Djilali Bounaama University,
Faculty of FSMI Department
of Mathematics

First Year Master in
Mathematics
Differential Equations
2025–2026

Final Exam

Exercise 01:(05 pts)

Let f and g be two C^1 functions on $\mathbb{R}^2$. We consider the following differential system

$$\begin{cases} x^{'}(t) = f(x(t), y(t)) \\ y^{'}(t) = g(x(t), y(t)). \end{cases} \quad (1)$$

Assume that this system admits an equilibrium $(x^*, y^*) \in \mathbb{R}^2$, and denote by A the Jacobian matrix of the system.

1. State, without proof, the properties relating the eigenvalues of the matrix A to the stability of the equilibrium (x^*, y^*) of system (1).

2. Let α and β be two real parameters, and let C be the square matrix defined by

   $$C = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

   Consider the following linear differential system

   $$X^{'}(t) = CX(t) \quad (2)$$

   Give an example of values of (α, β) for which the equilibrium $0_{\mathbb{R}^2} = (0, 0)^T$ is:

   a stable node, a stable focus, an unstable focus, and a center.

Exercise 02:(06 pts)

Let $A \in M_d(\mathbb{R})$ be a given square matrix.

1. Recall the necessary and sufficient condition on A for the equilibrium $x^* = 0$ to be asymptotically stable for the differential equation $x^{'} = Ax$.

2. Let P be a continuous matrix-valued function.

(a) Prove that, for any initial condition x_0, the Cauchy problem

$$\begin{cases} x'(t) = (A + P(t))x \\ x(0) = x_0 \end{cases} \tag{3}$$

admits a unique global solution.

(b) Prove that the solution x satisfies

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}P(s)x(s)\,ds, \quad \forall t \geq 0.$$

3. Assume that $\|e^{tA}\| \leq C_1 e^{-\lambda t}$ for all $t \geq 0$, with $\lambda > 0$, and that $\int_0^t \|P(t)\| < +\infty$.

   Prove that the function $\Psi(t) = e^{\lambda t}\|x(t)\|$ satisfies the inequality

$$\Psi(t) \leq C_1\|x_0\|e^{C_1 \int_0^t \|P(s)\|ds}, \quad \forall t \geq 0.$$

Exercise 03:(04 pts)

Consider the following differential system:

$$\begin{cases} (1+t^2)x' - tx - y = 2t^2 - 1 \\ (1+t^2)y' + x - ty = 3t. \end{cases} \tag{4}$$

1. Show that $\begin{pmatrix} 1 \\ -t \end{pmatrix}$ and $\begin{pmatrix} t \\ 1 \end{pmatrix}$ are two linearly independent solutions of

$$\begin{cases} (1+t^2)x' - tx - y = 0 \\ (1+t^2)y' + x - ty = 0. \end{cases} \tag{5}$$

2. Determine the resolvent matrix and deduce the solution of the homogeneous system.

3. Find a particular solution of system (4).

Exercise 04:(05 pts)

For $\theta \in \mathbb{R}$, define

$$M(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

1. Verify that $M_\theta^2 = I_2$, where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   is the identity matrix.

Djilali Bounaama University
Final Exam Solutions 2025-2026 (Semester 1) M1 Mathematics

Dr F. Chita

I. (05 points) **Exercise 01:** Let f, g be two C^1 functions on $\mathbb{R}^2$. Consider the following differential system:

$$\begin{cases} x_1'(t) = f(x_1(t), x_2(t)) \\ x_2'(t) = g(x_1(t), x_2(t)) \end{cases} \qquad (1) \tag{1}$$

Assume that this system admits an equilibrium $(x_1^*, x_2^*) \in \mathbb{R}^2$. Let A denote the Jacobian matrix of this system.

(1) State, without proof, the properties linking the eigenvalues of matrix A and the stability of the equilibrium (x_1^*, x_2^*) for system (1).

Answer: Let λ_1 and λ_2 be the eigenvalues of A (in $\mathbb{C}$). We have the following results:

1. If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, then the equilibrium (x_1^*, x_2^*) is **asymptotically stable** for system (1). 0.5 pt
2. If $Re(\lambda_1) > 0$ or $Re(\lambda_2) > 0$, then the equilibrium (x_1^*, x_2^*) is **unstable** for system (1). 0.5 pt
3. If $Re(\lambda_1) \leq 0$ and $Re(\lambda_2) = 0$ (or $Re(\lambda_2) \leq 0$ or $\lambda_1 = 0$), we cannot conclude the stability of (x_1^*, x_2^*) for system (1). 0.5 pt

(2) Let α and β be two real parameters. Let A be the square matrix defined by

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Consider the differential system:

$$X'(t) = AX(t) \quad \ldots (2) \tag{2}$$

Give an example of (α, β) for which the equilibrium $O_{\mathbb{R}^2} = (0,0)^T$ is:

Answer:

a) A stable node

For $\alpha = -1$ and $\beta = 0$, $A = -I_2$, $O_{\mathbb{R}^2}$ is a **stable node** for (2). 0.5 pt

b) A stable focus
For $\alpha = -1$ and $\beta = 1$,

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_{1,2} = -1 \pm i.$$

They have a non-zero imaginary part and a strictly negative real part, so $O_{\mathbb{R}^2}$ is a **stable focus**. [1 pt]

c) An unstable focus
For $\alpha = 1$ and $\beta = 1$,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_{1,2} = 1 \pm i.$$

They have a non-zero imaginary part and a strictly positive real part, so $O_{\mathbb{R}^2}$ is an **unstable focus**. [1 pt]

d) A center
For $\alpha = 0$ and $\beta = 1$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_{1,2} = \pm i.$$

They are purely imaginary, so $O_{\mathbb{R}^2}$ is a **center**. [1 pt]

II. (06pts points) **Exercise 02:** Let $A \in M_n(\mathbb{R})$ be a given square matrix.

(1) Recall the necessary and sufficient condition on A for the equilibrium $x^* = 0$ to be asymptotically stable for the equation $x' = Ax$.

Answer: The condition seen in class is that all eigenvalues of A must have a **strictly negative** real part. [0.5 pt]

(2) Let P be a continuous matrix-valued function.
a) Prove that for any initial condition x_0, the Cauchy problem

$$\begin{cases} x'(t) = (A + P(t))x, \\ x(0) = x_0 \end{cases}$$

admits a unique global solution.

Answer: This is a linear differential equation, so the global Cauchy-Lipschitz theorem applies, ensuring the existence and uniqueness of a global solution for any associated Cauchy problem. [1 pt]

(3) b) Given the initial condition x_0, show that x satisfies

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}P(s)x(s)\,ds, \quad \forall t \geq 0.$$

Answer: Let $b(t) = P(t)x(t)$. Then x satisfies

$$x'(t) = Ax + b(t).$$

Multiply by e^{-At}:

$$e^{-At}x' - e^{-At}Ax = e^{-At}b \quad \text{or equivalently} \quad \frac{d}{dt}(e^{-At}x(t)) = e^{-At}b(t).$$

Integrate from t_0 to t:

$$\int_{t_0}^t \frac{d}{ds}(e^{-As}x(s))ds = \int_{t_0}^t e^{-As}b(s)ds.$$

This gives

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^t e^{-As}b(s)ds.$$

Multiplying both sides by e^{At} and taking $t_0 = 0$ gives the integrated formula:

$$x(t) = e^{tA}x_0 + \int_0^t e^{A(t-s)}b(s)ds. \boxed{\text{2 pts}}$$

(4) Suppose that

$$\|e^{tA}\| \leq C_1 e^{-\lambda t}, \quad \lambda > 0, \ \forall t \geq 0, \quad \text{and} \quad \int_0^t \|P(s)\|ds < +\infty.$$

Define the function $\psi(t) := e^{\lambda t}\|x(t)\|$. Show that it satisfies the inequality

$$\psi(t) \leq C_1\|x_0\|\, e^{C_1 \int_0^{+\infty} \|P(s)\|ds}, \quad \forall t \geq 0.$$

Answer: For $t \geq 0$, taking the norm in the integral formula:

$$\|x(t)\| \leq \|e^{tA}\|\,\|x_0\| + \int_0^t \|e^{(t-s)A}\| \cdot \|P(s)\|\,\|x(s)\|ds.$$

Using the bound on $\|e^{tA}\|$, we get:

$$\|x(t)\| \leq C_1 e^{-\lambda t}\|x_0\| + C_1 \int_0^t e^{-\lambda(t-s)}\|P(s)\|\,\|x(s)\|ds.$$

Multiplying both sides by $e^{\lambda t}$, we have:

$$\psi(t) \leq C_1\|x_0\| + C_1 \int_0^t \|P(s)\|\,\psi(s)ds.$$

Applying Gronwall's lemma gives:

$$\psi(t) \leq C_1\|x_0\| \exp\left(C_1\int_0^t \|P(s)\|ds\right) \leq C_1\|x_0\| \exp\left(C_1\int_0^{+\infty} \|P(s)\|ds\right). \boxed{2 \text{ pts}}$$

III. (04 points) **Exercise 03:** Consider the following differential system

$$\begin{cases}(1+t^2)x' - tx - y = 2t^2 - 1, \\ (1+t^2)y' + x - ty = 3t\end{cases} \qquad (3)$$

(1) Show that

$$X_1 = \begin{pmatrix}1 \\ -t\end{pmatrix}, \quad X_2 = \begin{pmatrix}t \\ 1\end{pmatrix}$$

are two linearly independent solutions of the homogeneous system

$$\begin{cases}(1+t^2)x' - tx - y = 0, \\ (1+t^2)y' + x - ty = 0.\end{cases}$$

Answer: System (3) can be written in the form $X' = A(t)X + B(t)$, with

$$A(t) = \begin{pmatrix}\frac{t}{1+t^2} & \frac{1}{1+t^2} \\ -\frac{1}{1+t^2} & \frac{t}{1+t^2}\end{pmatrix}, \quad B(t) = \begin{pmatrix}\frac{2t^2-1}{1+t^2} \\ \frac{3t}{1+t^2}\end{pmatrix}.$$

