Inverse Trigonometric Functions

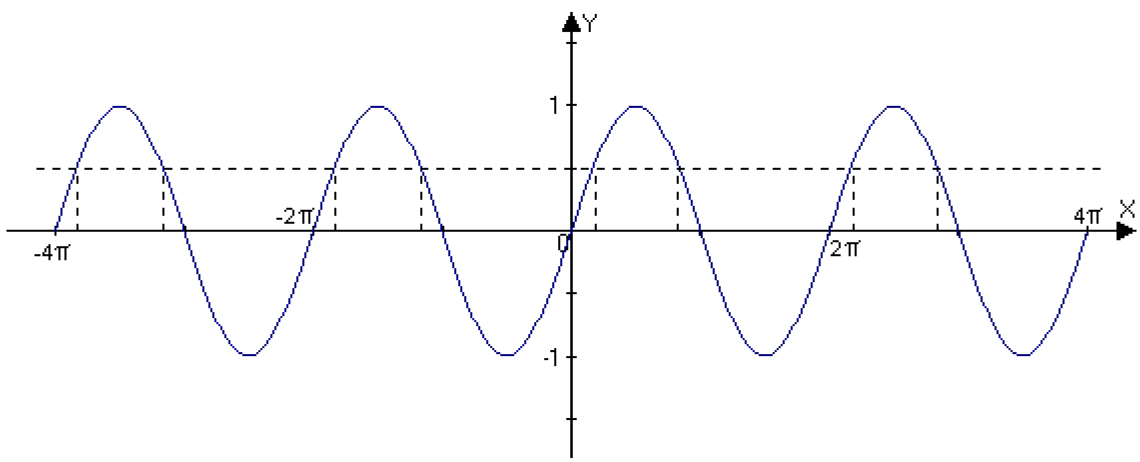
Review

First, let's review briefly inverse functions before getting into inverse trigonometric functions:

- $f \rightarrow f^{-1}$ is the inverse
- The range of f = the domain of f^{-1}, the inverse.
- The domain of f = the range of f^{-1} the inverse.
- $y = f(x) \rightarrow$ x in the domain of f.
- $x = f^{-1}(y) \rightarrow$ y in the domain of f^{-1}
- $f[f^{-1}(y)] = y \rightarrow$ y in the domain of f^{-1}
- $f^{-1}[f(x)] = x \rightarrow$ x in the domain of f

Trigonometry Without Restrictions

- Trigonometric functions are periodic, therefore each range value is within the limitless domain values (no breaks in between).



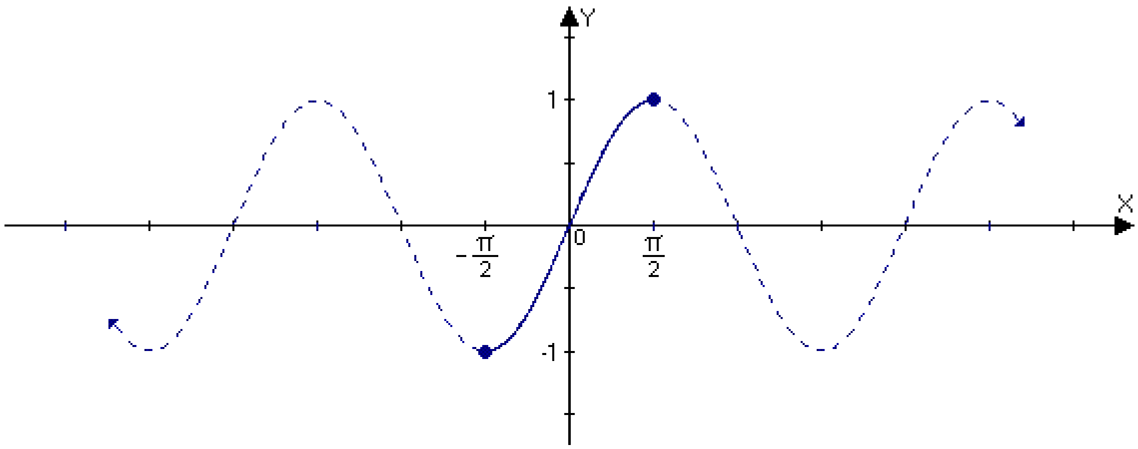
- Since trigonometric functions have no restrictions, there is no inverse.

