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Chapter 1

Some Elementary Logic

The study of logic is the study of the principles and methods used in distinguishing valid argu- ments from those that are not valid. The aim of this chapter is to help the student to understand the principles and methods used in each step of a proof. The starting point in logic is the term statement (or proposition) which is used in a technical sense. We introduce a minimal amount of mathematical logic which lie behind the concept of proof.

1.1 Mathematical Statements

When we prove theorems in mathematics, we are demonstrating the truth of certain statements. We therefore need to start our discussion of logic with a look at statements, and at how we recognize certain statements as true or false.

Definition 1.1.1. *A **statement** is any declarative sentence that is either true (T) or false (F), but not both. We refer to T or F as the **truth value** of the statement. Statements are usually denoted by lower case letters (for example: p, q, r, ...)*

Remark 1.1.1. *A statement is also called a **proposition.***

Example 1.1. *The following sentences are statements*

(a) *The world is flat.*

(b) $4 - 1$ *equals* 3.

(c) *The equation* $x^2 + 1 = 0$ *has two real solutions.*

(d) $(x + y)^2 = x^2 + 2xy + y^2$.

(e) *Gaza is a Palestinian city.*

Example 1.2. *The following sentences are NOT statements*

(a) *How are you ?*

(b) $x^2 = 9$.

(c) *I will come to school next week.*

(d) x is an even number

(e) Go to your room.

1.2 Connectives and Compound Statements

- A **simple statement** is a statement that conveys a single idea.
- A **compund statement** is a statement that conveys two or mor ideas.
- Connecting Simple statements with words and phrases such as and, or, if ...then, if and only if creates a comound statement.

Example 1.3. *The following sentences are compound statements*

1) *Gaza is Palestin city **and** Palastine is an arabic country.*
2) $2-1$ *equals* 3 ***or*** 7 *is divisible by* 2.
3) ***If*** 5 *is an integer,* ***then*** 5 *is a real number.*
4) *You will pass this course **if and only if** you learned how to construct a mathematical proof.*

Notation: We will denote simple statements by lowercase letters $p, q, r, \cdots$ and we will denote compound statements by uppercase letters $P, Q, R, \cdots$.

1.2.1 Truth tables

An important distinction must be made between a statement and the form of a statement. A statement form does not have a truth value. Instead, each form has a list of truth values that depend on the values assigned to its components. This list is displayed by presenting all possible combinations for the truth values of its components in a truth table. We will use **"T"(or** 1) for true and **"F"(or** 0) for false. If P denotes a statement then can be summarized neatly in the truth table

P
T
F

1.2.2 connectives

To form new compound statements out of old ones we use the following

ve fundamental connectives:

Definition 1.2.1. ***(Negation: " not ")***

*The **negation** of a statement P, denoted by $\overline{P}$ (read as **"not** P**" or "the negation of** P**"**) is the statement whose truth value is the opposite of the truth value of P.*

P	$\overline{P}$
T	F
F	T

Example 1.4. p: *thirteen is not a prime number.*

$\overline{P}$: *thirteen is a prime number.*

q: *Today is friday.* $\overline{q}$: *Today is not friday.*

Exercise 1.1. ***I.*** *Write the negation of each of the following:*

a: $\sqrt{2}$ *is a rational number.*

b: The sine function is continuous at $x = 0$.

c: An apple is not red.

d: 3 *divides* 12.

II. Give the truth values (True or False) of each of the above statements.

Definition 1.2.2. ***(Conjunction: "and")***

If p *and* q *are statements, then the conjunction of* p *and* q *is denoted by* $p \wedge q$ *(read:* ***"p and q"*** *or* ***"conjunction of p and q"****). The truth values for* $p \wedge q$ *are de*

ned as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Remark 1.2.1.

- *The statement* $p \wedge q$ *is true only when both* p *and* q *are true.*
- *In a compound statement with two components* p *and* q *there are* $2 \times 2 = 4$ *possibilities, called the logical possibilities. In general, if a compound statement has* n *components, then there are* 2^n *logical possibilities.*

Example 1.5. *Indicate which of the following statements is* T *and which is* F*:*

1) $1 + 1 = 2$ *and* $3 - 1 = 2$. [*the statement is* T]

2) 5 *is an integer and* $1 - 3 = 1$. [*the statement is* F]

Exercise 1.2. *Construct a truth table for the compound statement* $p \wedge \overline{p}$.

Definition 1.2.3. ***(Disjunction: "or")***

If p *and* q *are statements, then the disjunction of* p *and* q *is denoted by* $p \vee q$ *(read:* ***"p or q"*** *or* ***"the disjunction of p and q"****). The truth values for* $p \vee q$ *are defined as follows:*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.6. *Indicate which of the following statements is T and which is F:*

- $1 - 1 = 1$ *or* $3 + 3 = 6$. [*the statement is T*]
- $\sqrt{-1} = 2$ *or* $2^2 = -1$ [*the statement is F*]
- 7 *is a prime number or* 7 *is an odd number.*[*the statement is T*]

Exercise 1.3. *Construct a truth table for the compound statement* $p \vee \overline{p}$.

