UDBKM 2025-2026 | Differential Equations

Faculty of Mathematics | L3 Maths.

Exercise Series 03

Exercise 1:

Calculate the exponential of the following matrices:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Exercise 2:

Solve the following systems:

1. $\begin{cases} y_1' = y_1 + y_2, \\ \\ y_2' = y_1 + y_2. \end{cases}$

2. $Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y.$

4. $\begin{cases} Y' = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} Y \\ Y(t_0) = Y_0. \end{cases}$

Exercise 3:

Solve the following systems

1. $Y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{2t} \\ e^t \end{pmatrix}.$

2. $\begin{cases} Y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ Y(t_0) = Y_0. \end{cases}$

Exercise Series 03

Exercise 1:

Calculate The exponential of the following matrices

$A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$e^{A} = e^{\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}} = \begin{pmatrix} e^{3} & 0 \\ 0 & e^{4} \end{pmatrix}$ because ~~the matrix~~ A is diagonal matrix

let The matrix $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$B = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2 I_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$e^{B} = e^{2 I_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = e^{2} . I_2 . e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

Since The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an upper triangular matrix, it is nilpotent of index $m = 2$

Therefore,

$e^{B} = e^{2}\left(I_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e^{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e^{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2} & e^{2} \\ 0 & e^{2} \end{pmatrix}$

Exercise 02: Solve the following systems/

1) $\begin{cases} y_1' = y_1 + y_2 . \\ y_2' = y_1 + y_2 . \end{cases} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\det(A - \lambda I_2) = \left| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 - 1 = 0 \Rightarrow 1 - 2\lambda + \lambda^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$ are The two distinct eigenvalues of A.

V_1 and V_2 are The eigenvectors of A associated with λ_1 and λ_2, respectively, therefore
$AV_1 = \lambda_1 V_1$ and $AV_2 = \lambda_2 V_2$ which implies That :

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ x+y=0 \end{cases} \Rightarrow y = -x \quad V_1 = \begin{pmatrix} x \\ -x \end{pmatrix} x \in \mathbb{R}$$

We take $V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$(A - 2I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 2$ We have $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - 2I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$$E_{\lambda_2} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 / (A - \lambda_2 I_2) V_2 = 0_{\mathbb{R}^2} \right\} \Rightarrow \begin{cases} -x+y=0 \\ x-y=0 \Rightarrow x=y \end{cases}$$

we take $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore The general Solution is

$$Y(t) = C_1 V_1(t) e^{\lambda_1 t} + C_2 V_2(t) e^{\lambda_2 t} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{0} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$y_H(t) = \left\{ \begin{cases} C_1 + C_2 e^{2t} \\ -C_1 + C_2 e^{2t} \end{cases}, C_1, C_2 \in \mathbb{R} \right\}$$

2- We have $Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y$

let us compute The eigenvalues of the matrix A

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 2$$
$$= \lambda^2 - \lambda - 2 = 0$$

$\Delta = 1+8 = 9$, $\begin{cases} \lambda_1 = \frac{1-3}{2} = -1 \\ \lambda_2 = \frac{1+3}{2} = 2 \end{cases}$

V_1 and V_2 are the eigenvectors of A associated with λ_1 and λ_2, respectively

We have : $(A - \lambda_1 I_2) V_1 = 0_{\mathbb{R}^2} \Rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0_{\mathbb{R}^2}$

for $\lambda_1 = -1$

$\Rightarrow \begin{cases} 2x + y = 0 \Rightarrow y = -2x \\ 2x + y = 0 \end{cases} \Rightarrow V_1 = x \begin{pmatrix} 1 \\ -2 \end{pmatrix}, x \in \mathbb{R}$

for $\lambda_2 = 2$

we have $(A - \lambda_2 I_2) V_2 = 0_{\mathbb{R}^2} \Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} -x + y = 0 \\ 2x - 2y = 0 \end{cases} \Rightarrow y = x \Rightarrow V_2 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x \in \mathbb{R}$

$\det(V_1, V_2) = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}$

$= 1 + 2 = 3 \neq 0$

The generale Solution is given by

$y(t) = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$

$= C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} = \begin{cases} C_1 e^{-t} + C_2 e^{2t} \\ -2C_1 e^{-t} + C_2 e^{2t} \end{cases} C_1, C_2 \in \mathbb{R}$

3/ We have the system $\begin{cases} Y' = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} Y \\ Y(t_0) = Y_0 \end{cases}$

Step 1: Eigenvalues

$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -2$

Step 2: Eigenvectors.