- With that in mind, in order to have an inverse function for trigonometry, we restrict the domain of each function, so that it is one to one.

- A restricted domain gives an inverse function because the graph is one to one and able to pass the horizontal line test.

Trigonometry With Restrictions

- How to restrict a domain:
  - Restrict the domain of the sine function, y = sin x, so that it is one to one, and not infinite by setting an interval [-π/2, π/2]



  - The restricted sine function passes the horizontal line test, therefore it is one to one
  - Each range value (-1 to 1) is within the limited domain (-π/2, π/2).

- The restricted sine function benefits the analysis of the inverse sine function.

Inverse Sine Function

- $\sin^{-1}$ or arcsin is the inverse of the restricted sine function, y = sin x, [-π/2, π/2]

- The equations → $y = \sin^{-1} x$ or y = arcsin x

  which also means, sin y = x, where $-\pi/2 \leq y \leq \pi/2$, $-1 \leq x \leq 1$ (remember *f* range is f^{-1} domain and vice versa).

Restricted Sine vs. Inverse Sine

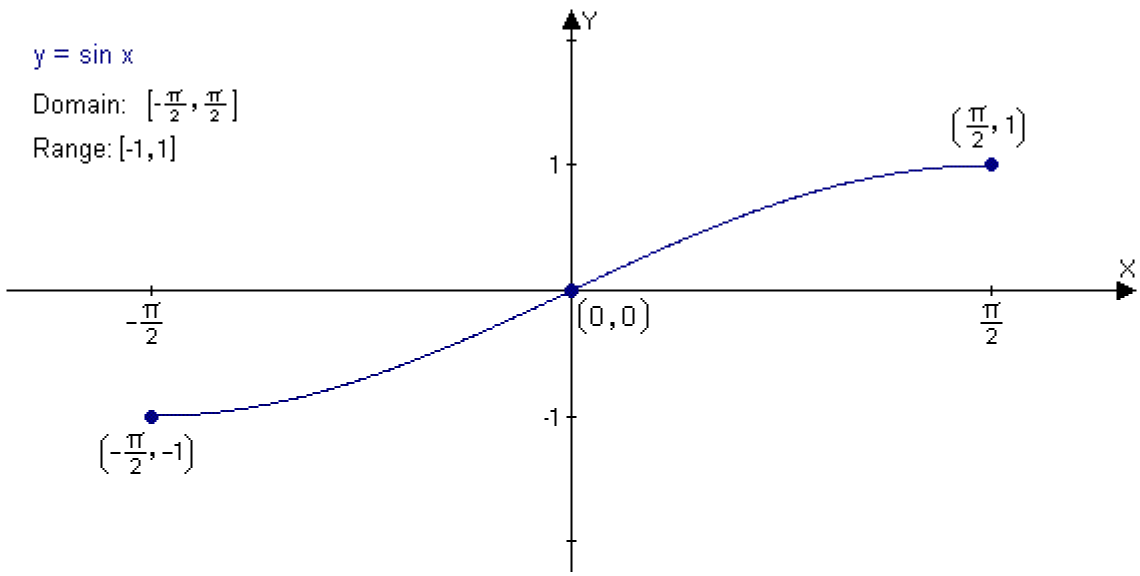
- As we established before, to have an inverse trigonometric function, first we need a restricted function.

- Once we have the restricted function, we take the points of the graph (range, domain, and origin), then switch the y's with the x's.

