Chapter 4

Introduction to notions of stability

4.0 Introduction

4.0 Autonomous Systems

In this chapter, we first recall the different definitions of stability, then we present the two Lyapunov methods, which are fundamental tools for determining the stability of dynamical systems described by ordinary differential equations.[19]

The nonlinear system given by relation (2.1) is said to be autonomous (or time-invariant) if f does not explicitly depend on time, that is:

$$\dot{x} = f(x). \tag{4.1}$$

Otherwise, the system is called non-autonomous (or time-varying).

In this section, we briefly review the results of Lyapunov theory for autonomous systems. Every non-autonomous system is equivalent to an autonomous system.

Proof. Consider the non-autonomous system:

$$\dot{x} = f(x, t).$$

We define the following system:

$$\begin{cases} \dot{x} = f(x, t), \\ \dot{t} = 1. \end{cases}$$

Let

$$y = \begin{pmatrix} x \\ t \end{pmatrix}, \quad \dot{y} = \begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix}.$$

Then,

$$\dot{y} = \begin{pmatrix} f(x,t) \\ 1 \end{pmatrix} = \begin{pmatrix} F_1(y) \\ F_2(y) \end{pmatrix} = F(y),$$

which is autonomous. □

4.0.1 Definitions

Consider a finite-dimensional continuous system described by a nonlinear first-order differential equation:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n.$$

[19]

Definition 1 (Equilibrium)**.** *A state x_e is called an equilibrium point of the autonomous system* (4.1) *if*

$$f(x_e) = 0.$$

Every equilibrium point can be shifted to the origin by a simple change of variables $x \mapsto x - x_e$. Therefore, without loss of generality, the following definitions and theorems will be established for the case $x_e = 0$.

Proof. Consider the two systems:

$$\dot{x} = f(x), \tag{4.2}$$

$$\dot{y} = f(y + a). \tag{4.3}$$

If $x_e = a$ is an equilibrium point of system (4.2), then $y_e = 0$ is an equilibrium point of system (4.3).

Indeed,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s) + a)\, ds,$$
$$y(t) = x(t_0) - a + \int_{t_0}^{t} f(x(s))\, ds,$$
$$y(t) + a = x(t_0) + \int_{t_0}^{t} f(x(s))\, ds.$$

Thus, $x(t)$ is a solution of (4.2) if and only if $x(t) - a$ is a solution of (4.3). In other words, $x_e = a$ is a stable equilibrium of (4.2) if and only if $y(t) \equiv 0$ is a stable equilibrium of (4.3). □

Definition 2 (Stability). *[19] The equilibrium point $x_e = 0$ is said to be stable if, for every $\varepsilon > 0$, there exists $\eta(\varepsilon, t_0) > 0$ such that if $\|x(t_0)\| < \eta$, then*

$$\|x(t)\| < \varepsilon, \qquad \forall t > t_0.$$

Otherwise, the equilibrium point is unstable [?].

Graphically, the stability of x_e means that the trajectory $x(t)$ in the state space remains inside the ball $B(x_e, \varepsilon)$ if its initial point belongs to a ball $B(x_e, \eta)$.

Equivalently, we can reformulate this definition as:

$$\forall \varepsilon > 0,\ \exists\ \eta(\varepsilon, t_0) > 0 \quad \text{such that} \quad x(t_0) \in B_\eta \implies x(t) \in B_\varepsilon, \quad \forall t > t_0.$$

The domain B_η is called the domain of attraction of the equilibrium state. Stability can be represented as shown in Figure (2.1).

Definition 3 (Asymptotic Stability). *[19] The equilibrium point $x_e = 0$ is asymptotically stable if:*

1. *It is stable.*
2. *And if η can be chosen such that:*

$$\|x(t_0)\| < \eta \quad \Rightarrow \quad \lim_{t \to +\infty} x(t) = 0$$

It can be represented schematically as shown in Figure (2.1).

It should be noted that the second previous condition does not imply the stability of the equilibrium point.
Asymptotic stability includes the property of stability, but specifies in addition that any trajectory initialized in the ball $B(x_e(t_0), \eta)$ converges towards x_e [3].

Definition 4. *(Marginal stability)*
An equilibrium point which is stable but not asymptotically stable is said to be marginally stable.

