Chapter 1

Sets, Relations and Maps

1.1 Basic Concepts of Set Theory.

1.1.1 Sets and elements

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. The notion of set is taken as “undefined”, “primitive”, or “basic”, so we don’t try to define what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Définition 1.1.1. *A **set** is a collection of objects together with some rule to determine whether a given object belongs to this collection. Any object of this collection is called an **element**. of the set.*

• Sets are usually denoted by capital letters and the members by lower case letters. We usually write all elements in curly brackets. The notation

$$A = \{a, b, c\}$$

means that the set A consists of 3 elements: a, b and c. We can say that the element a belongs to the set A, write $a \in A$, or that d is not a member of A, write $d \notin A$. Sets can be finite or infinite.

• Sets can be finite or infinite.

• There is exactly one set, the **empty set**, or null set, which has no members at all. The symbol ϕ represents the empty set.

• A set with only one member is called a singleton or a singleton set. (“Singleton of a”)

1.1.2 Specification of sets

There are three main ways to specify a set:

(1) by listing all its members (list notation);

(2) by stating a property of its elements (predicate notation);

(3) by defining a set of rules which generates (defines) its members (recursive rules).

(1) List notation: The first way is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces:

Examples:$\{1, 12, 45\}$, $\{$ George Washington, Bill Clinton$\}$, $\{a, b, d, m\}$.

"Three-dot abbreviation":$\{1, 2, \cdots, 100\}$.

$\{1, 2, 3, 4, \cdots, \}$ – this is not a real list notation, it is not a finite list, but it's common practice as long as the continuation is clear.

Note that we do not care about the order of elements of the list, and elements can be listed several times.$\{1, 12, 45\}$, $\{12, 1, 45, 1\}$ and $\{45, 12, 45, 1\}$ are different representations of the same set (see below the notion of identity of sets). List notation. The first way is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces: Examples: 1, 12, 45, George Washington, Bill Clinton, a,b,d,m. "Three-dot abbreviation": 1,2, ..., 100. (See xeroxed "preliminaries", pp xxii-xxiii) 1,2,3,4,... – this is not a real list notation, it is not a finite list, but it's common practice as long as the continuation is clear. Note that we do not care about the order of elements of the list, and elements can be listed several times. 1, 12, 45, 12, 1, 45,1 and 45,12, 45,1 are different representations of the same set (see below the notion of identity of sets).

Predicate notation. Example

$\{x \mid x$ is a natural number and $x < 8\}$

Reading:" the set of all x such that x is a natural number and is less than 8".

So the second part of this notation is a property the members of the set share (a condition or a predicate which holds for members of this set).

Other examples:

$\{x \mid x$ is a letter of Russian alphabet $\}$

$\{y \mid y$ is a student of UDBKM and y is older than $25\}$.

General form:

$\{x \mid P(x)\}$, where P is some predicate (condition, property).

The language to describe these predicates is not usually fixed in a strict way.

Recursive rules: (Always safe.) Example – the set E of even numbers greater than 3:

a) $4 \in E$

b) if $x \in E$, then $x + 2 \in E$

c) nothing else belongs to E.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules a and b. (The third rule should always be there; sometimes in practice it is left implicit. It's best when you're a beginner to make it explicit.)

1.1.3 Subsets and power sets

Définition 1.1.2. *If A is a **subset** of B (write $A \subseteq B$), then all elements of A are also elements of B;*

$$(A \subseteq B) \Longleftrightarrow (\forall x, x \in A \Longrightarrow x \in B)$$

. So A is "contained" in B.

If you want to say that A is **NOT** subset of B, write mathematically $A \nsubseteq B$.

Exemple 1.1. *$\{a,b\} \subseteq \{d,a,b,e\}$ and $\{a,b\} \subset \{d,a,b,e\}$, $\{a,b\} \subseteq \{a,b\}$ but $\{a,b\} \not\subset \{a,b\}$*

Notice that the empty set is a subset of any set.
To show that $A \subseteq B$, you need to show that every element of A is also an element of B.

