Chapter 3

Linear differential system

Let I be a non-empty open interval of $\mathbb{R}$. For all $i, j = \overline{1, n}$, let a_{ij} and b_i be functions defined on I with values in $\mathbb{R}$.

Consider the system of n linear differential equations:

$$\begin{cases} y_1'(t) = a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + b_1(t) \\ y_2'(t) = a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + b_2(t) \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n'(t) = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + b_n(t) \end{cases} \qquad (S)$$

The unknowns $y_1, y_2, \ldots, y_n$ are functions defined on I with values in $\mathbb{R}$.

Lemma 3.0.1.

The system (S) is equivalent to the system:

$$y'(t) = A(t)y(t) + B(t), \quad \forall t \in I \quad (S')$$

with

$$y : I \to \mathbb{R}^n, \quad t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

$$B : I \to \mathbb{R}^n, \quad t \mapsto B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

$$A : I \to M_n(\mathbb{R}), \quad t \mapsto A(t) = (a_{ij}(t))_{i,j=\overline{1,n}}.$$

[?]

Proof 3.0.1.

Indeed:

$$(S) \iff \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + \cdots + a_{1n}(t)y_n(t) \\ a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + \cdots + a_{2n}(t)y_n(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}(t)y_1(t) + a_{n2}(t)y_2(t) + \cdots + a_{nn}(t)y_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix} \iff y'(t) = A(t)y$$

Example 3.0.1.

Consider the system:

$$\begin{cases} y_1'(t) = t^2 y_1(t) - y_2(t) + 1 \\ y_2'(t) = \cos t\, y_1(t) - e^t y_2(t) \end{cases}, \quad t \in I = \mathbb{R}.$$

It can be written in the form:

$$y'(t) = A(t)y(t) + B(t), \quad \forall t \in I$$

with

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} t^2 & -1 \\ \cos t & -e^t \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Lemma 3.0.2.

Every second-order equation can be written in the form:

$$y'(t) = A(t)y(t) + B(t).$$

Proof 3.0.2.

Consider a second-order equation:

$$y''(t) + a(t)y'(t) + b(t)y(t) = c(t)$$

and set

$$y_1(t) = y(t), \quad y_2(t) = y'(t)$$

then, for all $t \in I$:

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y''(t) = -a(t)y_2(t) - b(t)y_1(t) + c(t) \end{cases}$$

which is equivalent to $y'(t) = A(t)y(t) + B(t)$, with

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ c(t) \end{pmatrix}, \quad \forall t \in I.$$

Definition 3.1.

The system (S') is called a first-order linear differential system with variable coefficients and a nonzero second member.

Remark 3.0.1.

If $B(t) = 0$ for all $t \in I$, then the system

$$y'(t) = A(t)y(t) \tag{3.1}$$

is called a homogeneous first-order linear differential system (without a second member). Equation (3.1) is called the homogeneous system associated with (S').

• <u>Notation:</u>

For simplicity, we write:

$$(S') = \begin{cases} y' = A(t)y & \text{(homogeneous)} \\ y' = A(t)y + B(t) & \text{(nonhomogeneous)} \end{cases}$$

3.0.1 The existence of a Cauchy problem

Theorem 3.0.1.

If A and B are continuous on the interval I, then for every $(t_0, y_0) \in I \times \mathbb{R}^n$, the system

$$\begin{cases} y' = A(t)y + B(t) \\ y(t_0) = y_0 \end{cases}$$

admits a unique global solution.

3.0.2 Homogeneous and non-homogeneous systems

Theorem 3.0.2.

The set of solutions of (3.1), denoted by S_H, is a vector space of dimension n.

Theorem 3.0.3.

[?] Let y_p be a particular solution of (E). Then the set of solutions of (E), denoted by S_E, is given by:

$$S_E = S_H + S_p,$$

where S_H is the set of solutions of the homogeneous system.[?]