Definition 1.2.4. ***(CONDITIONAL:" if ....then")***

If p *and* q *are statements, then the statement "*$p \Rightarrow q$*" is a conditional statement (read:* ***"p implies q"*** *or* ***"if p then q")****. If* p *is true and* q *is false then* $p \Rightarrow q$ *is false, and in all other cases* $p \Rightarrow q$ *is true.*

p is called the ***antecedent*** *and q is called the* ***consequent****. If* $p \Rightarrow q$ *is true, then sometimes p is called the* ***hypothesis*** *and q is called the* ***conclusion****.*

The truth values of $p \Rightarrow q$ *is*

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.7. *Determine whether the following statements are T or F:*

- *If* $2 - 4 = 2$, *then* $2 - 2 = 4$. [*the statement is T*]
- *If* $7 < 9$, *then* $7 < 8$. [*the statement is T*]
- *If* $3 > 3$, *then* $4 > 3$. [*the statement is T*]
- *If* $5 < 6$, *then* 5 *is even.*[*the statement is F*]

Exercise 1.4.

- *Construct the truth table for the compound statement* $(p \vee q) \Rightarrow r$.
- *Is the statement* $p \Rightarrow q$ *equivalent to the statement* $\overline{q} \Rightarrow \overline{p}$. *Explain.*

Remark 1.2.2. *We use* $p \Rightarrow q$ *to translate the following statements:*

1) *If p, then q.*
2) *p only if q.*
3) *q if p.*
4) *p is sufficient to q.*
5) *q is necessary for p.*
6) *q whenever p.*

Definition 1.2.5. ***(BICONDITIONAL: "if and only if")***

If p *and* q *are statements, then the statement* $p \Leftrightarrow q$ *is called the biconditional (read :* p *if and only if* q*") and* *is abbreviated to "p iff q", or 'p is equivalent to q".*

If both p and q are true, or if both are false, then $p \Leftrightarrow q$ is true. It is false if (p is true and q is false), and it is also false if (p is false and q is true).

The truth values of $p \Leftrightarrow q$ is

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.8. *Determine whether the following statements are T or F:*

- 1 *is odd if and only if* 3 *is even.[the statement is F]*
- $\mid 5 \mid = -5$ *if and only if* $5 > 0$. *[the statement is F]*
- $\sqrt{4} = 2$ *if and only if* $2^2 = 4$. *[the statement is T]*
- $5 > 6$ *if and only if* 5 *is even.[the statement is T]*

Exercise 1.5. *Construct the truth table for the compound statement* $(p \wedge q) \Leftrightarrow p$.

Remark 1.2.3. *We use $p \Leftrightarrow q$ to translate the following statements:*

1) p if and only if q.

2) p is equivalent to q.

3) p is necessary and sufficient for q.

Exercise 1.6. *Translate the given compound statements into a symbolic form using the suggested symbols.*

(a) *"A natural number is even if and only it is divisible by 2." (E, D)*

(b) *"A matrix has an inverse whenever its determinant is not zero." (I, Z)*

(c) *"A function is differentiable at a point only if it is continuous at that point." (D, C).*

Remark 1.2.4. *Using truth tables, we can show that:*

1) $\overline{\overline{P}} \Leftrightarrow P$.

2) $(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow Q)]$..

3) $(P \Rightarrow Q) \Rightarrow (\overline{P} \vee Q)$.

4) $(P \Rightarrow Q) \Rightarrow (\overline{Q} \Rightarrow \overline{Q})$.

1.3 THEOREMS OF LOGIC

Theoreme 1.3.1. ***(Morgan's Rules)*** *If P and Q are statements,then*

1) $(\overline{P \wedge Q}) \Leftrightarrow (\overline{P} \vee \overline{Q})$.

2) $(\overline{P \vee Q}) \Leftrightarrow (\overline{P} \wedge \overline{Q})$.

Proof. By using the tables of truths we have the required results. □

Theoreme 1.3.2. *Let P, Q, and R be any three statements.*

a. ***Commutative Properties:*** $(P \vee Q) \Leftrightarrow (Q \vee P)$ *and* $(P \wedge Q) \Leftrightarrow (Q \wedge P)$.

b. ***Associative Properties:*** $[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)]$ *and* $[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)]$.

c. ***Distributive Properties:***$[(P \vee Q) \wedge R] \Leftrightarrow [(P \wedge R) \vee (Q \wedge R)]$ *and* $[(P \wedge Q) \vee R] \Leftrightarrow [(P \vee R) \wedge (Q \vee R)]$.

Theoreme 1.3.3. *Let P, Q, and R be any three statements.*

a. ***Reflexive Property:*** *For any statement P, $P \Rightarrow P$.*

b. ***Antisymmetric Property:*** *If $P \Rightarrow Q$ and $Q \Rightarrow P$, then $P \Leftrightarrow Q$.*

c. ***Transitive Property:*** *If $P \Rightarrow Q)$ and $Q \Rightarrow R$, then $P \Leftrightarrow R$.*

Definition 1.3.4. *From the implication $P \Rightarrow Q$ we can define the following propositions:*

a. The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

b. The contrapositive of $P \Rightarrow Q$ is $\overline{Q} \Rightarrow \overline{P}$.

c. The negation of $P \Rightarrow Q$ is $P \wedge \overline{Q}$.