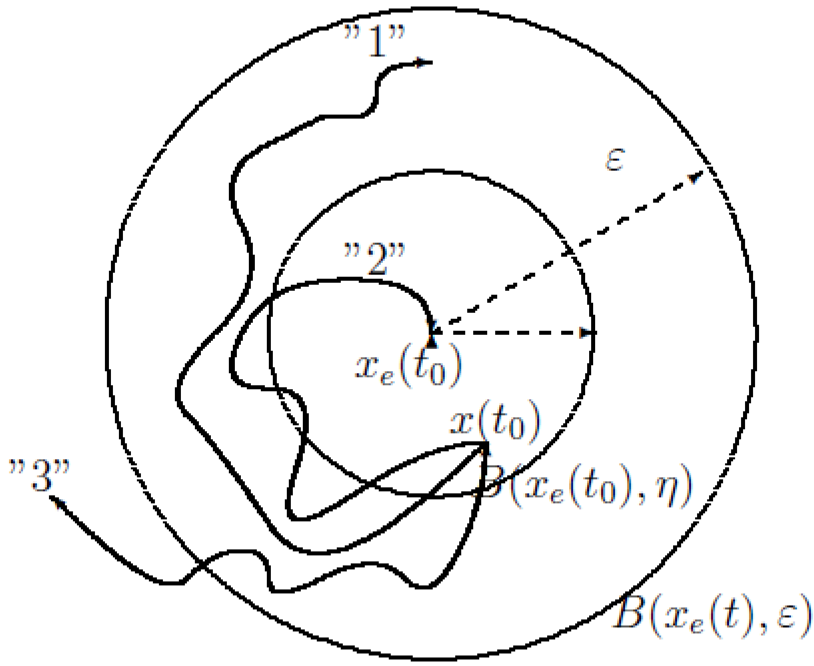

Figure 4.1: Illustration of Stability.

Definition 5. *(Exponential stability)*
An equilibrium point is said to be exponentially stable if there exist two strictly positive numbers α and λ, independent of time and initial conditions, such that:

$$\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\lambda t}, \quad \forall t > t_0 \tag{4.4}$$

for $x(t_0) \in B_\eta$. The scalar λ represents the rate of convergence of the solution $x(t)$.

Exponential stability implies asymptotic stability, the converse is not true; but for time-invariant systems given in the form $\dot{x} = Ax$, asymptotic stability implies exponential stability [7].

Definition 6. *(Uniform stability)*
The equilibrium point $x_e = 0$ is uniformly stable if it is stable with $\eta = \eta(\varepsilon)$ chosen independently of time t_0.

Definition 7. *(Uniform asymptotic stability)*
The equilibrium point $x_e = 0$ is uniformly asymptotically stable if it is uniformly stable and there exists an attraction ball B, independent of t_0, such that $x(t_0) \in B$ implies $x(t) \to 0$ when $t \to \infty$.

Uniform stability implies stability, the converse is not true; but for invariant systems given in the form $\dot{x} = f(x)$, the stability of constant solutions implies uniform stability.

Proof:

Let $y(t) = a$ be a constant stable solution of the autonomous system given by relation (2.2), then

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } x(t) \text{ solution of } \dot{x} = f(x):$$
$$\|x(t_0) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_0$$

We show that

$$\forall \varepsilon > 0, \exists \eta(\varepsilon) > 0 \text{ such that for all } x(t) \text{ solution of } \dot{x} = f(x):$$
$$\|x(t_1) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_1$$

We know that if $x(t)$ is a solution of an autonomous system (2.2), then $x(t+T)$ is also a solution of (2.2). Hence, from the stability of the solution $y(t) = a$, we have:

$$\|x(t_0 + T) - a\| < \eta \Rightarrow \|x(t+T) - a\| < \varepsilon, \forall t \geq t_0$$

which is equivalent to:

$$\|x(t_1) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_1 \text{ with } t_1 = t_0 + T$$

This completes the proof.

Disadvantages of these definitions

The definitions of stability present some important disadvantages:

• It is necessary to be able to explicitly compute each solution corresponding to each initial condition.

• The manipulation of these definitions is tedious.