3.1 Matrix exponential method

Let $A \in M_n(\mathbb{R})$. We define:

$$\begin{cases} A^k = \underbrace{A \cdots A}_{\text{k times}}, \quad \text{if } k \in \mathbb{N}^*, \\ A^0 = I_n \end{cases}$$

where $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ is the identity matrix.

Consider the series defined by:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{l \to +\infty} \left(I_n + \frac{A}{1!} + \cdots + \frac{A^l}{l!} \right)$$

with

$$\begin{cases} k! = 1 \cdot 2 \cdot \cdots \cdot k, & \text{if } k \in \mathbb{N}^*, \\ 0! = 1. \end{cases}$$

[?]

Theorem 3.1.1.

The series:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is convergent.

Proof 3.1.1.

We have

$$\left\| \frac{A^k}{k!} \right\| \leq \frac{\|A^k\|}{k!}.$$

Moreover, $\frac{A^k}{k!}$ represents the general term of a convergent numerical series, hence:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is normally convergent. Therefore, it is convergent.

Definition 3.2.

The exponential of a matrix A, denoted e^A, is defined as the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Example 3.1.1.

Consider:

$$e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \quad e^{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}$$

We compute:

$$A_1^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2^2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $A_1^2 = A_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, hence $A_1^n = A_2^n = 0$ for all $n \geq 2$.

$$\begin{aligned} e^{A_1} &= \sum_{k=0}^{\infty} \frac{A_1^k}{k!} = \sum_{k=0}^{1} \frac{A_1^k}{k!} = \frac{A_1^0}{0!} + \frac{A_1}{1!} \\ &= I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Theorem 3.1.2.

Let $A, B \in M_n(\mathbb{R})$.

(1) $e^{0_n} = I_n$, where 0_n is the zero matrix.

(2) In general, $e^{A+B} \neq e^A \cdot e^B$, but if A and B commute, i.e., $AB = BA$, then

$$e^{A+B} = e^A \cdot e^B.$$

(3) e^A is invertible and $(e^A)^{-1} = e^{-A}$.

(4) The function

$$F : \mathbb{R} \to M_n(\mathbb{R}), \quad t \mapsto e^{tA}$$

is differentiable, and we have $(e^{tA})' = Ae^{tA}$ for all $t \in \mathbb{R}$.

- Indeed:

$$e^{0_n} = \sum_{k=0}^{\infty} \frac{(0_n)^k}{k!} = \frac{0_n^0}{0!} + \sum_{k=1}^{\infty} \frac{(0_n)^k}{k!} = I_n + 0_n = I_n.$$

- Also,

$$\begin{aligned} (e^{tA})' &= \Big(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\Big)' = \sum_{k=0}^{\infty} \Big(\frac{(tA)^k}{k!}\Big)' \\ &= A \cdot \sum_{k=1}^{\infty} \frac{t^{k-1}A^{k-1}}{(k-1)!}. \end{aligned}$$

Setting $p = k - 1$, we find:

$$(e^{tA})' = A \cdot \sum_{p=0}^{\infty} \frac{t^p A^p}{p!} = A \cdot e^{tA}.$$

Remark 3.1.1.

There exist matrices A and B such that

$$e^{A+B} \neq e^A \cdot e^B.$$

For example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From the definition of the matrix exponential:

$$e^{A+B} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}.$$

Meanwhile,

$$e^A = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}, \quad e^B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad e^A \cdot e^B = \begin{pmatrix} e & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus, $e^{A+B} \neq e^A \cdot e^B$.

Lemma 3.1.1.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then

$$e^{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{bmatrix}.$$

Proof 3.1.2.