Exercice 1.1. *Let $E = \left]1, \frac{5}{2}\right]$, $F = [-5,4]$. Prove that $E \subseteq F$.*

Proof. Start by choosing an arbitrary element $x \in E$,then

$$\begin{aligned} x \in E &\Longrightarrow 1 < x \leq \frac{5}{2} \\ &\Longrightarrow -5 \leq 1 < x \leq \frac{5}{2} < 3 \\ &\Longrightarrow x \in F. \end{aligned}$$

□

If A is a subset of B but it they are not equal, then we say that A is **a proper subset** of B and write it $A \subset B$ (or $A \subsetneq B$ or $A \underset{\neq}{\subseteq} B$).
To show that A is a proper subset of B, you need to show that $A \subseteq b$ and find at least one element of B which is not an element of A.

Exemple 1.2.

$$\{a,b,c\} \subset \{a,b,c,d\},$$

$$\{1,2,3\} \not\subset \{1,2,3\} \text{ but } \{1,2,3\} \subseteq \{1,2,3\}$$

.

Définition 1.1.3. *The **power set** of a set A consists of all subsets of A and is usually denoted by $\mathcal{P}(A)$ (some writers use 2^A).*

Exemple 1.3. *if $A = \{a,b\}, \mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$.*

From the example above: $a \in A$; $\{a\} \subseteq A$; $\{a\} \in \mathcal{P}(A)$.
$\phi \notin A$; $\phi \subseteq A$; $\phi \in \mathcal{P}(A); \phi \subseteq \mathcal{P}(A)$.

1.1.4 Cardinality and equality

Définition 1.1.4. *In mathematics, the **cardinality** of a set A ($card(A) or \mid A \mid$) is a measure of the " number of the elements of the set".*

Exemple 1.4. *If $\{a,b,c\}$, then $\mid A \mid = 3$*

Exemple 1.5. *Important to notice:*

$$\mid \{\phi\} \mid = 1, \quad \textit{while} \quad \mid \phi \mid = 0.$$

$$\mid \{0\} \mid \neq 0, \quad \textit{but} \quad \mid \{0\} \mid = 1.$$

Notice that the repetitions are ignored when we are counting the members of the set. The convention is to list each element only once,the same number can be weritten in different forms.

$$F = \{2, -1, 0, 1, 2, \cos\pi, \cos\frac{\pi}{2}\} = \{-1, 0, 1, 2\}, \quad \text{as } \cos\pi = -1, \cos\frac{\pi}{2} = 0, \text{Hence}, \mid F \mid = 4.$$

Définition 1.1.5. *Two sets A and B are* **equal** *when they have exactly the same elements, i.e. every element of A is an element of B and every element of B is an element of A. So*

$$A = B \Longleftrightarrow (A \subseteq B \wedge B \subseteq A).$$

To show that two sets A and B are equal, pick an arbitrary $x \in A$ and show that $x \in B$ and vice versa.

Exemple 1.6. *Let $A = \{1, 2, 3, 4\}$ and $B = \{x : x \in \mathbb{N}, x^2 < 17\}$, where $\mathbb{N}$ is the set of natural numbers. Show that $A = B$.*

Proof. To prove the equality of the sets, we must show that for every x,

$$x \in B \Longrightarrow x \in A \quad (B \subseteq A)$$

and

$$x \in A \Longrightarrow x \in B \quad (A \subseteq B).$$

So if $x \in B$, then $x^2 < 17$, which implies $x < \sqrt{17}$. Therefore $x \leq 4$. Since x is a positive integer, therefore for every $x \in B$ we have that $0 < x \leq 4$. Hence, $x \in B \Longrightarrow x \in A$.