Example 1.9. *Let $n \geq 2$, and consider the statement (I)*

$$I : [(n \text{ is prime and } n \neq 2) \;\Rightarrow\; (n \text{ is } \; \text{odd})].$$

The converse of (I) is

$$[(n \text{ is } \; \text{odd}) \Rightarrow n \text{ is prime and } n \neq 2)]$$

The contrapositive of(I) is

$$[(n \text{ is } \; \text{even }) \Rightarrow (n \text{ is not prime or } n = 2].$$

The negation of (I) is

$$[(n \text{ is prime and } n \neq 2) \text{ and } (n \text{ is even})].$$

Exercise 1.7. *Let P and Q are statements. Determine $(\overline{P \Rightarrow Q})$ and $(\overline{Q \Rightarrow P})$ then Deduce the negation of $\overline{P \Leftrightarrow Q}$.*

Solution Using Morgan's rules we have:

$$\begin{aligned}(\overline{P \Rightarrow Q}) &\Leftrightarrow \overline{\overline{P} \vee Q} \\ &\Leftrightarrow \overline{\overline{P}} \wedge \overline{Q} \\ &\Leftrightarrow P \wedge \overline{Q}\end{aligned}$$

So $(\overline{Q \Rightarrow P}) \Leftrightarrow Q \wedge \overline{P}$. Deduces that:

$$\begin{aligned}\overline{P \Leftrightarrow Q} &\Leftrightarrow \overline{(P \Rightarrow Q) \wedge (Q \Rightarrow P)} \\ &\Leftrightarrow (\overline{P \Rightarrow Q}) \vee (\overline{Q \Rightarrow P}) \\ &\Leftrightarrow (P \wedge \overline{Q}) \vee (Q \wedge \overline{P}).\end{aligned}$$

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1.4 Quantification Rules

Some sentences depend on some variables and become statements when the variables are replaced by a certain values.

definition:(Open sentence)

A sentence containing one or more variables and which becomes a statement only when the variables are replaced by certain values is called an **open sentence** (or **a propositional predicate**).
Notation: An open sentence P with variables $x_1, x_2, \cdots, x_n$ will be denoted by $P(x_1, x_2, \cdots, x_n)$.

Example 1.10. $P(x) : x + 1 = 0$ *is an open sentence.*
$P(0)$ *is false but* $P(-1)$ *is true.*

Example 1.11. $P(x, y) : x + 2y = -1$ *is an open sentence.*
$P(-3, 1)$ *is true but* $P(-1, 1)$ *is false.*

Definition: (Universal and existence quantifi ers)

▶ The expression for all (or for every, or for each, or (sometimes) for any), is called the **universal quantifier** and is often written ∀.
The following all have the same meaning (and are true)

- for all x and for all y, $(x + y)^2 = x^2 + 2xy + y^2$.
- for any x and y, $(x + y)^2 = x^2 + 2xy + y^2$.
- for each x and each y, $(x + y)^2 = x^2 + 2xy + y^2$.
- $\forall x \forall y \left((x + y)^2 = x^2 + 2xy + y^2 \right)$.

It is implicit in the above that when we say "for all x" or $\forall x$, we really mean for all real numbers x, etc. In other words, the quantifier ∀ "ranges over" the real numbers. More generally, we always quantify over some set of objects, and often make the *abuse* of *language* of suppressing this set when it is clear from context what is intended. If it is not clear from context, we can include the set over which the quantifier ranges. Thus we could write

$$\text{for all } x \in \mathbb{R} \text{ and for all} y \in \mathbb{R},\ (x + y)^2 = x^2 + 2xy + y^2.$$

which we abbreviate to

$$\forall x \in \mathbb{R} \forall\ y \in \mathbb{R},\ (x + y)^2 = x^2 + 2xy + y^2.$$

▶ The expression there exists (or there is, or there is at least one, or there are some), is called the **existential quantifier** and is often written ∃.
The following statements all have the same meaning (and are true)

- there exists an irrational number.
- there is at least one irrational number.
- some real number is irrational.
- irrational numbers exist.
- $\exists x$(x is irrational).

The last statement is read as "there exists x such that x is irrational".

It is implicit here that when we write $\exists x$, we mean that there exists a *real number* x. In other words, the quantifier $\exists$ " ranges over" the real numbers.

Order of Quantifiers

The order in which quanti¯ers occur is often critical. For example, consider the statements

$$\forall x \exists y (x < y). \tag{1.1}$$

and

$$\exists y \forall x (x < y). \tag{1.2}$$

We read these statements as

for all x there exists y such that $x < y$

and

there exists y such that for all $x, x < y$,

respectively. Here (as usual for us) the quanti¯ers are intended to range over the real numbers. Note once again that the meaning of these statments is *unchanged* if we replace x and y by, say, u and v.