Consequently, results that allow determining stability without having to integrate the system equations would be welcome.

4.0.2 Results on Homogeneous Linear Systems

Fundamental Theorem:

We consider the following homogeneous linear system:

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n$$

The fundamental phenomenon is that all the solutions are of the same nature for homogeneous linear systems.

One can therefore speak of stable or unstable systems [8].

Let $\quad \dot{x} = A(t)x.$

From the stability point of view, all the solutions of $\dot{x} = A(t)x$ are of the same nature (stable, uniformly stable, uniformly asymptotically stable, $\cdots$).

It is therefore enough to know the nature of the zero solution $x(t) \equiv 0$, $\forall t > 0$.

Proof:

Let $x(t)$, $y(t)$ be two solutions of $\dot{x} = A(t)x$.

Suppose $x(t)$ is stable, we show that $y(t)$ is also stable.

If $x(t)$ is stable, then

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } \bar{x}(t) \text{ solution of } \dot{x} = A(t)x :$$
$$\|x(t_0) - \bar{x}(t_0)\| < \eta \Rightarrow \|x(t) - \bar{x}(t)\| < \varepsilon, \forall t \geq t_0$$

But

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &= \|R(t,t_0)x(t_0) - R(t,t_0)\bar{x}(t_0)\| \\ &= \|R(t,t_0)(x(t_0) - \bar{x}(t_0))\| \end{aligned}$$

where $R(t, t_0)$ is the resolvent of the equation $\dot{x} = A(t)x$.

$y(t)$ is stable if:

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } \bar{y}(t) \text{ solution of } \dot{x} = A(t)x :$$
$$\|y(t_0) - \bar{y}(t_0)\| < \eta \Rightarrow \|y(t) - \bar{y}(t)\| < \varepsilon, \forall t \geq t_0$$

We have:

$$\begin{aligned}\|y(t)-\bar{y}(t)\| &= \|R(t,t_0)y(t_0)-R(t,t_0)\bar{y}(t_0)\| \\ &= \|R(t,t_0)(y(t_0)-\bar{y}(t_0))\|\end{aligned}$$

and thanks to the linearity and continuity of the resolvent, $y(t)$ is stable.

The system $\dot{x}=A(t)x$ is stable for $t\geq t_0$ if and only if:

$$\exists K(t_0) \text{ such that } \quad \|R(t,t_0)\|\leq K, \forall t\geq t_0$$

Proof:

Suppose that $\|R(t,t_0)\|\leq K$, $\forall t\geq t_0$, (which implies that all solutions are bounded), and we show that $x\equiv 0$ is stable, i.e., we show:

$$\begin{aligned}&\forall\varepsilon>0, \exists\eta(\varepsilon,t_0)>0 \text{ such that for all } \bar{x}(t) \text{ solution of } \dot{x}=A(t)x: \\ &\qquad \|\bar{x}(t_0)\|<\eta\Rightarrow\|R(t,t_0)\bar{x}(t_0)\|<\varepsilon, \forall t\geq t_0\end{aligned}$$

Let ε be arbitrary. From the previous hypothesis, we have:

$$\|R(t,t_0)\bar{x}(t_0)\|<K\,\|\bar{x}(t_0)\|$$

thus, if $\|\bar{x}(t_0)\|\leq\dfrac{\varepsilon}{K}$, we obtain $\|R(t,t_0)\bar{x}(t_0)\|\leq\varepsilon$. Therefore, it suffices to take $\eta=\dfrac{\varepsilon}{K}$.

Conversely: suppose $x\equiv 0$ is stable, and we show that $\|R(t,t_0)\|$ is bounded. Indeed:

$$\begin{aligned}\|R(t,t_0)\| &= \sup_{\|x_0\|\leq 1}\|R(t,t_0)x_0\| \\ &= \sup_{\|z\|\leq\eta}\left\|R(t,t_0)\frac{1}{\eta}z\right\| \\ &= \frac{1}{\eta}\sup_{\|z\|\leq\eta}\|R(t,t_0)z\|\end{aligned}$$