Restricted Sine vs. Inverse Sine Continued …

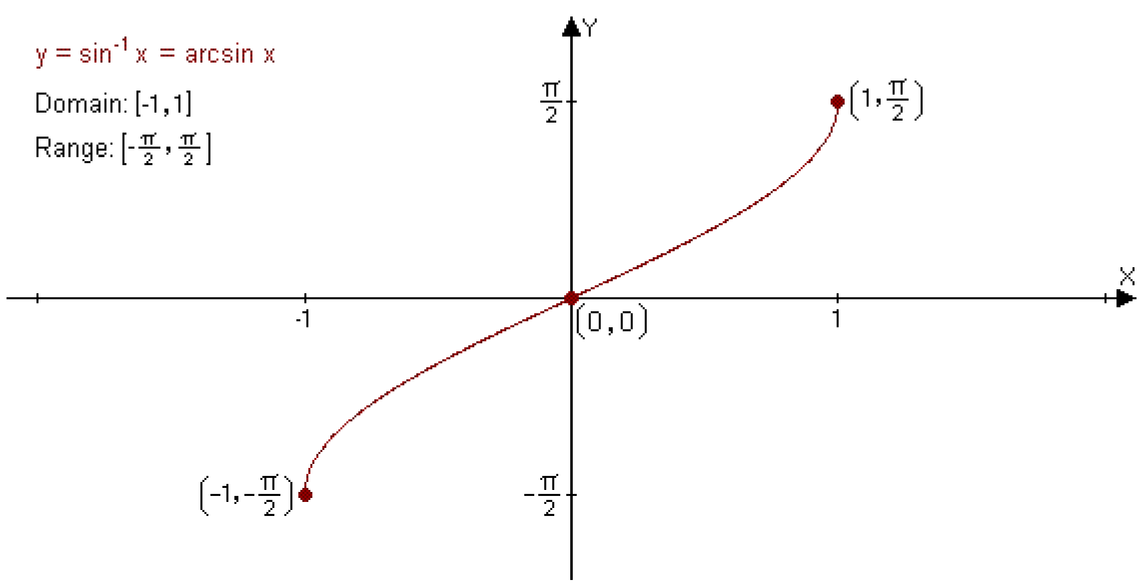
- For example:
  - These are the coordinates for the restricted sine function.

    (- π/2, -1), (0, 0), (π/2, 1)



  - Reverse the order by switching x with y to achieve an inverse sine function.

    (-1, - π/2), (0, 0), (1, π/2)



Sine-Inverse Sine Identities

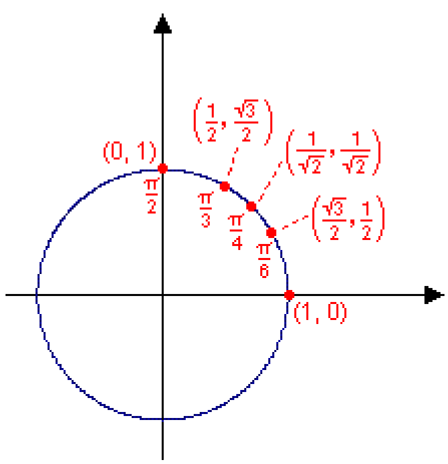
- $\sin(\sin^{-1} x) = x$, where $-1 < x < 1$
  - Example: $\sin(\sin^{-1} 0.5) = 0.5$
    $\sin(\sin^{-1} 1.5) \neq 1.5$
    (not within the interval or domain of the inverse sine function)

- $\sin^{-1}(\sin x) = x$, where $-\pi/2 \leq x \leq \pi/2$
  - Example: $\sin^{-1}[\sin(-1.5)] = -1.5$
    $\sin^{-1}[\sin(-2)] \neq -2$
    (not within the interval or domain of the restricted sine function)

Without Calculator

- To attain the value of an inverse trigonometric function without using the calculator requires the knowledge of the Circular Points Coordinates, found in Chapter 5, the Wrapping Function section.

- Here is quadrant I of the Unit Circle



- The Unit Circle figure shows the coordinates of Key Circular Points.

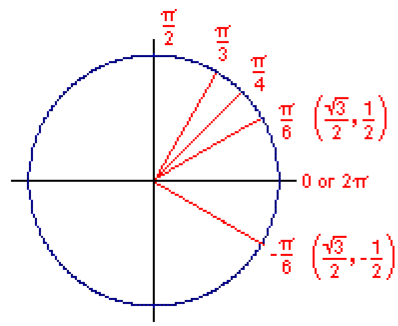
- These coordinates assist with the finding of the exact value of an inverse trigonometric function.