Statement (1.1) is true. We can justify this as follows5 (in somewhat more detail than usual!):

Let x be an arbitrary real number. Then $x < x + 1$, and so $x < y$ is true if y equals (for exemple) $x + 1$. Hence the statement $\exists y, (x < y)$is true.

But x was an *arbitrary* real number, and so the statement

for all x there exists y such that $x < y$

is true. That is, (1.1) is true. On the other hand, statement (1.2) is false.

It asserts that there exists some number y such that $\forall x(x < y)$.

But $\forall x(x < y)$ means "there exists y such that y is an upper bound for $\mathbb{R}$."

We know this last assertion is false.

Alternatively, we could justify that (1.2) is false as follows:

Let y be an arbitrary real number.

Then $y + 1 < y$ is false

Hence the statement $\exists x(x < y)$ is false. Since y is an arbitrary real number, it follows that the statement *there exists* y such that for all $x, x < y$,

is false.

Exercise 1.8. *Rewrite each of the following quantified statements in symbolic form.*

1. *Some books are not novels.*
2. *Not all apples are red.*
3. *All diamonds have brilliance.*
4. *Some cereals contain vitamins.*

Solution:

1. **Some books are not novels**. This statement can be rewritten:

   There exists an x such that x is a book and x is not a novel.

   Let B be the predicate *"is a book. "*

   Let N be the predicate *"is a novel. "*

   Symbolically this statement is written $(\exists x)(B(x) \wedge \overline{N(x)})$.

2. **Not all apples are red.** This statement can be rewritten:

   There exists an x such that x is an apple and x is not red.

   Let A be the predicate *"is an apple."*

   Let R be the predicate *"is red."*

   Symbolically, this statement is written $(\exists x)(A(x) \wedge \overline{R(x)})$.

3. **All diamonds have brilliance.** This statement can be rewritten:

   All diamonds are brilliant objects.

   Therefore, you can rewrite the statement in the form: For all$x$, if x is a diamond, then x is a brilliant object.

   Let D be the predicate *"is a diamond. "*

   Let B be the predicate *"is a brilliant object. "*

   Symbolically, this statement is written $(\forall x)(D(x) \Rightarrow B(x)$.

4. **Some cereals contain vitamins.** This statement can be written:

   Some tortillas are products which are made of flour.

   Therefore, you can rewrite the statement this way:

   There exists an x such that x is a tortilla and x is a product which is made of flour.

   Let T be the predicate *"is a tortilla. "*

   Let F be the predicate *"is a product which is made of flour. "*

   Symbolically, the statement is written $(\exists x)(T(x) \wedge F(x))$.

Rulr of quantifier negation

Remark 1.4.1. *Let the domain of discourse be* $U = \{a_1, a_2, \cdots, a_n\}$*. Then*

1. *The statement* $(\forall x \in U)(P(x))$ *means* $P(a_1) \wedge P(a_1) \wedge \cdots P(a_n)$.

2. *The statement* $(\exists x \in U)(P(x))$ *means* $P(a_1) \vee P(a_1) \vee \cdots P(a_n)$.

definition: (Quantifier negation)

1. $\overline{(\forall x \in U)(P(x))} \Leftrightarrow (\exists x \in U)\overline{(P(x))}$.
2. $\overline{(\exists x \in U)(P(x))} \Leftrightarrow (\forall x \in U)\overline{(P(x))}$.

Example 1.12.

- $\overline{\forall x \in \mathbb{R}\ x^2 \geq 0} \Leftrightarrow \exists x \in \mathbb{R}, x^2 < 0$.
- $\overline{\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}, x + y = 0} \Leftrightarrow \exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}, x + y \neq 0$.

1.5 Methods of Mathematical Proof

A proof is a complete justification of the truth of a statement called a theorem. It generally begins with some hypotheses stated in the theorem and proceeds by correct reasoning to the claimed state- ment. It is nothing more than an argument that presents a line of reasoning explaining why the statement follows from known facts. There are several methods of proof .Here we introduce the basic methods of proof.

1.5.1 DIRECT PROOF:

A direct proof is a logical step-by-step argument from the given conditions to the conclusion.

▶ **Proving conditional statements** $p \Rightarrow q$

The most famous example is the direct proof of statement of the form $p \Rightarrow q$ which proceeds in a step-by-step fashion from the condition p to the conclusion q. Since $p \Rightarrow q$ is false only when p is true and q is false, it suffices to show that this situation cannot happen. The direct way to proceed is to assume that p is true and show (deduce) that q is also true.

A direct proof of $p \Rightarrow q$ will have the following form:

Direct proof of $p \Rightarrow q$:
Assume p.
⋮
Therefore,q.

▶ **Proving biconditional statements** $p \Leftrightarrow q$

Proofs of biconditional statements $p \Leftrightarrow q$ often make use of the tautology

$$(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)].$$

Proofs of $p \Leftrightarrow q$ generally have the following two-part form:

Two-Part Proof Of $p \Leftrightarrow q$
(i) Show $p \Rightarrow q$.
(ii) Show$q \Rightarrow p$.
Therefore,$p \Leftrightarrow q$.