And from the stability of the zero solution:

$$\|z\|\leq\eta\Rightarrow\|R(t,t_0)z\|\leq\varepsilon$$

hence:

$$\begin{aligned} \frac{1}{\eta}\|R(t,t_0)z\| &\leq \frac{\varepsilon}{\eta} \\ \Rightarrow \|R(t,t_0)\| &\leq \frac{\varepsilon}{\eta} = K \end{aligned}$$

The system $\dot{x} = A(t)x$ is asymptotically stable if and only if

$$\lim_{t\to+\infty} R(t,t_0) = 0$$

Proof: Suppose that

$$\lim_{t\to+\infty} R(t,t_0) = 0$$

We show that $x(t)$ is stable (or bounded) and

$$\lim_{t\to+\infty} \|x(t)\| = 0$$

Since $x(t) = R(t,t_0)x_0$, then

$$\lim_{t\to+\infty} \|x(t)\| = 0$$

$R(t,t_0)$ is continuous and therefore bounded for any finite t, and as $t \to \infty$

$$R(t,t_0) \longrightarrow 0$$

Thus the resolvent is bounded, which implies that all solutions are stable.

Hence the system $\dot{x} = A(t)x$ is asymptotically stable.

Conversely: suppose $x \equiv 0$ is asymptotically stable, i.e.,

1. $\forall \varepsilon > 0$, $\exists \eta(\varepsilon, t_0) > 0$ such that for all $\bar{x}(t)$ solution of $\dot{x} = A(t)x$:
   $\|\bar{x}(t_0)\| < \eta \Rightarrow \|R(t,t_0)\bar{x}(t_0)\| < \varepsilon$, $\forall t \geq t_0$
2. and η can be chosen such that:

$$\lim_{t\to+\infty} \|\bar{x}(t)\| = 0.$$

Therefore:

$$\lim_{t\to+\infty} \|\bar{x}(t)\| = 0 \quad \Rightarrow \quad \lim_{t\to+\infty} \|R(t,t_0)\bar{x}_0\| = 0$$
$$\Rightarrow \quad \lim_{t\to+\infty} \|R(t,t_0)\| = 0$$

The system $\dot{x} = A(t)x$ is uniformly asymptotically stable for $t_0 \geq T$ if and only if :

$$\exists K > 0,\ \exists \sigma > 0,\ \forall T \leq s \leq t < +\infty \ / \quad \|R(t,s)\| \leq Ke^{-\sigma(t-s)}.$$

Special Case of the Homogeneous Linear System

Consider the system $\dot{x} = A(t)x$, with $A(t) = A$, $\forall t \geq 0$.

We suppose that:

$$\forall i \in \{1, 2, \cdots, n\}, \quad \Re(\lambda_i) < 0$$

where $\lambda_i = \alpha_i + i\beta_i$ ($i \in \{1, 2, \cdots, n\}$) are the eigenvalues of the matrix A.

Then

$$\exists K > 0, \exists \sigma > 0 \quad / \quad \|R(t,0)\| = \left\|e^{tA}\right\| \leq Ke^{-\sigma t}.$$

If $\forall i \in \{1, 2, \cdots, n\}$, $\Re(\lambda_i) < 0$,

then the system is asymptotically stable.

For systems with time-varying coefficients, there is no criterion based on eigenvalues.

Definition 8. *Consider the polynomial P_n given by*

$$P_n(\lambda) = \det(A - \lambda I_n) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

We say that the polynomial P_n is uniformly asymptotically stable if, for every root λ of P_n, one has $\Re(\lambda) < 0$.

(*Hurwitz Criterion*)

Let P_n be a polynomial of the form

$$P_n(\lambda) = \det(A - \lambda I_n) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0, \quad a_n > 0.$$

For P_n to be uniformly asymptotically stable, it is necessary and sufficient that the leading principal minors of the Hurwitz matrix of the characteristic equation ($P_n(\lambda) = 0$) be strictly positive [8].