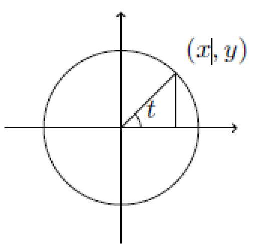
Now assume $x \in A$, so $x \in \{1, 2, 3, 4\}$. To prove that $x \in B$, it suffices to show that the largest element $x \in A$ satisfier $x^2 < 17$. Then it is also true for the smaller values since they all belong to $\mathbb{N}$.
Since $\forall x \in A; x^2 \leq 4^2 \leq 16 < 17$, we have that $x \in A \Longrightarrow x \in B$ □

Exercice 1.2. *Show that $\{(\cos t, \sin t) : t \in \mathbb{R}\} = \{(x, y) : x^2 + y^2 = 1\}$.*

Proof. Let $A = \{(\cos t, \sin t) : t \in \mathbb{R}\}$ and $B = \{(x, y) : x^2 + y^2 = 1\}$. Now, to show that $A = B$ we need to show that $A \subseteq B$ and $B \subseteq A$.
Let $x = \cos t$ and $y = \sin t$. Then

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

because $\cos^2 t + \sin^2 t = 1$ is a known identity. Hence we have that $A \subseteq B$.
Now, to show that $B \subseteq A$ we appeal to geometry. Let $(x, y) \in B$, hence $x^2 + y^2 = 1$. So (x, y) lies on the unit circle.



Therefore, we have that $\cos t = x$ and $\sin t = y$. As $x^2 + y^2 = 1$, substituting in for x and y gives

$$\cos^2 t + \sin^2 t = 1$$

and hence we have shown that $B \subseteq A$ and so $A = B$.

□

1.1.5 Common sets of numbers

The commonly used sets of numbers are:

- The set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$;
- The set of integers,$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$;
- The set of rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$;
- The set of real numbers $\mathbb{R}$, which is the union of both rational $\mathbb{Q}$ and irrational numbers (which cannot be expressed as a fraction, for example $\log 2$; $\sqrt{2}$; π; e).
- the complex numbers $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Notice that one set is a subset of another, in the following order: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

1.1.6 SET OPERATIONS

a. **Intersection**

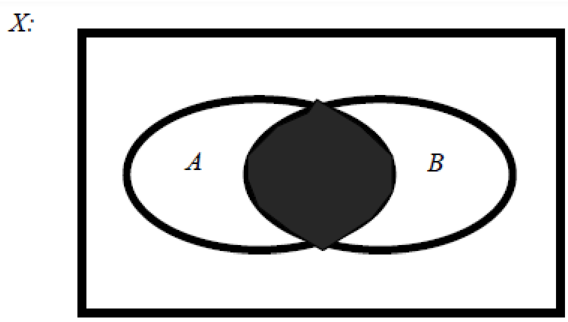
Définition 1.1.6. *Let A and B be subsets of a set X. The intersection of A and B is the set of all elements in X common to both A and B.*

Notation: *"$A \cap B$" denotes "A intersection B" or the intersection of sets A and B.*

Thus, $A \cap B := \{x \in X \mid (x \in A)$ and $(x \in B)\}$, or $A \cap B := \{x \in X \mid (x \in A) \wedge (x \in B)\}$,

Exemple 1.7.

(a) Given that the box below represents X, the shaded area represents $A \cap B$:



(b) Let $A = \{2, 4, 5\}$ and $B = \{1, 4, 6, 8\}$. Then, $A \cap B = \{4\}$.

(c) Let $A = \{2, 4, 5\}$ and $B = \{1, 3\}$. Then, $A \cap B = \phi$

Remark 1.1.1. *If, as in the above example, A and B are two sets such that $A \cap B$ is the empty set, we say that A and B are* ***disjoint.***

b. **Union**

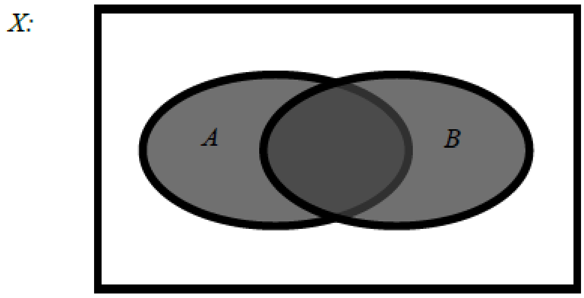
Définition 1.1.7. *Let A and B be subsets of a set X. The **union** of A and B is the set of all elements belonging to A or B.*

Notation:"$A \cup B$" denotes " A union$B$" or the union of sets A and B.