Remark: some cases it is possible to prove a biconditional sentence $p \Leftrightarrow q$ that uses the connective throughout. This amounts to starting with p and then replacing it with a sequence of equivalent statements, the last one being q. With$n$ intermediate statements $R_1, R_2, \cdots, R_n$, a biconditional proof of $p \Leftrightarrow q$ has the form:

Biconditional Proof Of $p \Leftrightarrow q$

$$\begin{aligned} p &\Leftrightarrow R_1 \\ &\Leftrightarrow R_2 \\ &\vdots \\ &\Leftrightarrow R_n \\ &\Leftrightarrow q \end{aligned}$$

Example 1.13. *Let n and m be integers. Then*

(i) if n and m are both even, then $n+m$ is even,

(ii) if n and m are both odd, then $n+m$ is even,

(i) if one of n and m is even and the other is odd, then $n+m$ is odd.

Rough notes: This is a warm-up theorem to make us comfortable with writing mathematical arguments. Start with the hypothesis, which tells you that both n and$m$ are even integers (for part (i)). Use your knowledge about the even and odd numbers, writing them in forms $2k$ or $2k+1$ for some integer$k$.

Proof. (i) If n and m are even, then there exist integers k and j such that $n = 2k$ and $m = 2j$. Then

$$n + m = 2k + 2j = 2(k + j);$$

And since $k, j \in \mathbb{Z}; (k+j) \in \mathbb{Z} : n+m$ is even.
(ii) and (iii) are left for a reader as an exercise. □

Exercise 1.9. *Let $n \in \mathbb{N}; n > 1$. Suppose that n is not prime $\Rightarrow 2^n - 1$ is not a prime.*

Rough notes: Notice that this statement gives us a starting point; we know what it means to be a prime, so it is reasonable to begin by writing n as a product of two natural numbers $n = a \times b$.
To

nd the next step, we have to "play" with the numbers so we receive the expression of the required form. We are looking at $2^{ab} - 1$ and we want to factorise this. We know the identity

$$t^m - 1 = (t-1)(1 + t + t^2 + \cdots + t^{m-1}).$$

Apply this identity with $t = 2^b$ and $m = a$ to obtain

$$2^a - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}).$$

Always keep in mind where you are trying to get to - it is a useful advice here!

Proof. Since n is **not a prime**, $\exists a, b \in \mathbb{N}$ such that $n = a \times b,\ 1 < a, b < n$. Let $x = 2^b - 1$ and $y = 1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}$. Then

$$\begin{aligned} xy &= (2^b - 1)(1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b} && \text{(substituting for } x \text{ and} y) \\ &= 2^b + 2^{2b} + 2^{3b} + \cdots + 2^{ab} - 1 - 2^b - 2^{2b} - 2^{3b} - \cdots - 2^{(a-1)b} && \text{(multiplying out the brackets)} \\ &= 2^{ab} - 1 && \text{(taking away the similar items)} \\ &= 2^n - 1 && (\text{as } n = ab) \end{aligned}$$

Now notice that since $1 < b < n$, we have that $1 < 2^b - 1 < 2^n - 1$, so $1 < x < 2^n - 1$. Therefore, x is a positive factor, hence $2^n - 1$ is **not prime number.** ☐

Note: It **is not true** that:$n \in \mathbb{N}$, if n is prime $\Rightarrow 2^n - 1$ is prime.(exemple;$2^{11} - 1 = 23 \times 89$).

1.5.2 Indirect proof

There are another method of proof called indirect proof or the proof by reduction ad absurdum. An indirect proof of validity is done by including, as an additional hypotheses, the negation of the conclusion, and then deriving a contradiction. As soon as a contradiction is obtained, the proof is complete.

Proof by Counterexamples:

Having in mind a little ”writer - reader battle”, we should be sceptical about any presented statement and try to fi

nd a counterexample, which will disprove the conjecture. It may happen that the theorem is true, so it is not obvious in which direction to go - trying to prove or disprove? One counterexample is enough to say that the statement is not true, even though there will be many examples in its favour.
(i.e. to prove that a proposition of the form $\forall x \in E, P(x)$ is false. The idea is to find at least one $x_0 \in E$ for which the proposition is false.)

Example 1.14. *Conjecture: every man is Chinese.*

Counterexample: it suffices to find at least one man who is not Chinese.