for $\lambda_1 = 2 \Rightarrow (A-2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} -2x + 4y = 0 \\ x - 2y = 0 \Rightarrow x = 2y \end{cases}$ we take $\Rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for $\lambda_2 = -2 \Rightarrow (A+2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} 2x + 4y = 0 \\ x + 2y = 0 \Rightarrow x = -2y \end{cases} \Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

General Solution

$\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 2+2 = 4 \neq 0$

$Y(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$Y(0) = \begin{cases} 2C_1 + 2C_2 = x_0 \\ C_1 + C_2 = y_0 \end{cases} \Rightarrow \begin{cases} 2C_1 - 2C_2 = x_0 \\ 2C_1 + 2C_2 = 2y_0 \end{cases} \quad 4C_1 = x_0 + 2y_0$

$\Rightarrow C_1 = \frac{x_0}{4} + \frac{y_0}{2} \Rightarrow C_2 = y_0 - C_1$

$= y_0 - \left(\frac{x_0}{4} + \frac{y_0}{2}\right)$

$C_2 = \frac{1}{2}y_0 - \frac{x_0}{4}$

$Y(t) = \left(\frac{x_0 + 2y_0}{4}\right)e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \left(\frac{2y_0 - x_0}{4}\right)e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Exercise 03: Solve the following systems

$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix}$$

the general solution is $y_G = y_H + y_p$

Step 1: Eigenvalues

We have $\lambda_1 = 2$, $\lambda_2 = 1$

Step 2:

for $\lambda_1 = 2$ $(A-2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left\{ -y = 0 \right. \Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $\lambda_2 = 1$ $(A-I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = 0 \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General Solution

$$Y(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 e^{2t} \\ C_2 e^{t} \end{pmatrix}$$

te^{2t}

te^{t}

the particular Solution $Y_p(t)$

We use

$$\boxed{Y_p(t) = C_1(t) V_1 e^{\lambda_1 t} + C_2(t) V_2 e^{\lambda_2 t}}$$ with $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}' = \begin{pmatrix} V_1 e^{\lambda_1 t} & V_2 e^{\lambda_2 t} \end{pmatrix}^{-1} B(t)$

$$Y_p(t) = \begin{pmatrix} C_1(t) e^{2t} \\ C_2(t) e^{t} \end{pmatrix} \Big/ \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}\right]^{-1} B(t)$$

$$= \begin{bmatrix} e^{2t} & \\ & e^{t} \end{bmatrix}^{-1} B(t)$$

(5)

$$= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} \int 1\,dt \\ \int 1\,dt \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Therefore, The Particular Solution is

$$y_p(t) = \begin{pmatrix} t e^{2t} \\ t e^{t} \end{pmatrix}$$

General solution is given by

$$y_G(t) = \begin{cases} c_1 e^{2t} + t e^{2t} \\ c_2 e^{t} + t e^{t} \end{cases}$$

method 2 :

Let us compute Y_p :

We use the method of variation of constants.

There existe a particular Solution of the form

$$Y_p(t) = e^{tA} \cdot C(t), \quad \forall t \in \mathbb{R}$$

$$\text{with } \boxed{C'(t) = e^{-tA} \cdot B(t), \quad \forall t \in \mathbb{R}}$$

2/ let $\begin{cases} Y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ Y(t_0) = Y_0 \end{cases}$

We have : $e^{(t-t_0)A} = e^{(t-t_0)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = e^{\begin{pmatrix} 0 & t-t_0 \\ 0 & 0 \end{pmatrix}}$

with $\begin{bmatrix} 0 & t-t_0 \\ 0 & 0 \end{bmatrix}$ it is a nilpotent matrix because

it is an upper triangular matrix whose

diagonal entries are zero. One easily checks that

$$\begin{bmatrix} 0 & t-t_0 \\ 0 & 0 \end{bmatrix}^2 = 0$$

Therefore

$$e^{(t-t_0)A} = e^{\begin{bmatrix} 0 & t-t_0 \\ 0 & 0 \end{bmatrix}} = I_2 + \frac{\begin{bmatrix} 0 & t-t_0 \\ 0 & 0 \end{bmatrix}}{1!} = \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix}$$

The general solution is

$$Y(t) = e^{(t-t_0)A} Y_0 + \int_{t_0}^{t} e^{(t-u)A} B(u)\, du.$$

$$= \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix} Y_0 + \int_{t_0}^{t} \begin{bmatrix} 1 & t-u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} du$$

$$= \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix} Y_0 + \int_{t_0}^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} du$$

$$\boxed{Y(t) = \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix} Y_0 + \begin{pmatrix} t-t_0 \\ 0 \end{pmatrix}}$$