Thus,$A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$, or $A \cup B := \{x \in X \mid x \in A \wedge x \in B\}$.

Exemple 1.8.

(a) Given that the box below represents X, the shaded area represents $A \cup B$:



(b) Let $A = \{2, 4, 5\}$ and $B = \{1, 4, 6, 8\}$. Then, $A \cup B = \{1, 2, 4, 5, 6, 8\}$.

c. **Difference**

Définition 1.1.8. *Let A and B be subsets of a set X. The set $B - A$ (or $B \setminus A$), called the **difference** of B and A, is the set of all elements in B which are not in$A$.*

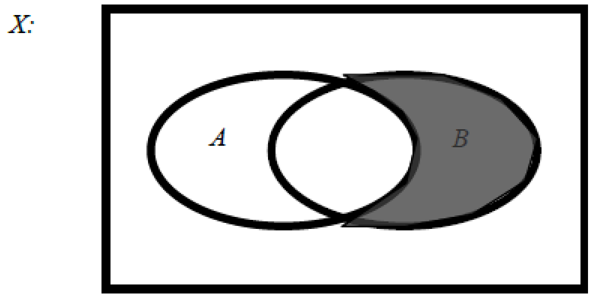
Thus, $B - A = \{x \in X \mid x \in B$ and $x \notin A\}$.

Notation: symmetric difference of A and B is given by

$$A \triangle B := (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Exemple 1.9.

(a) Given that the box below represents X, the shaded area represents $B - A$.



(b) Let $B = \{2, 3, 6, 10, 13, 15\}$ and $A = \{2, 10, 15, 21, 22\}$. Then $B - A = \{3, 6, 13\}$.

d. **complement**

Définition 1.1.9. *If $A \subset X$, then $X - A$ is sometimes called the **complement** of A with respect to X.*

Notation: The following symbols are used to denote the complement of A with respect to X:

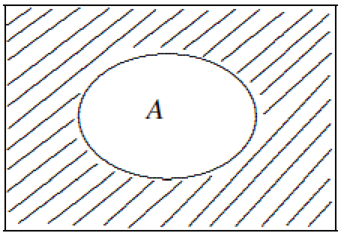
$C_X A, C_X^A, A^c, \overline{A}$.

Thus, $A^c = \{x \in X \mid x \notin A\}$.

Exemple 1.10.

(a) Given that the box below represents X, the shaded area represents A^c.

X :



(b) Let $X = \{1, 2, \cdots, 10\}$ be the universal set, $A = \{2, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{4, 8, 10\}$. Then

$C_X^A = \overline{A} = \{1, 4, 6, 8, 9, 10\}$, $C_X^B = \overline{B} = \{1, 3, 5, 7, 9\}$ *and* $C_X^C = \overline{C} = \{1, 2, 3, 5, 6, 7, 9\}$.

$C_B^C = \{2, 6\}$ *and* C_A^C *is not defined because* $C \not\subset A$.

e. **Cartesian Product**

Définition 1.1.10. *The **Cartesian product** (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Remark 1.1.2. *We've already mentioned that when a set A is described by listing its elements, the order in which the elements of A are listed doesn't matter. That is, if the set A consists of two elements x and y, then $A = \{x, y\} = \{y, x\}$. When we speak of the ordered pair (x, y), however, this is another story. The ordered pair (x, y) is a **single element** consisting of a pair of elements in which x is the first element (or first coordinate) of the ordered pair (x, y) and y is the second element (or second coordinate). Moreover, for two ordered pairs (x, y) and (w, z) to be equal, that is, $(x, y) = (w, z)$, we must have $x = w$ and $y = z$. So, if $x \neq y$, then $(x, y) \neq (y, x)$.*

Exemple 1.11. *If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then*

$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$,

while

$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$.

