to star: Let $\mathbf{u}_0 = (\mathbf{A},\mathbf{B})$, then $\mathbf{U}_T = \{(\mathbf{A},\mathbf{B})\}$.

Add a new arc that does not form a cycle: Choose $\mathbf{u}_1 = (\mathbf{A},\mathbf{C})$.

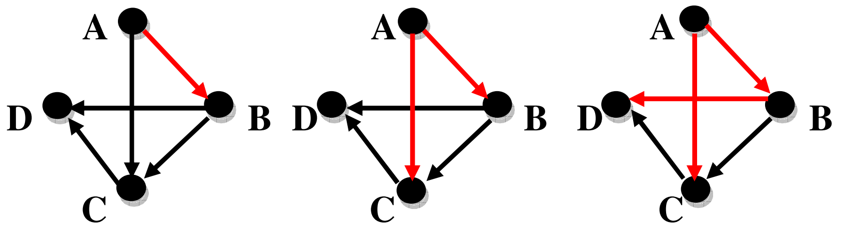
Adding $(\mathbf{A},\mathbf{C})$ (does not form a cycle). Now $\mathbf{U}_T = \{(\mathbf{A},\mathbf{B}),(\mathbf{A},\mathbf{C})\}$.

Continue adding: **Choose** $\mathbf{u}_2 = (\mathbf{B},\mathbf{D})$. Adding $(\mathbf{B},\mathbf{D})$ (does not form a cycle). Now $\mathbf{U}_T = \{(\mathbf{A},\mathbf{B}),(\mathbf{A},\mathbf{C}),(\mathbf{B},\mathbf{D})\}$.

Stop when no more edges can be added without creating a cycle**:** Remaining edges: $(\mathbf{B},\mathbf{C})$ and $(\mathbf{C},\mathbf{D})$.

Adding either of them creates a cycle. So the process stops.

Result: The spanning tree is $\mathbf{T} = (\mathbf{X},\{(\mathbf{A},\mathbf{B}),(\mathbf{A},\mathbf{C}),(\mathbf{B},\mathbf{D})\})$.



$\mathbf{U}_T = \{(\mathbf{A},\mathbf{B})\}$ $\quad$ $\mathbf{U}_T = \{(\mathbf{A},\mathbf{B}),(\mathbf{A},\mathbf{C})\}$ $\quad$ $\mathbf{U}_T = \{(\mathbf{A},\mathbf{B}),(\mathbf{A},\mathbf{C}),(\mathbf{B},\mathbf{D})\}$

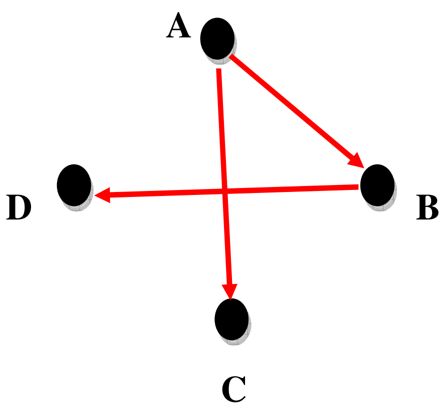


Figure 14: A spanning tree.

<u>Cotree:</u>

Given a connected graph $\mathbf{G} = (\mathbf{X},\ \mathbf{U})$ and one of its spanning trees $\mathbf{T} = (\mathbf{X},\ \mathbf{U_T})$. The **cotree** associated with **T** is the partial graph $\mathbf{T^*} = (\mathbf{X},\ \mathbf{U_{T^*}})$ where $\mathbf{U_{T^*}} = \mathbf{U} \setminus \mathbf{U_T}$ is the set of edges of **G** not belonging to the spanning tree.

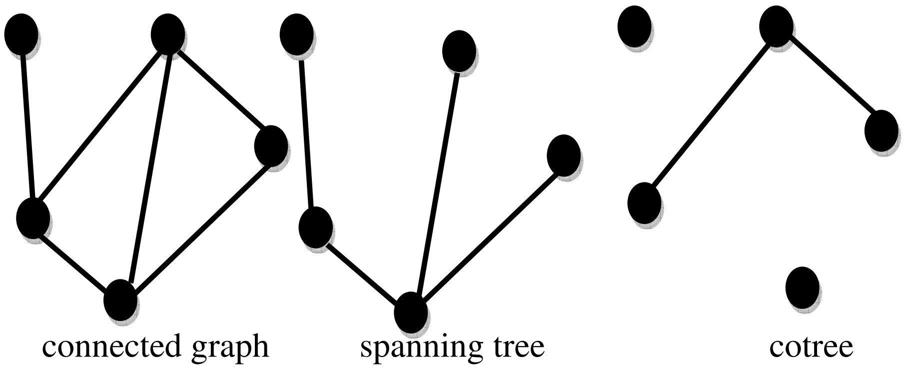


Figure 15: A cotree.