Without Calculator

Example 1: Find the value for → $\sin^{-1}(-1/2)$

Answer:

- $\sin^{-1}(-1/2)$, is the same as $\sin y = -1/2$, where $-\pi/2 \le y \le \pi/2$



- Since the figure displays a mirror image of π/6 on the IV quadrant, the answer is:
  $y =$ **$-\pi/6$** $= \sin^{-1}(-1/2)$

- Although $\sin(11\pi/6) = -1/2$, y must be within the interval $[-\pi/2, \pi/2]$.

- Consequently, $y = -\pi/6$, which is between the interval, meets the conditions for the inverse sine function.

With Calculator

- There are different types of brands on calculators, so read the instructions in the user's manual.

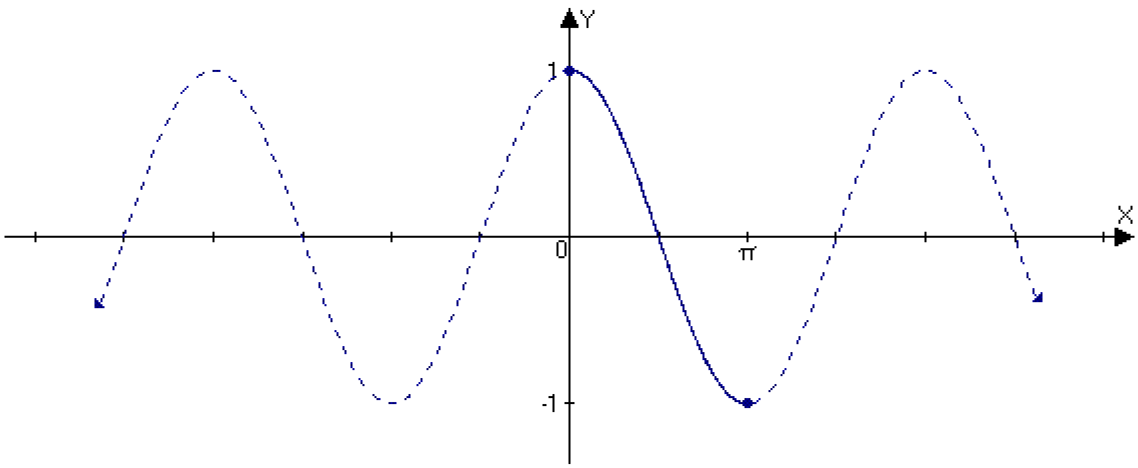
- Make sure to set the calculator on radian mode.

- If the calculator displays an error, then the values or digits used are not within the domain of the trigonometry function

  – For example:
  If you punch in $\sin^{-1}(1.548)$ on your calculator, the device will state that there is an error because 1.548 is not within the domain of $\sin^{-1}$.

Restrict Cosine Function

- The restriction of a cosine function is similar to the restriction of a sine function.
- The intervals are [0, π] because within this interval the graph passes the horizontal line test.
- Each range goes through once as *x* moves from 0 to π.



Inverse Cosine Function

- Once we have the restricted function, we are able to proceed with defining the inverse cosine function, $\cos^{-1}$ or arccos.
- The inverse of the restricted cosine function y= cos *x*, $0 \leq x \leq \pi$, is y= $\cos^{-1} x$ and y = arccos *x*.
- Which also means, cos y = *x*, where $0 \leq y \leq \pi$, $-1 < x < 1$ (Remember, the domain of *f* is the range of f^{-1}, and vice versa).