Consider the polynomial

$$P_4(\lambda) = \det(A - \lambda I_4) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \quad a_4 > 0.$$

The associated Hurwitz matrix is:

$$\begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix}$$

For P_4 to be uniformly asymptotically stable, it is necessary and sufficient that the leading principal minors of the Hurwitz matrix of the characteristic equation ($P_4(\lambda) = 0$) be strictly positive, i.e.:

$$\begin{aligned} \Delta_1 &= a_3 > 0, \\ \Delta_2 &= \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} > 0, \\ \Delta_3 &= \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \\ \Delta_4 &= \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix} > 0. \end{aligned}$$

Consider the system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

The matrix of this system is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) = a_2\lambda^2 + a_1\lambda + a_0. \end{aligned}$$

We now determine the stability region of this system using Hurwitz's criterion.

$$\begin{aligned} \Delta_1 &= a_1 > 0, \\ \Delta_2 &= \begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix} = a_1 a_0 > 0. \end{aligned}$$

$$\begin{aligned} a_1 > 0 &\Leftrightarrow -(a+d) > 0 \\ &\Leftrightarrow a < -d, \end{aligned}$$

$$\begin{aligned} a_1 a_0 > 0 &\Leftrightarrow -(a+d)(ad-bc) > 0 \\ &\Leftrightarrow a < -d \ \text{ and } \ ad > bc. \end{aligned}$$

Therefore, the stability domain is

$$D = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; a+d < 0 \ \text{ and } \ ad - bc > 0 \right\}.$$

4.0.3 Results on Non-Homogeneous Linear Systems

$$\dot{x} = A(t)x + b(t). \tag{4.5}$$

For non-homogeneous systems, there is no automatic link between the fact that the system is stable and the fact that the system is bounded; everything depends on the non-homogeneous term.

- All solutions of the equation $\dot{x} = A(t)x + b(t)$ are of the same nature.
- The nature of the system $\dot{x} = A(t)x$ is the same as that of the system $\dot{x} = A(t)x + b(t)$.

Proof:

Let $x(t)$ and $y(t)$ be two solutions of $\dot{x} = A(t)x$, and let $X(t), Y(t)$ be two solutions of

$\dot{x} = A(t)x + b(t)$ such that

$$X(t) = x(t) + \int_{t_0}^{t} R(t,s)b(s)\,ds,$$
$$Y(t) = y(t) + \int_{t_0}^{t} R(t,s)b(s)\,ds.$$

Then

$$X(t) - Y(t) = x(t) - y(t),$$

which is a solution of $\dot{x} = A(t)x$. This implies that $X(t) - Y(t)$ has the same behavior as $x(t) - y(t)$; therefore, the result is proved.

1. If the system $\dot{x} = A(t)x + b(t)$ is stable and one solution is bounded, then all solutions are bounded.
2. If two solutions of $\dot{x} = A(t)x + b(t)$ are bounded, then the system is stable, and therefore all solutions are bounded.

Proof:

Let $X(t)$ be any solution of $\dot{x} = A(t)x+b(t)$, and $Y(t)$ a bounded solution of $\dot{x} = A(t)x+b(t)$. We show (1). Suppose the system $\dot{x} = A(t)x+b(t)$ is stable. Since $X(t)-Y(t)$ is a solution of the homogeneous system $\dot{x} = A(t)x$, which is stable, all solutions of $\dot{x} = A(t)x$ are bounded. Thus,

$$X(t) - Y(t) \text{ bounded} \quad \Rightarrow \quad X(t) \text{ bounded}.$$

To prove (2), suppose there exist two bounded solutions $X(t)$ and $Y(t)$ of $\dot{x} = A(t)x+b(t)$. Then $X(t)-Y(t)$ is bounded, so the system $\dot{x} = A(t)x$ is stable, and consequently the system $\dot{x} = A(t)x + b(t)$ is stable.

4.0.4 Lyapunov's direct method

We consider the following definitions:

Definition 9. *(Positive definite and semi-definite function)*

A continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be ***positive definite*** *if:*

1. $V(0) = 0$
2. $V(x) > 0$ *for* $x \neq 0$

and it is ***positive semi-definite*** *if:*

1. $V(0) = 0$

2. $V(x) \geq 0$ *for* $x \neq 0$

Similarly, V *is said to be* ***negative definite*** *(resp.* ***negative semi-definite****) if:*

1. $V(0) = 0$

2. $-V(x) > 0$ *(resp.* $-V(x) \geq 0$*)*

Definition 10. *(Lyapunov function)*

V *is called a* ***Lyapunov function*** *for the autonomous system given by relation (2.2) if in a ball* B *we have:*