Since, for example, $(x, 1) \in A \times B$ and $(x, 1) \neq B \times A$, these two sets do not contain the same elements; so $A \times B \neq B \times A$.Also, $\mid A \mid= 2$ and $\mid B \mid= 3$; while $\mid A \times B \mid= 6$. Indeed, for all finite sets A and B,

$$\mid A \times B \mid=\mid A \mid \cdot \mid B \mid .$$

1.1.7 Laws of the algebra of sets

Théorème 1.1.11. *Let X be an arbitrary set and let $P(X)$ the power set of X. Let A, B, and C be arbitrary elements of $P(X)$ then*

Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Identity laws	$A \cup \phi = A$ $A \cup X = X$	$A \cap \phi = \phi$ $A \cap X = A$
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Complement laws	$A \cup A^c = X$ $X^c = \phi$ $(A^c)^c = \overline{\overline{A}} = A$	$A \cap A^c = \phi$ $\phi^c = X$ *If* $A \subset B$ *then* $B^c \subset A^c$
De Morgan's laws	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$

Exercice 1.3.

1.2 Introduction to Relations

Sometimes it is necessary not to look at the full Cartesian product of two sets A and B, but rather at a subset of the Cartesian product. This leads to the following

1.2.1 Binary relations

Définition 1.2.1. *A binary relation $\mathcal{R}$ from a set A to a set B is a subset $R \subseteq A \times B$.*

In other words, a binary relation from A to B is a set $\mathcal{R}$ of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\mathcal{R}b$ to denote that $(a, b) \in \mathcal{R}$ and $a \not\mathcal{R}$ to denote that $(a, b) \notin \mathcal{R}$. Moreover, when (a, b) belongs to $\mathcal{R}$, a is said to be related to b by $\mathcal{R}$.

Binary relations represent relationships between the elements of two sets.

The set of pairs $(a, b) \in A \times B$ which satisfy $(a, b) \in \mathcal{R}$ is called the **graph** of the relation $\mathcal{R}$ we denote it by $G_{\mathcal{R}}$, and we write

$$G_{\mathcal{R}} = \{(a, b) \in A \times B \mid a\mathcal{R}b\}$$

Exemple 1.12. *Let $A = \{0, 1, 2, 3, 5, 8, 10, 16\}, B = \{4, 10, 16, 20, 23, 27\}$. Then $\mathcal{R}\{(x, y) \mid y = 2x\}$ is a relation from A to B and we write*

$$\forall (x, y) \in A \times B, x\mathcal{R}y \Leftrightarrow y = 2x$$

.

This means, for instance, that

$$G_{\mathcal{R}} = \{(x, y) \in A \times B \mid x\mathcal{R}y\} = \{(2; 4), (5; 10), (8; 16), (10, 20)\}$$

$2\mathcal{R}4$, *but that* $3 \not\mathcal{R}4$.

1.2.2 Binary relation on a set

Définition 1.2.2. *A binary relation $\mathcal{R}$ from a set A is a subset of $A \times A$ or from A to A.*

Exemple 1.13. *Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $\mathcal{R} = \{(a, b) \mid a$ divides $b\}$?*

Solution: Because $(a; b)$ is in $\mathcal{R}$ if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$G_{\mathcal{R}} = \{(1; 1), (1; 2), (1; 3), (1; 4), (2; 2), (2; 4), (3; 3), (4; 4)\}.$$

The pairs in this relation are displayed graphically form in Figure 2.