Example 1.15. *Disprove the statement:*

$$\textit{If } x \in \mathbb{Z}, \textit{then } \frac{x^2 + x}{x^2 - x} = \frac{x + 1}{x - 1}.$$

Proof. if $x = 0$, then $x^2 - x = 0$ and so $\dfrac{x^2 + x}{x^2 - x}$ is not defined. On the other hand, if $x = 0$, then $\dfrac{x + 1}{x - 1} = -1$;
So the expresions $\dfrac{x^2 + x}{x^2 - x}$ and $\dfrac{x + 1}{x - 1}$ are certainl not equal whene $x = 0$.
Thus, $x = 0$ is a conterexemple to the statement holds. ☐

Exercise 1.10. *Disprove the statement:*

$$\textit{Let } n \in \mathbb{N}. \textit{If } n^2 + 3n \textit{ is even, then } n \textit{odd .}$$

Proof. If $n = 2$, then $n^2 + 3n = 2^2 + 3.2 = 10$ is even and 2 is even. thus, $n = 2$ is a conterexemple. □

In the preceding example, not only is 2 a counterexample, every even integer is a counterexample

Proof by cases:

While attempting to give a proof of a mathematical statement concerning an element x in some set S, it is sometimes useful to observe that x possesses one of two or more properties. A common property that x may possess is that of belonging to a particular subset of S. If we can verify the truth of the statement for each property that x may have, then we have a proof of the statement. Such a proof is then divided into parts called cases, one case for each property that$x$ may possess or for each subset to which x may belong. This method is called **proof by cases**. Indeed, it may be useful in a proof by cases to further divide a case into other cases, called **subcases**.
For example, in a proof of $\forall n\ Z, R(n)$, it might be convenient to use a proof by cases whose proof is divided into the two cases
Case $1.n$ *is even. and Case* $2.n$ *is odd.*

Example 1.16. *The square of any integer is of the form* $3k$ *or* $3k + 1$.

Rough notes: This is a simple example of the proof, where at some point it is easier to split the problem into 2 cases and consider them separately - otherwise it would be hard to find a conclusion. Start by expressing an integer a as $3q + r, (q, r \in \mathbb{Z})$ and then square it. Then split the problem and show that the statement holds for both cases.

Proof. We know that every integer can be written in the form: $3q + 1$ or $3q + 2$ or $3q$.
So let $a = 3q + r$, where $q \in \mathbb{Z}, r \in \{0, 1, 2\}$ Then

$$a^2 = (3q + r)^2 = 9q^2 + 6qr + r^2 = 3\underbrace{(3q^2 + 2qr)}_{\in \mathbb{Z} \text{as} q, r \in \mathbb{Z}} + r^2$$

So let $3q^2 + 2qr := k, k \in \mathbb{Z}$ We have $a^2 = 3k + r^2$: Now,
case I: if $r = 0$ or $r = 1$, we are done;
case II: if $r = 2 \Rightarrow r^2 = 4$ and then $a^2 = 3k + 4 = 3k + 3 + 1 = 3(k + 1) + 1$ which is in the required form. □

Example 1.17. *Let* $n \in \mathbb{Z}$. *Then* $n^2 + n$ *is even.*

Rough notes: To show that the expression is even, it may be helpful to consider the cases when n is even and odd - what does it mean?

- CASE I: n is even (express it mathematically);
- CASE II: n is odd; now, the simple alegabra should bring us to the required conclusion.

Proof. Exercice for a reader. □

Exercise 1.11. *(Triangle Inequality): Suppose* $x, y \in \mathbb{R}$*. Then* $| x + y | \leq | x | + | y |$.

Notes: To split the proof into small problems, we need to recall the modulus function, which is de

ned using cases:

$$| x | = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}.$$

Then, using the defi

nition, carefully substitute x or $(-x)$ for $| x |$, depending on the case.

Proof.

Case I: $x \geq 0, y \geq 0$. So, $midx | = x$ and $| y | = y$. Hence, $x + y \geq 0$.

SO

$$| x + y | = x + y = | x | + | y |$$

Case II: $x < 0, y < 0$. So $midx | = -x$ and $| y | = -y$. Then $x + y < 0$.

SO

$$| x + y | = -(x + y) = (-x) + (-y) = | x | + | y |$$

Case III: One of x and y is positive and the other is negative. Without loss of generality, assume that x is positive ($x \geq 0$so $| x | = x$) and y is negative ($y < 0; | y | = -y$). Now we need to split the problem into 2 subcases:

i. $x + y \geq 0$. So

$$| x + y | = x + y \leq x + (-y) = | x | + | y |$$

ii. $x + y < 0$. So

$$| x + y | = -x + (-y) \leq x + (-y) = | x | + | y |$$

□

Proof by Mathematical Induction

Proof by mathematical induction is a very useful method in proving the validity of a mathematical statement $(\forall n)P(n)$ involving integers n greater than or equal to some initial integer n_0.

Principle of Mathematical Induction:

Let $P(n)$ be an infinite collection of statements with $n, n_0 \in \mathbb{N}$ and $n_0 \leq n$. Suppose that

(i) $P(n_0)$ is true, and

(ii) $P(n) \Rightarrow P(n + 1), \forall n \geq n_0$.

Then, $P(n)$ is true $\forall n \in \mathbb{N}, n \geq n_0$.

When constructing the proof by induction, you need to present the statement $P(n)$ and then follow three simple steps (simple in a sense that they can be described easily; they might be very complicated for some examples though, especially the induction step):

- **INDUCTION BASE** check if $P(n_0)$ is true, i.e. the statement holds for $n = n_0$,
- **INDUCTION HYPOTHESIS** assume $P(n)$ is true, i.e. the statement holds for n,
- **INDUCTION STEP** show that if $P(n)$ holds, then $P(n+1)$ also does.