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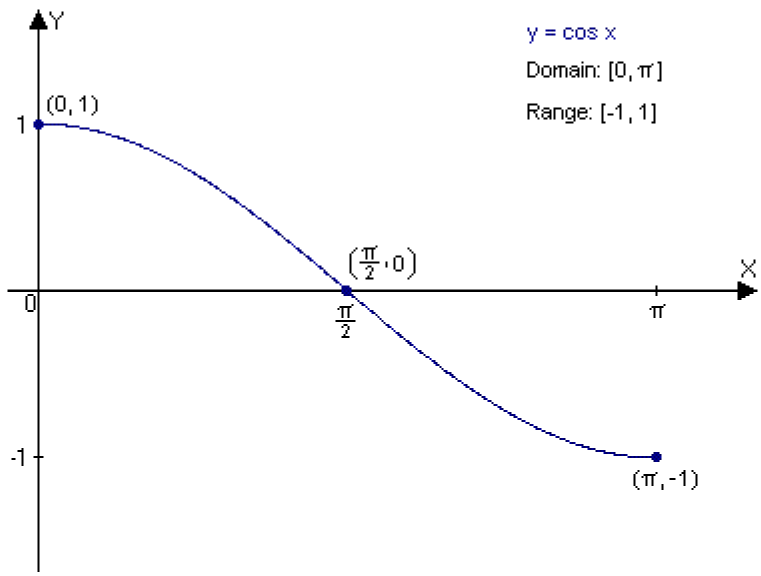
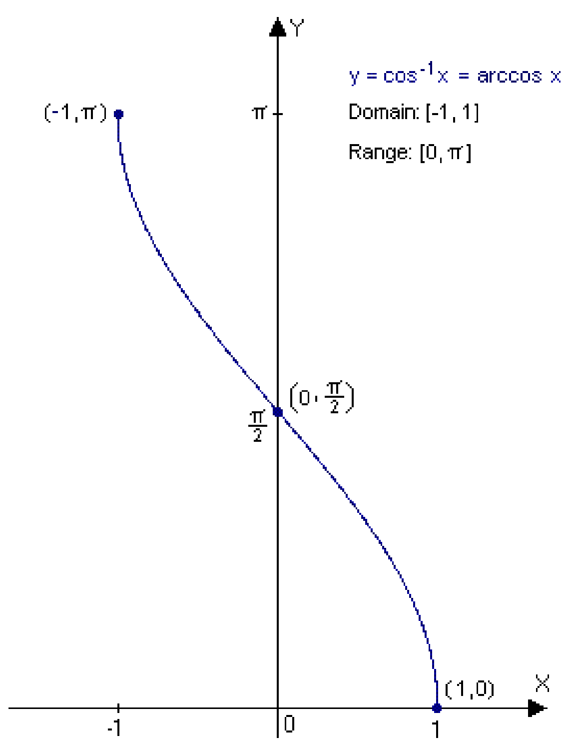
Restricted Cosine vs. Inverse Cosine

- The restricted cosine function has the domain, range, and x-intercept coordinates: (0,1) (π/2, 0) (π, -1)



- The inverse cosine function switched the coordinates of the restricted function, x is now y, and y is now x: (1, 0) (0, π/2) (-1, π)



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Cosine-Inverse Cosine Identities

- $\cos(\cos^{-1} x) = x$, where $-1 \le x \le 1$
  - Example: $\cos(\cos^{-1} 0.5) = 0.5$
    $\cos(\cos^{-1} 1.5) \ne 1.5$
    (not within the interval or domain of the inverse cosine function)

- $\cos^{-1}(\cos x) = x$, where $0 \le x \le \pi$
  - Example: $\cos^{-1}[\cos(0.5)] = 0.5$
    $\cos^{-1}[\cos(-2)] \ne -2$
    (not within the interval or domain of the restricted cosine function)

Cosine Inverse Solving Without Calculator:

Example 2: $\cos(\cos^{-1} 0.6)$

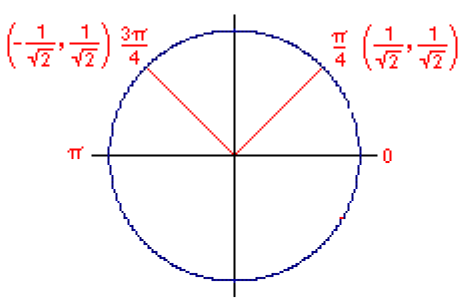
Answer:

Since $-1 \le 0.6 \le 1$, then $\cos(\cos^{-1} 0.6) =$ **0.6** because the form is following the cosine-inverse cosine identities.

