Exercise Series 03

- **Exercise 01:**

Find the equilibrium points of the following differential equation

$$\frac{dx}{dt} = \sin x$$

- **Exercise 02:**

Consider the following function

$$V(x,y) = x^2 + y^2$$

Use two methods to study the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -x + y \\ \dfrac{dy}{dt} = -x - y \end{cases} \qquad (1)$$

- **Exercise 03:**

Consider the following system

$$\begin{cases} \dfrac{dx}{dt} = -y^3 \\ \dfrac{dy}{dt} = x \end{cases} \qquad (2)$$

1. Can we study the stability of System (2) using the linearization method?

2. Consider the function $V(x,y) = 2x^2 + y^4$. Study the stability of System (2).

- **Exercise 04:**

1. Cite three methods for studying the stability of nonlinear autonomous systems.

2. Let $a \in \mathbb{R}_+^*$. Study the stability of the system

$$Y' = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} Y.$$

3. Use the Lyapunov function defined on $\mathbb{R}^2$ by $V(x,y) = x^2 + y^2$ to study the stability of the system

$$\begin{cases} x' = y - x(x^2 + y^2) \\ y' = -x - y(x^2 + y^2) \end{cases}$$

- **Exercise 05:**

Consider the system

$$\begin{cases} x' = x - x^2 + 2y \\ y' = -2xy + 2x \end{cases} \qquad (E)$$

1. Study the nature of the origin of system (E).

Correction of the Series

- **Exercise 01:**

The equilibria satisfy $\sin(x) = 0$.

Thus, there exists an infinite number of equilibria:

$$x_k^* = k\pi, \quad k \in \mathbb{Z}.$$

- **Exercise 02:**

Consider the function $V(x,y) = x^2 + y^2$. Use two methods to study the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -x + y \\ \dfrac{dy}{dt} = -x - y \end{cases} \qquad (1)$$

Solution: Lyapunov Method

- First, $V(0,0) = 0$.

- $V(x,y) = x^2 + y^2 > 0$, for all $(x,y) \neq (0,0)$.

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$$\frac{dV}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x(-x+y) + 2y(-x-y)$$
$$= -2x^2 + 2xy - 2xy - 2y^2 = -2(x^2+y^2) < 0.$$

Therefore, the system is asymptotically stable.

- **Exercise 03:**

Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -y^3 = f_1(x,y) \\ \dfrac{dy}{dt} = x = f_2(x,y) \end{cases} \tag{2}$$

1. Can we study the stability of System (2) using linearization?

2. Consider the function $V(x,y) = 2x^2 + y^4$. Study the stability of System (2).

   We have

$$Df(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -3y^2 \\ 1 & 0 \end{pmatrix}.$$

   Thus,

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   The eigenvalues satisfy $\lambda_1 = \lambda_2 = 0$.

   Since both eigenvalues of $Df(0,0)$ are zero, the linearized system gives no conclusion.

   Hence, we cannot study the stability of System (2) using the linearization method.

   On the other hand,

$$V(0,0) = 0, \qquad V(x,y) = 2x^2 + y^4 > 0 \quad \text{for } (x,y) \neq (0,0).$$

   Moreover,

$$V'(x,y) = 4x\frac{dx}{dt} + 4y^3\frac{dy}{dt} = 4x(-y^3) + 4y^3(x) = 0 \leq 0.$$

   Therefore, the system is stable.

- **Exercise 04:**

1. The methods are: the definition of stability, the linearization method, and the Lyapunov function method.

2. The eigenvalues of

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

are $\lambda_1 = a$ and $\lambda_2 = -a$.

Since one eigenvalue has strictly positive real part, the system is unstable.

3. Study of V:

(a) $V(0,0) = 0$ and $V(x,y) = x^2 + y^2 > 0$ for $(x,y) \neq (0,0)$.

(b)

$$\begin{aligned} V'(x,y) &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= 2x(y - x(x^2+y^2)) + 2y(-x - y(x^2+y^2)) \\ &= -2(x^2+y^2)^2 < 0. \end{aligned}$$

Thus, the system is asymptotically stable.

- **Exercise 05:**

The matrix of the linearized system of (E) near the origin is

$$\begin{aligned} A &= Df(0,0) \\ &= \begin{pmatrix} \frac{\partial}{\partial x}(x - x^2 + 2y) & \frac{\partial}{\partial y}(x - x^2 + 2y) \\ \frac{\partial}{\partial x}(-2xy + 2x) & \frac{\partial}{\partial y}(-2xy + 2x) \end{pmatrix}_{(0,0)} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$