We finish the proof with the conclusion " since $P(n_0)$ is true and $P(n) \Rightarrow P(n+1)$, the statement $P(n)$ holds is true by the Principle of Mathematical Induction".

Example 1.18. *Show that* $2^{3n+1} + 5$ *is always a multiple of* 7.

Proof. The statement $P(n)$: "$2^{3n+1} + 5$ is always a multiple of 7".

- **BASE** $(n = 0)$

$$2^{3\times 0+1} + 5 = 2 + 5 = 7 = 7 \times 1$$

  then $P(0)$ is true.

- **INDUCTION HYPOTHESIS:** Assume $P(n)$ is true, so

$$2^{3n+1} + 5 \text{ is always a multiple of } 7, n \in \mathbb{N}.$$

- **INDUCTION STEP:** Now, we want to show that $P(n) \Rightarrow P(n+1)$, where

$$P(n+1) : 2^{3(n+1)+1} + 5 = 2^{3n+4} + 5 \text{ is always a multiple of } 7$$

  We know from induction hypothesis that $2^{3n+1} + 5$ is always a multiple of 7, so we can write

$$\begin{aligned} 2^{3n+1} + 5 = 7 \times k \quad \text{for some} k \in \mathbb{Z} &\Longrightarrow (2^{3n+1} + 5) \times 2^3 = 7 \times k \times 2^3 \\ &\Longrightarrow 2^{3n+4} + 40 = 7 \times k \times 8 \\ &\Longrightarrow 2^{3n+4} + 5 = 56 \times k - 35 \\ &\Longrightarrow 2^{3n+4} + 5 = 7(8 \times k - 5) \\ &\Longrightarrow 2^{3n+4} + 5 = 7 \times k", \text{ where } k" = (8 \times k - 5) \in \mathbb{Z}. \end{aligned}$$

  We have shown that $P(0)$ holds and if $P(n)$, then $P(n+1)$ is also true. Hence by the Principle of Mathematical Induction, it follows that $P(n)$ holds for all natural n.

□

Exercise 1.12. *Let* $a_{n+1} = \frac{1}{5}(a_n^2 + 6)$ *and* $a_1 = \frac{1}{5}$ *is decreasing.*

Notes:(definition) A sequence (a_n) is decreasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

We will use the definition to prove the statement. Notice that we need to show $a_n \leq a_{n+1}$ for all n - this should suddenly bring to your mind induction.

Proof. We will show that the statement $P(n)$ holds for all n.

$$P(n) : a_{n+1} < a_n \text{ for all} n.$$

- BASE: $(n = 2)$
$$a_2 = \frac{1}{5}\left((\frac{5}{2})^2 + 6\right) = \frac{1}{5}\left(\frac{25}{4} + 6\right) = \frac{49}{20}.$$
Note: $a_2 = \frac{49}{20} < \frac{5}{2} = a_1$. Hence, $P(2)$ is true.
- HYPOTHESIS: Suppose for some $n \geq 2$, $a_{n+2} \leq a_{n+1}$.
- INDUCTION STEP:
$$\begin{aligned} a_{n+2} &= \frac{(a_{n+1})^2}{5} + \frac{6}{5} \\ &\leq \frac{(a_n)^2}{5} + \frac{6}{5} \\ &= a_{n+1}. \end{aligned}$$
Hence $a_{n+2} \leq a_{n+1}$.
Since $P(2)$ is true and $P(n) \implies P(n+1)$, it follows that the sequence is decreasing by the Mathematical Induction.

□

Exercise 1.13. *Use mathematical induction to prove that:*

1) $\forall n \in \mathbb{N}, \sum_{j=1}^{n}(3j-2) = \frac{1}{2}(3n-1).$

2) $\forall n \in \mathbb{N}, 3 + 11 + 19 + \cdots + (8n-5) = 4n^2 - n.$

Proof by contradiction

Suppose, as usual, that we would like to show that a certain mathematical statement R is true. If R is expressed as the quantified statement $\forall x \in E, P(x) \implies Q(x)$, then we have already introduced two proof techniques, namely direct proof and proof by contrapositive, that could be used to establish the truth of R. We now introduce a third method that can be used to establish the truth of R, regardless of whether R is expressed in terms of an implication.

Suppose that we assume R is a false statement and, from this assumption, we are able to arrive at or deduce a statement that contradicts some assumption we made in the proof or some known fact. (The known fact might be a definition, an axiom or a theorem.) If we denote this assumption or known fact by P, then what we have deduced is $\overline{P}$ and have thus produced the contradiction $C : P \wedge (\overline{P})$. We have therefore established the truth of the implication

$$\overline{R} \implies C$$

However, because $\overline{R} \implies C$ is true and C is false, it follows that $\overline{R}$ is false and so R is true, as desired. This technique is called **proof by contradiction**.