Example 3: $\arccos(-1/\sqrt{2})$

Answer:

- $\arccos(-1/\sqrt{2})$, is the same as $\cos y = -1/\sqrt{2}$, where $0 < y < \pi$.



- Due to the fact, that the figure displays a mirror image of $\pi/4$ on the II quadrant, ($3\pi/4$), the **answer** is $\mathbf{y = 3\pi/4} = \arccos(-1/\sqrt{2})$.

- Even though $\cos(-3\pi/4) = -1/\sqrt{2}$, $y \ne -3\pi/4$. The y must be within the interval $[0, \pi]$.

Solving Cosine Inverse With Calculator

- There are different types of brands on calculators, so read the instructions in the user's manual.

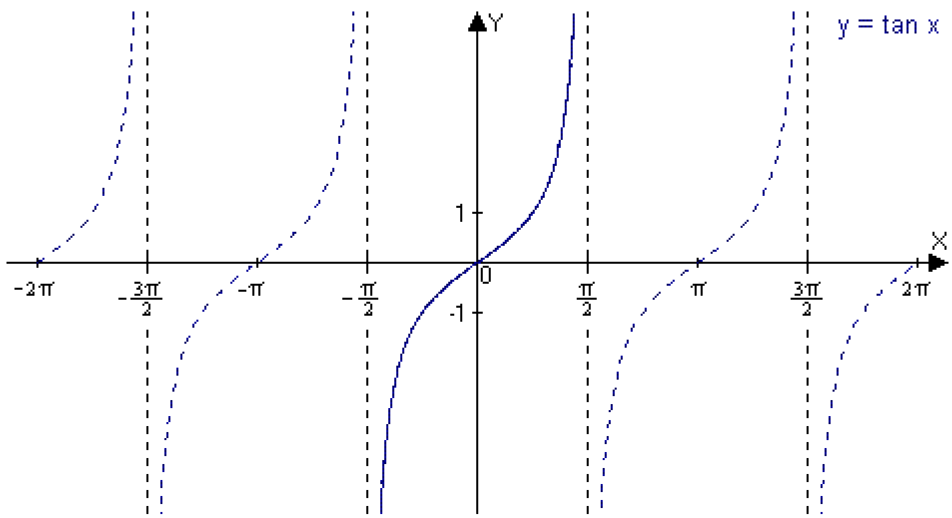
- Make sure to set the calculator on radian mode.

- If the calculator displays an error, then the values or digits used are not within the domain of the trigonometry function

  - For example:
    If you punch in $\cos^{-1}$ (1.238) on your calculator, the device will state that there is an error because 1.238 is not within the domain of $\cos^{-1}$.

Restriction of Tangent Function

- To become a one-to-one function, we choose the interval (-π/2, -π/2), thus a restricted function is formed.



- The restricted tangent function passes the horizontal line test.

- Each range value (y) is given exactly once as x proceeds across the restricted domain.

- Now, that we have the function restricted we will use it to formulize the inverse tangent function.