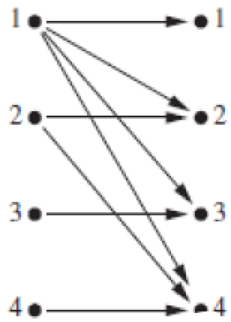


FIGURE 2

1.2.3 Properties of Relations

Let $\mathcal{R}$ be a binary relation defined on A.Then

1) $\mathcal{R}$ is **reflexive** iff $(\forall x \in A, (x\mathcal{R}x))$,

2) $\mathcal{R}$ is **symetric** iff $[\forall x, y \in A, (x\mathcal{R}y) \Longrightarrow (y\mathcal{R}x)]$,

3) $\mathcal{R}$ is **antisymetric** iff $[\forall x, y \in A, ((x\mathcal{R}y) \wedge (y\mathcal{R}x)) \Longrightarrow x = y]$,

4) $\mathcal{R}$ is **transitive** iff $[\forall x, y, z \in A, ((x\mathcal{R}y) \wedge (y\mathcal{R}z)) \Longrightarrow x\mathcal{R}z]$,

Exemple 1.14.

1.2.4 EQUIVALENCE RELATION

Définition 1.2.3. *A relation $\mathcal{R}$ on a set A is called an* ***equivalence relation*** *if it is* ***reflexive, symmetric,*** *and* ***transitive****.*

Exemple 1.15. *We consider the relation* $\mathcal{R}$ *defined on* $\mathbb{Z}$ *by:*

$$\forall x, y \in \mathbb{Z}, x\mathcal{R}y \Leftrightarrow x - y \text{ is divisible by } 3.$$

Solution: We show that $\mathcal{R}$ satisfies the conditions of an equivalence relation.

1) **Reflexive** : $(\forall x \in \mathbb{Z}, (x\mathcal{R}x))$, $\forall x \in \mathbb{Z}$ we have $x - x = 0$ and 0 is divisible by 3. So $x\mathcal{R}x$ is true. Then $\mathcal{R}$ is reflixive.

2) **Symetric** : $[\forall x, y \in \mathbb{Z}, (x\mathcal{R}y) \Longrightarrow (y\mathcal{R}x)]$,

   $\forall x, y \in \mathbb{Z}$ we have

$$\begin{aligned} x\mathcal{R}y &\Rightarrow x - y \text{is divisible by 3} \\ &\Rightarrow \exists k \in \mathbb{Z}, x - y = 3k \\ &\Rightarrow \exists k \in \mathbb{Z}, -(y - x) = 3k \\ &\Rightarrow \exists k \in \mathbb{Z}, (y - x) = -3k = 3k' \text{where} k' = -k \\ &\Rightarrow y\mathcal{R}x \text{is true} \end{aligned}$$

   Then $\mathcal{R}$ is symitric.

3) **Transitive** : $[\forall x, y, z \in A, ((x\mathcal{R}y) \ \wedge \ (y\mathcal{R}z)) \Longrightarrow x\mathcal{R}z]$,

   $\forall x, y \in \mathbb{Z}$ we have $x\mathcal{R}y \Rightarrow \exists k \in \mathbb{Z}, x - y = 3k$................(1)

   $y\mathcal{R}z \Rightarrow \exists k' \in \mathbb{Z}, y - z = 3k'$................(2)

   from (1) and (2) we find

$$x - y + y - z = 3k + 3k' \Rightarrow x - z = 3(k + k') = 3k'' \text{ where } \ k'' = k + k' \in \mathbb{Z}$$

   So $x\mathcal{R}z$ is true, then $\mathcal{R}$ is transitive.

4) Since $\mathcal{R}$ is reflexive, symetric and transitive then $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Exercice 1.4. *We consider the relation* $\mathcal{R}$ *defined on* $\mathbb{R}$ *by:*

$$\forall x, y \in \mathbb{Z}, x\mathcal{R}y \Leftrightarrow xe^y = ye^X.$$

Show that $\mathcal{R}$ *is an equivalence relation on* $\mathbb{R}$*.*

1.2.5 Equivalence classes

Définition 1.2.4. *Let* $\mathcal{R}$ *be an equivalence relation on a set* A*.*

1) The set of all elements that are related to an element a *of* A *is called the* ***equivalence class*** *of* a*, denoted by*

$$\overline{a} = \dot{a} = \{x \in A \mid x\mathcal{R}a\} = \{x \in A \mid a\mathcal{R}x\}$$