If R is the quantified statement $\forall x \in E, P(x) \implies Q(x)$, then a proof by contradiction of this statement consists of verifying the implication

$$\overline{(\forall x \in E, P(x) \implies Q(x))} \implies C$$

for some contradiction$C$. However, since

$$\begin{aligned}\overline{(\forall x \in E, P(x) \implies Q(x))} &\iff \exists x \in E, \overline{(P(x) \implies Q(x)}\\ &\iff \exists x \in E, (P(x) \wedge \overline{Q(x)}\end{aligned}$$

it follows that a proof by contradiction of $\forall x \in E, P(x) \implies Q(x)$ would begin by assuming the existence of some element $x \in E$ such that $P(x)$ is true and $Q(x)$ is false. That is, **a proof by contradiction** of $\forall x \in E, P(x) \implies Q(x)$ begins by assuming the existence of a counterexample of this quantified statement. Often the reader is alerted that a proof by contradiction is being used by saying (or writing)

Suppose thar R is false.

or

Assume, to the contrary, that R is false.

Therefore, if$R$ is the quantified statement $\forall x \in E, P(x) \implies Q(x)$, then a proof by contradiction might begin with: *Assume, to the contrary, that there exists some element $x \in E$ for which $P(x)$ is true and $Q(x)$ is false.*

(or something along these lines). The remainder of the proof then consists of showing that this assumption leads to a contradiction.

Example 1.19. *Prove that $\sqrt{2} + \sqrt{6} < \sqrt{15}$.*

Proof. Assume for a contradiction that $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. So

$$\begin{aligned}\sqrt{2} + \sqrt{6} \geq \sqrt{15} &\implies (\sqrt{2} + \sqrt{6})^2 \geq 15\\ &\implies 8 + 2\sqrt{12} \geq 15\\ &\implies 2\sqrt{12} \geq 7\\ &\implies 48 \geq 49\end{aligned}$$

The last statement is clearly p not true, hence we reached the contradiction. Therefore, we proved that $\sqrt{2} + \sqrt{6} < \sqrt{15}$. □

Exercise 1.14. *If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.*

Proof. Assume, to the contrary, that there exist an even integer a and an odd integer b such that $4 \mid (a^2 + 2b^2)$. Thus, $a = 2x, b = 2y + 1$ and $a^2 + 2b^2 = 4z$ for some integers x, y and z. Hence,

$$a^2 + 2b^2 = (2x)^2 + 2(2y + 1)^2 = 4z.$$

Simplifying, we obtain $4x^2 + 8y^2 + 8y + 2 = 4z$ or, equivalently,

$$2 = 4z - 4x^2 - 8y^2 - 8y = 4(z - x^2 - 2y^2 - 2y).$$

Since $(z - x^2 - 2y^2 - 2y)$ is an integer, $4 \mid 2$,which impossible. □

PROOF BY CONTRAPOSITION:

A proof by contraposition or contrapositive proof for a conditional sentence $p \Rightarrow q$ makes use of the tautology

$$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p}).$$

It is an indirect proof method in wich we first give a direct proof of $(\overline{q} \Rightarrow \overline{p})$ and then conclude by replacement that$(p \Rightarrow q)$.

Example 1.20. *Let $n \in \mathbb{Z}$. if n^2 is odd, then n is odd.*

Proof. Let n be even (which is "not B"). So

$$\begin{aligned} n\text{is even } &\Longrightarrow n = 2k,\ \ k \in \mathbb{Z} \\ &\Longrightarrow n^2 = (2k)^2 = 4k^2 = 2 \times 2k^2\ \ k \in \mathbb{Z} \\ &\Longrightarrow n \text{ is even} \end{aligned}$$

So we proved that n is even $\Longrightarrow n^2$ is even. Now using the contrapositive we conclude that

$$n^2 \text{ not even (odd)} \Longrightarrow n \text{ not even (odd)},$$

which proves the statement. □

Exercise 1.15.

Friendly reminder

' The importance of proofs goes well beyond a university degree. It is eventually about using reason in everyday life. This could contribute to solving major and global problems."

You have seen many methods of proofs presented in previous sections and they are all used in different areas of mathematics. It has been underlined many times that writing proofs is not easy, but with a lot of practice and open mind, pure mathematics is not as scary. Here are some final tips to keep in your head when starting the next proof. Good luck!

- Experiment! If one method does not work, try a different one. Lots of practice allows for an "educated guess" in the future;
- do not start with what you are trying to prove;
- use correct English with full punctuation;
- begin by outlining what is assumed and what needs to be proved; do not skip this step!
- remove initial working when writing up the final version of the proof, but include all steps of reasoning.

Practice! "Look at proofs in lecture notes and textbooks to get a good idea of how proofs should be written."

"**R**ead your proofs aloud - if it doesn't make sense to someone listening, then you haven't written enough".

Organise your work! "Students often struggle to present their work in a logical order - the classic example is starting from the conclusion and deducing the premise".

Obtain more examples! " Construct own examples on which you can run proofs (this is only a tool for better understanding and does not replace the proofs)".

Feedback. "Consider it carefully - understanding how you could have phrased the argument better will improve future work".