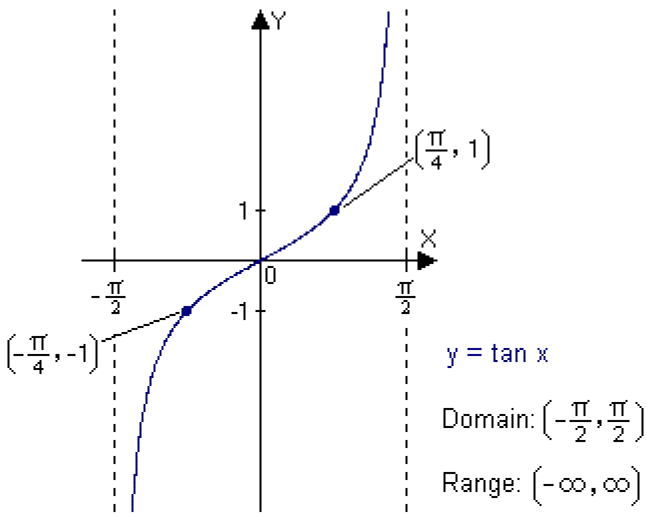
Inverse Tangent Function

- Signified by tan -1 or arctan → y= tan -1 or y= arctan x

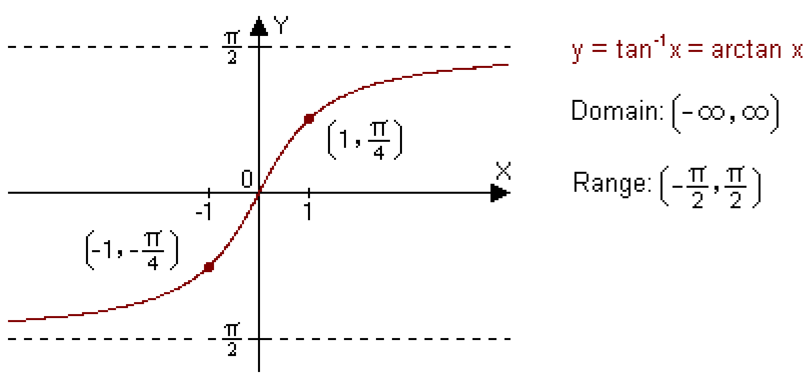
- The definition, undifferentiated to sine and cosine, is the inverse of the restricted tan function (y= tan x), in the interval -π/2 ≤ x ≤ π/2

- The inverse is equivalent to tan y= x, where -π/2 ≤ y ≤ π/2

- Here is the graph of restricted tangent function



- Here is the graph of inverse tangent function



- The coordinates on the restricted function (- π/4, -1), (0, 0), and (π/4, 1) are reversed on the inverse function.

- The vertical asymptotes on the restricted function become horizontal on the inverse.

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Tangent-Inverse Tangent Identities

- $\tan(\tan^{-1} x) = x$, where $-\infty < x < \infty$
  - Example: $\tan(\tan^{-1} 2) = 2$
    $\tan(\tan^{-1} -1.5) = -1.5$

- $\tan^{-1}(\tan x) = x$, where $-\pi/2 < x < \pi/2$

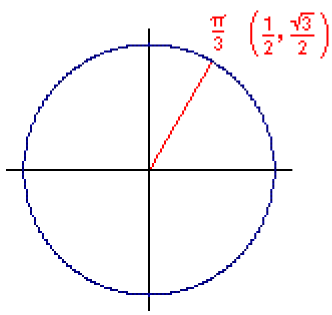
  $\tan^{-1}[\tan(-0.5)] = -0.5$
  $\tan^{-1}[\tan(-2)] \neq -2$
  (not within the interval or domain of the restricted tangent function)

Solving Inverse Tangent Problem Without Calculator

Example 4: $y = \tan^{-1}(\sqrt{3})$

Answer:

- $\tan^{-1}(\sqrt{3})$, is the same as $\tan y = \sqrt{3}$, where $-\pi/2 < y < \pi/2$. Therefore, $y = \pi/3 = \tan^{-1}(\sqrt{3})$:



- Since $\tan x = b/a = \sqrt{3}/2 \div \frac{1}{2} = \sqrt{3}/2 \times 2/1 = \sqrt{3}/2$, then the **answer** to $\tan^{-1}(\sqrt{3}) = y = \boldsymbol{\pi/3}$

Example 5: $\tan[\tan^{-1}(56)]$

Answer:

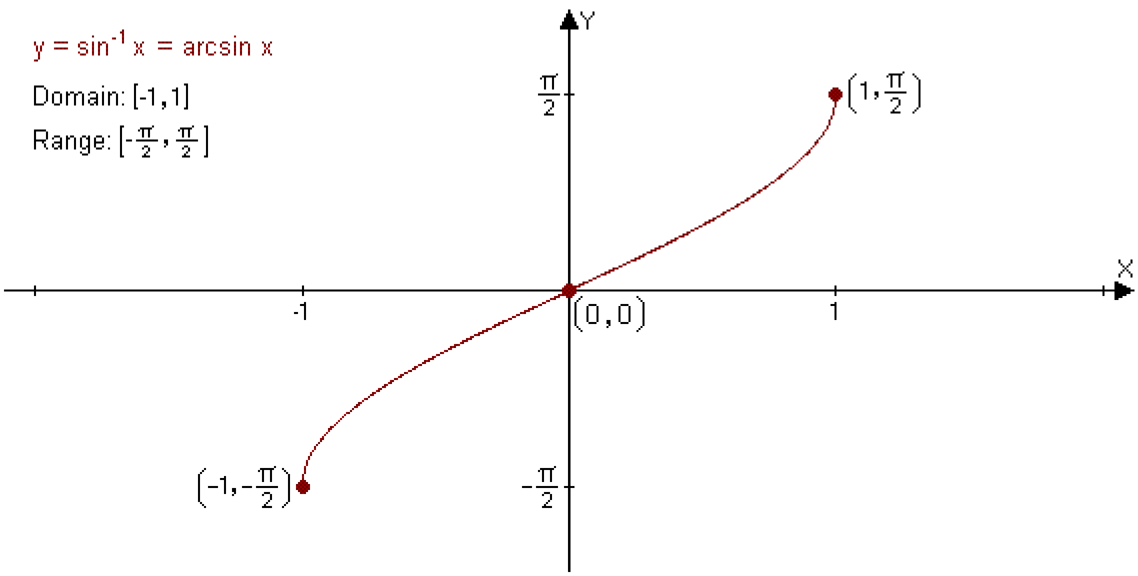
- According to the Tangent-Inverse Tangent Identities, $\tan(\tan^{-1} x) = x$, where $-\infty < x < \infty$. Consequently, any number x will equal number x because the domain is infinite, no limits.

- So, the **answer**: $\tan[\tan^{-1}(56)] = \mathbf{56}$

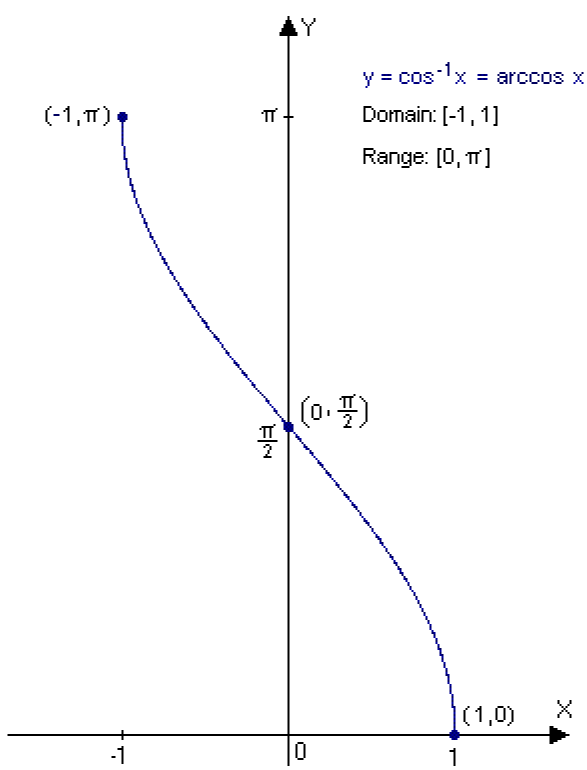
Summary

Let us summarize all the different inverse trigonometric functions.

- $y = \sin^{-1} x \rightarrow x = \sin y$, where $-1 < x < 1$, and $-\pi/2 < y < \pi/2$



- $y = \cos^{-1} x \rightarrow x = \cos y$, where $-1< x <1$, and $0 < y < \pi$



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Summary Continued …

- $y = \tan^{-1} x \rightarrow x = \tan y$, where $-\infty < x < \infty$, and $-\pi/2 < y < \pi/2$