2) a *is a* ***representative*** *of equivalence class* $\dot{a}$*.*

3) If $b \in \dot{a}$*, then* b *is a* ***representative*** *of this equivalence class. Any element of a class can be used as representative.*

4) The set of equivalence classes of all the elements of A is called ***set quotient*** *of A by the equivalence relation $\mathcal{R}$, we not it $A_{/\mathcal{R}}$, and we write*

$$A_{/\mathcal{R}} = \{\dot{a}, a \in A\}.$$

Exemple 1.16. *We consider the equivalente relation $\mathcal{R}$ defined on $\mathbb{Z}$ by:*

$$\forall x, y \in \mathbb{Z}, x\mathcal{R}y \Leftrightarrow x - y \text{ is divisible by } 3.$$

1) Determine $\dot{0}, \dot{1}, \dot{2}, \dot{3}, \dot{4}$.

2) Determine $\mathbb{Z}_{/\mathcal{R}}$.

Solution:

1) We give the equivalence classes. $\dot{0}, \dot{1}, \dot{2}, \dot{3}, \dot{4}$.

$$\begin{aligned}\dot{0} &= \{x \in \mathbb{Z} \mid x\mathcal{R}0\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x = 3k\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x - 0 = 3k\},\\ &= \{3k, k \in \mathbb{Z}\},\\ &= \{\cdots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \cdots\}.\end{aligned}$$

$$\begin{aligned}\dot{1} &= \{x \in \mathbb{Z} \mid x\mathcal{R}1\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x - 1 = 3k\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x = 3k + 1\},\\ &= \{3k + 1, k \in \mathbb{Z}\},\\ &= \{\cdots, -13, -10, -7, -4, 1, 4, 7, 10, 13, \cdots\}.\end{aligned}$$

$$\begin{aligned}\dot{2} &= \{x \in \mathbb{Z} \mid x\mathcal{R}2\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x - 2 = 3k\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x = 3k + 2\},\\ &= \{3k + 2, k \in \mathbb{Z}\},\\ &= \{\cdots, -14, -11, -8, -5, 2, 5, 8, 11, 14, \cdots\}.\end{aligned}$$

$$\begin{aligned}\dot{3} &= \{x \in \mathbb{Z} \mid x\mathcal{R}3\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x - 3 = 3k\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x = 3k + 3 = 3(k + 1) = 3k' \text{where} k' = k + 1 \in \mathbb{Z}\},\\ &= \{3k', k' \in \mathbb{Z}\},\\ &= \{\cdots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \cdots\}\\ &= \dot{0}.\end{aligned}$$

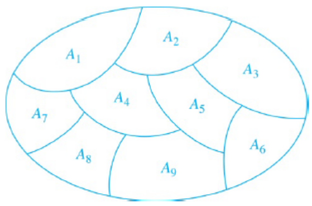
$$\begin{aligned}\dot{4} &= \{x \in \mathbb{Z} \mid x\mathcal{R}4\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x - 4 = 3k\},\\ &= \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}; x = 3k + 4 = 3(k+1) + 1 = 3k' + 1 \text{where} k' = k + 1 \in \mathbb{Z}\},\\ &= \{3k' + 1, k' \in \mathbb{Z}\},\\ &= \{\cdots, -13, -10, -7, -4, 1, 4, 7, 10, 13, \cdots\}\\ &= \dot{1}.\end{aligned}$$

we can see also that $\dot{5} = \dot{2}$.

2) According to question 1) the quotient set is

$$\mathbb{Z}_{/\mathcal{R}} = \{\dot{0}, \dot{1}, \dot{2}\}.$$

Remark 1.2.1. *An equivalence relation $\mathcal{R}$ on A, will divide the set A into an equivalence classes, and they are called* ***portion*** *of X.*



A partition of a set

1.2.6 Order relation

Définition 1.2.5.