1. V *is positive definite (*$V(0) = 0,\ x \neq 0 \Longrightarrow V(x) > 0$*).*

2. V *has continuous partial derivatives* $\left(\dfrac{\partial V}{\partial x} \text{ is continuous}\right)$.

3. *The time derivative of* V *is negative semi-definite:*

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)\dot{x} = \frac{\partial V}{\partial x} f(x) \leq 0, \quad \text{if } x \neq 0, \quad \dot{V}(0) = 0$$

(*Stability*)

If for system (2.2) there exists a positive definite scalar function V whose time derivative $\dfrac{dV}{dt}$ is negative definite (resp. negative semi-definite), then system (2.2) is asymptotically stable (resp. stable).

(*Instability*)

Suppose that for system (2.2) there exists a function V, differentiable in a neighborhood of the origin and such that $V(0) = 0$. If its derivative $\dfrac{dV}{dt}$ is positive definite and if there exist points around the origin where V takes positive values, then the equilibrium point $x_e = 0$ is unstable.

1. A system may admit several Lyapunov functions. For example, if V is a Lyapunov function for a given system, then the function

$$V_{\alpha,\rho}(x) = \rho V^{\alpha} \quad \text{for } \rho > 0,\ \alpha > 1$$

is also a Lyapunov function for the original system.

2. The stability conditions are only sufficient. (However, one may state that the equilibrium point is unstable if there exists a positive definite function V for which $\dot{V}(x) > 0$ at least along one trajectory of the system.)

4.0.5 Lyapunov's indirect method (Linearization around an equilibrium point)

Assume that the function f of the autonomous system given by relation (2.2) is continuously differentiable, and that $x_e = 0$ is an equilibrium point. Then, using Taylor expansion around the equilibrium point, relation (2.2) can be written as:

$$\dot{x} = \left[\frac{\partial f}{\partial x}\right]_{x=0} x + f_s(x) \tag{4.6}$$

where $f_s(x)$ represents the higher-order terms in x.

Thus, the linearization of the original nonlinear system at the equilibrium point is given by:

$$\dot{x} = Ax \tag{4.7}$$

where A denotes the Jacobian matrix of f with respect to x at the equilibrium $x_e = 0$, i.e.,

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \tag{4.8}$$

which is called the *linear approximation* of the autonomous system given by relation (2.2) around the origin.

The characteristic polynomial of the linearized system is:

$$P_A(\lambda) = \det(\lambda I - A)$$

and the eigenvalues of A are the solutions of

$$P_A(\lambda) = \det(\lambda I - A) = 0.$$

Three cases must then be distinguished:

1. If all eigenvalues have strictly negative real parts, the system $\dot{x} = Ax$ is asymptotically stable in the neighborhood of the equilibrium point. That is, the trajectories return to it after a small perturbation.

2. If at least one eigenvalue has a strictly positive real part, the system $\dot{x} = Ax$ is unstable.
3. If at least one eigenvalue is purely imaginary, no conclusion can be drawn [?].

Results derived from Lyapunov's indirect method:

1. If the linearized system is asymptotically stable, then the equilibrium point of the original system given by relation (2.2) is asymptotically stable.
2. If the linearized system is not stable, then the equilibrium point of the original system given by relation (2.2) is unstable.
3. If the linearized system is marginally stable, then the equilibrium point of the original system given by relation (2.2) may be stable or unstable (no conclusion can be drawn).

The above results form the basis of linear control theory, which is generally used in practice. As a consequence, the stability of linear time-invariant systems can be determined by the following theorem:

The equilibrium point of the linearized system is asymptotically stable if and only if, for every symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P such that:

$$A^T P + PA = -Q.$$

If Q is only positive semi-definite ($Q \geq 0$), then only stability can be concluded, not asymptotic stability [?].

The local stability of the nonlinear system can be deduced from the stability of the linearized system as stated in the following theorem:

If the linearized system is strictly stable (the real parts of the eigenvalues of the matrix A are strictly negative), then the equilibrium point of the nonlinear system is locally asymptotically stable. Otherwise, if the linearized system is unstable, then the nonlinear system is also unstable.

This theorem does not allow any conclusion about the marginal stability of the linearized system.

4.1 Solved exercises

Exercise 1: