Exercise Series 02

- **Exercise 01:** Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Determine all tangent vectors to Ω

- **Exercise 02:**

Let $\Omega \subset \mathbb{R}^2$ and $\mathcal{E} \in \Omega$. Calculate $T_x(\mathcal{E})$ in the following cases:

Case 1: $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y = x\}$

Let $x \in \Omega$ be arbitrary.

Case 2: $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

- **Exercise 03:**

We consider the ball $\mathbb{B}$ centered at $(0, 0)$ with radius 1, i.e.:

$$\mathbb{B} = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

1. Compute $T_{\mathbb{B}}((x_0, y_0))$ in the case $x_0^2 + y_0^2 < 1$. (Recall that $T_{\mathbb{B}}((x_0, y_0))$ is the tangent cone to $\mathbb{B}$ at (x_0, y_0).)

2. Compute $T_{\mathbb{B}}((x_0, y_0))$ in the case $x_0^2 + y_0^2 = 1$.

3. Using Nagumo's theorem, give a necessary and sufficient condition for the Cauchy problem

$$(y_1', y_2') = f(y_1, y_2); \quad (y_1(t_0), y_2(t_0)) = (a, b) \in \mathbb{B}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(y_1, y_2) = (y_1, 0),$$

to have at least one local solution.

Solution to Exercise 1 We want to determine all vectors v tangent to Ω at a point x_0.

A vector $v = (v_1, v_2)$ is tangent to Ω at x_0 if:

$$\lim_{h \to 0^+} \frac{\text{dist}(x_0 + hv, \Omega)}{h} = 0.$$

Here, $x_0 + hv = (1 + hv_1, hv_2)$. The distance between $(1 + hv_1, hv_2)$ and the circle is:

$$\text{dist}((1 + hv_1, hv_2), \Omega) = \left| \sqrt{(1 + hv_1)^2 + (hv_2)^2} - 1 \right|.$$

Hence:

$$\frac{1}{h}\text{dist}(x_0 + hv, \Omega) = \frac{\sqrt{(1 + hv_1)^2 + (hv_2)^2} - 1}{h}.$$

Taking the limit as $h \to 0$, we get:

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}(x_0 + hv, \Omega) = v_1 = 0.$$

Thus, all tangent vectors at the point $(1, 0)$ are of the form $v = (0, v_2)$.

Solution to Exercise 2

Reminder: A vector $v \in T_x(\mathcal{E})$ if and only if:

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}(\mathcal{E} + hv, \Omega) = 0.$$

Let $\mathcal{E} = (\epsilon_1, \epsilon_2) \in \Omega$, so $\epsilon_1 = \epsilon_2$. Let $v = (v_1, v_2) \in T_{\mathcal{E}}(\Omega)$.

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}((\epsilon_1 + hv_1, \epsilon_2 + hv_2), \Omega) = 0$$

The distance from a point (a, b) to the line $y = x$ is:

$$\text{dist}((a, b), \Omega) = \frac{|a - b|}{\sqrt{2}}.$$

So:

$$\text{dist}(\mathcal{E} + hv, \Omega) = \frac{|(\epsilon_1 + hv_1) - (\epsilon_2 + hv_2)|}{\sqrt{2}} = \frac{|h(v_1 - v_2)|}{\sqrt{2}}.$$

Then:

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}(\mathcal{E} + hv, \Omega) = \frac{|v_1 - v_2|}{\sqrt{2}} = 0 \implies v_1 = v_2.$$

Thus:

$$T_{\mathcal{E}}(\Omega) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 = v_2\}.$$

Case 2: $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Let $\mathcal{E} = (0, 0)$ be the origin. Let $v = (v_1, v_2) \in T_{\mathcal{E}}(\Omega)$.

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}(\mathcal{E} + hv, \Omega) = 0$$

Here, $\mathcal{E} + hv = (hv_1, hv_2)$. The distance from (hv_1, hv_2) to the circle is:

$$\text{dist}((hv_1, hv_2), \Omega) = \left| \sqrt{(hv_1)^2 + (hv_2)^2} - 1 \right| = |h\sqrt{v_1^2 + v_2^2} - 1|.$$

Then:

$$\frac{1}{h}\text{dist}((hv_1, hv_2), \Omega) = \frac{|h\sqrt{v_1^2+v_2^2}-1|}{h} = \frac{|1-h\sqrt{v_1^2+v_2^2}|}{h} \to \infty \text{ as } h \to 0^+.$$

This limit is never zero for any nonzero vector v.

Conclusion:

$$T_{(0,0)}(\Omega) = \{(0,0)\}.$$

Solution to Exercise 03

We consider the unit ball

$$\mathbb{B} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}.$$

1. **Case $x_0^2 + y_0^2 < 1$ (interior point)**

   If (x_0, y_0) is strictly inside the ball, then the tangent cone at (x_0, y_0) is the whole space $\mathbb{R}^2$, since we can move in any direction without leaving the ball. Hence,

   $$T_{\mathbb{B}}((x_0,y_0)) = \mathbb{R}^2, \quad \text{if } x_0^2+y_0^2 < 1.$$

2. **Case $x_0^2 + y_0^2 = 1$ (boundary point)**

   If (x_0, y_0) is on the boundary (the unit circle), the tangent cone consists of all vectors pointing *inwards or tangent to the circle*, i.e., not pointing outside. For a convex set, the tangent cone at a boundary point (x_0, y_0) is given by

   $$T_{\mathbb{B}}((x_0,y_0)) = \{(v_1,v_2) \in \mathbb{R}^2 \mid x_0v_1 + y_0v_2 \le 0\}.$$

3. **Nagumo's theorem and the Cauchy problem**

   Consider the Cauchy problem

   $$(y_1', y_2') = f(y_1,y_2) = (y_1, 0), \quad (y_1(t_0), y_2(t_0)) = (a,b) \in \mathbb{B}.$$

   According to **Nagumo's theorem** (Theorem **??**), a necessary and sufficient condition for the existence of a local solution that remains in $\mathbb{B}$ is

   $$f(y_1,y_2) \in T_{\mathbb{B}}(y_1,y_2), \quad \forall (y_1,y_2) \in \partial\mathbb{B}.$$

   At a boundary point (x_0, y_0), we have

   $$f(x_0,y_0) = (x_0, 0) \in T_{\mathbb{B}}((x_0,y_0)) \implies x_0^2 + 0\cdot y_0 \le 0 \implies x_0 = 0.$$

   Therefore, the necessary and sufficient condition for a solution to remain inside $\mathbb{B}$ is

   $$(a,b) \in \mathbb{B}, \quad \text{and if } a^2+b^2 = 1 \text{ (boundary), then } a = 0.$$

$$\text{Summary:} \quad \begin{cases} T_{\mathbb{B}}((x_0, y_0)) = \mathbb{R}^2, & \text{if } x_0^2 + y_0^2 < 1, \\ T_{\mathbb{B}}((x_0, y_0)) = \{(v_1, v_2) \mid x_0 v_1 + y_0 v_2 \leq 0\}, & \text{if } x_0^2 + y_0^2 = 1, \\ \text{Nagumo condition: } (a, b) \in \mathbb{B}, & \text{and if } a^2 + b^2 = 1, \text{ then } a = 0. \end{cases}$$