1. *A relation $\mathcal{R}$ on a set A is called a* ***order relation*** *iff it is* ***refexive, antisymmetric,*** *and* ***transitive.***
2. *Let $\mathcal{R}$ be an order relation on A and $a, b \in A$. The elements a and b are comparable if either $a\mathcal{R}b$* ***or*** *$b\mathcal{R}a$ holds.*
3. *Let $\mathcal{R}$ be an order relation on A and if any two elements of A are comparable, then $\mathcal{R}$ is called a* ***total order***
4. *If the order is not total it is said to be* ***partial.***

Exemple 1.17. *the relation $\mathcal{R}$ defined on $\mathbb{R}$ by $(\forall x, y \in \mathbb{R}; x\mathcal{R}y \Leftrightarrow x \leq y)$ is a relation of total order on $\mathbb{R}$. We say that all the elements of $\mathbb{R}$ are comparable.*

Exemple 1.18. *Let $\mathcal{R}$ is a relation defined on $\mathbb{N}$ by:*

$$\forall x, y \in \mathbb{N}, x\mathcal{R}y \Leftrightarrow \exists n \in \mathbb{N}; y = nx.$$

1. *Show that $\mathcal{R}$ is an order relation.*
2. *The order is it total?*

Solution:

1.3 Functions

1.3.1 Basic Definitions

Définition 1.3.1. *Let A, B be nonempty sets. A **function** from A to B is a relation from A to B which assigns every element $x \in A$ at most one element in $y \in B$ such that $x\mathcal{R}y$.*
Generally , the fenctions are denoted by $f, g, h, k, \cdots$. and we write:

$$f : A \longrightarrow B$$
$$x \longmapsto y = f(x)$$

Terminology about Functions

Let $f : A \to B$ be a function from A to B.

- A is called the set of **input** of f .
- B is called the set of **output** or **codomain** of f .
- If $f(a) = b$ then b is the **image** of a under f and a is the **preimage** of b.
- The **domain** of f is the set

$$D_f = \{x \in A : \exists y \in B; y = f(x)\}$$

- The set $f(A) := \{f(x) \mid x \in A\}$ is called the **range** of f . (Note the difference between the range and the codomain.)
- Two functions $f : A \to B$ and $g : A' \to B'$ are equal iff $A = A'$, $B = B'$ and $\forall a \in A; f(a) = g(a)$.
- The **graph** of f is the set of ordered pairs

$$G_f = \{(x, y) \in A \times B \mid y = f(x)\}$$

Définition 1.3.2. *We call **application** or **mapping** of a set A into a set A, any correspondence f between the elements of A and those of B which to each $x \in A$ corresponds to a **unique** element $y \in B$ and which satisfies the relation $y = f(x)$.*

Remark 1.3.1. *An application f defined from A to B is a function whene $D_f = A$.*

Exemple 1.19. *the function*
$f : \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x) = \dfrac{1}{x}$
it is a function because 0 *has no image by f, whereas* $f : \mathbb{R}^* \to \mathbb{R}$
$x \mapsto f(x) = \dfrac{1}{x}$
it is an application because $D_F = \mathbb{R}^*$

1.3.2 Injection-Surjection-Bijection

Injections, Surjections, Bijections

Definition

A function $f : A \to B$ is **injective** ("one-to-one") iff $f(a) = f(b) \to a = b$. Then f is called an **injection**.

Definition

A function $f : A \to B$ is **surjective** ("onto") iff $\forall b \in B \exists a \in A\ f(a) = b$. Then f is called a **surjection**.

A function $f : A \to B$ is surjective iff $f(A) = B$, i.e., the range is equal to the codomain.

Definition

A function $f : A \to B$ is **bijective** iff it is injective and surjective. Then f is called a **bijection** or **one-to-one correspondence**.

Reasoning about Injections, Surjections

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.