We check that X_1 and X_2 are solutions of the homogeneous system:

- $X_1' - AX_1 = \begin{pmatrix}0 \\ -1\end{pmatrix} - \begin{pmatrix}\frac{t}{1+t^2} & \frac{1}{1+t^2} \\ -\frac{1}{1+t^2} & \frac{t}{1+t^2}\end{pmatrix}\begin{pmatrix}1 \\ -t\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$ $\boxed{0.5 \text{ pt}}$
- $X_2' - AX_2 = \begin{pmatrix}0 \\ 1\end{pmatrix} - \begin{pmatrix}\frac{t}{1+t^2} & \frac{1}{1+t^2} \\ -\frac{1}{1+t^2} & \frac{t}{1+t^2}\end{pmatrix}\begin{pmatrix}t \\ 1\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$ $\boxed{0.5 \text{ pt}}$
- The Wronskian:

$$W(X_1, X_2) = \det\begin{pmatrix}1 & t \\ -t & 1\end{pmatrix} = 1 + t^2 \neq 0, \quad \forall t \in \mathbb{R}. \boxed{0.5 \text{ pt}}$$

(2) Determine the resolvent matrix and deduce the solution of the homogeneous system.

Answer: a) The fundamental matrix is

$$M(t) = \begin{pmatrix}1 & t \\ -t & 1\end{pmatrix}.$$

b) The resolvent matrix is

$$R(t, t_0) = M(t)M(t_0)^{-1}, \quad \text{with} \quad M(t_0)^{-1} = \frac{1}{1+t_0^2}\begin{pmatrix}1 & -t_0 \\ t_0 & 1\end{pmatrix}. \boxed{0.5 \text{ pt}}$$

Thus

$$R(t,t_0) = \frac{1}{1+t_0^2}\begin{pmatrix} 1+tt_0 & -t_0+t \\ -t+t_0 & 1+tt_0 \end{pmatrix}. \boxed{0.5 \text{ pt}}$$

The homogeneous solution is then:

$$X_H(t) = R(t,t_0)C = \frac{1}{1+t_0^2}\begin{pmatrix} C_1 - t_0C_2 + t(t_0C_1 + C_2) \\ -t(C_1 - t_0C_2) + C_2 + t_0C_1 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \boxed{0.5 \text{ pt}}$$

(3) Find a particular solution of (3).

Answer: A particular solution is

$$X_P(t) = \int_{t_0}^{t} R(t,s)B(s)\,ds.$$

After calculation, we obtain

$$X_P(t) = \begin{pmatrix} t^3 - t - tt_0^2 + t_0 \\ 2t^2 - t_0^2 + tt_0 \end{pmatrix}. \boxed{0.5 \text{ pt}}$$

IV. (05 points) **Exercise 04:**

(1) **Answer:** We have

$$\begin{aligned} M_\theta^2 &= M_\theta \cdot M_\theta \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. \qquad (0.1 \text{ pt}) \end{aligned}$$

(2) **Answer:** 2.1) Show that $\forall n \in \mathbb{N},\ M_\theta^{2n} = I_2$.
For $n = 0$, we have $M_\theta^0 = I_2$.
Assume for a fixed n that $M_\theta^{2n} = I_2$, then

$$M_\theta^{2(n+1)} = M_\theta^{2n} \cdot M_\theta^2 = I_2 \cdot I_2 = I_2.$$

Hence, $\forall n \in \mathbb{N},\ M_\theta^{2n} = I_2$. (0.1 pt)
2.2) Show that $\forall n \in \mathbb{N},\ M_\theta^{2n+1} = M_\theta$.
For $n = 0$, we have $M_\theta^{2\cdot 0+1} = M_\theta$.
Assume for a fixed n that $M_\theta^{2n+1} = M_\theta$, then

$$\begin{aligned} M_\theta^{2n+3} &= M_\theta^{2n+1} \cdot M_\theta^2 \\ &= M_\theta \cdot I_2 = M_\theta. \end{aligned}$$

Hence, $\forall n \in \mathbb{N},\ M_\theta^{2n+1} = M_\theta.$ (0.1 pt)

(3) **Answer:** Let $t \in \mathbb{R}$. We have

$$\begin{aligned}
e^{tM_\theta} &= \sum_{n=0}^{+\infty} \frac{(tM_\theta)^n}{n!} = \sum_{n=0}^{+\infty} \frac{t^n M_\theta^n}{n!} \\
&= \sum_{n=0}^{+\infty} \frac{t^{2n} M_\theta^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{t^{2n+1} M_\theta^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{+\infty} \frac{t^{2n} I_2}{(2n)!} + \sum_{n=0}^{+\infty} \frac{t^{2n+1} M_\theta}{(2n+1)!} \\
&= \left(\sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}\right) I_2 + \left(\sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!}\right) M_\theta \\
&= \cosh t \cdot I_2 + \sinh t \cdot M_\theta. \qquad \text{(0.2 pt)}
\end{aligned}$$