[**?**]. We have:

$$e^{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^k}{k!}.$$

But one can show by induction that for all $k \in \mathbb{N}$, we have:

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix}.$$

Therefore:

$$\begin{aligned}
e^{\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}} &= \sum_{k=0}^{\infty}\frac{\begin{bmatrix}\lambda_1^k & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n^k\end{bmatrix}}{k!}\\
&= \lim_{l\to+\infty}\sum_{k=0}^{l}\begin{bmatrix}\frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \lim_{l\to+\infty}\begin{bmatrix}\sum_{k=0}^{l}\frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sum_{k=0}^{l}\frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \begin{bmatrix}\lim_{l\to+\infty}\sum_{k=0}^{l}\frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lim_{l\to+\infty}\sum_{k=0}^{l}\frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \begin{bmatrix}e^{\lambda_1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_n}\end{bmatrix}.
\end{aligned}$$

Example 3.1.2.

We have:

$$e^{\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}} = \begin{pmatrix}e^2 & 0\\ 0 & e^3\end{pmatrix}.$$

Also:

$$e^{\begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -3\end{pmatrix}} = \begin{pmatrix}e^1 & 0 & 0\\ 0 & e^4 & 0\\ 0 & 0 & e^{-3}\end{pmatrix}.$$

Theorem 3.1.3.

Let $A \in M_n(\mathbb{R})$.

(a) If P is an invertible matrix, then

$$e^{PAP^{-1}} = P\, e^{A}\, P^{-1}.$$

(b) For $\lambda \in \mathbb{R}$, we have

$$e^{\lambda I_n + A} = e^{\lambda} \cdot e^{A}.$$

Proof 3.1.3.

(a) We compute:

$$e^{PAP^{-1}} = \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!}.$$

By induction, one can show that for all $k \in \mathbb{N}$,

$$(PAP^{-1})^k = PA^kP^{-1}.$$

Thus,

$$\begin{aligned} e^{PAP^{-1}} &= \lim_{l \to +\infty} \sum_{k=0}^{l} \frac{PA^kP^{-1}}{k!} \\ &= \lim_{l \to +\infty} \Big[P \Big(\sum_{k=0}^{l} \frac{A^k}{k!} \Big) P^{-1} \Big] \\ &= P \Big(\lim_{l \to +\infty} \sum_{k=0}^{l} \frac{A^k}{k!} \Big) P^{-1} \\ &= P\, e^{A}\, P^{-1}. \end{aligned}$$

(b) For $e^{\lambda I_n + A}$: since $(\lambda I_n)A = A(\lambda I_n)$, we know that λI_n and A commute. Thus,

$$e^{\lambda I_n + A} = e^{\lambda I_n} \cdot e^{A}.$$

Now,

$$e^{\lambda I_n} = e^{\lambda\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}} = e^{\begin{pmatrix}\lambda & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda\end{pmatrix}}$$

$$= \begin{pmatrix}e^{\lambda} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda}\end{pmatrix} = e^{\lambda}\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix} = e^{\lambda} I_n.$$

Therefore,

$$e^{\lambda I_n + A} = (e^{\lambda} I_n)e^A = e^{\lambda}(I_n e^A) = e^{\lambda}e^A.$$

Example 3.1.3.

Let us compute e^A where $A = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$. We have:

$$\det(A - \lambda I_2) = \left|\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| = \left|\begin{pmatrix}-\lambda & -1\\ -1 & -\lambda\end{pmatrix}\right| = \lambda^2 - 1 = 0$$

which implies that $\lambda_1 = 1$ and $\lambda_2 = -1$ are the two distinct eigenvalues of A. Thus, A is diagonalizable:

$$A = P \cdot D \cdot P^{-1}, \quad D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

where $P = (v_1, v_2)$, with v_1 and v_2 being the eigenvectors of A associated with λ_1 and λ_2, respectively.

The eigenspaces:

$$E_{\lambda_1} = \{v_1 \in \mathbb{R}^2 \mid (A - \lambda_1 I)v_1 = 0\} = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^2 \middle| \begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0_{\mathbb{R}^2}\right\}$$

$$= \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^2 \middle| -x - y = 0\right\} = \left\{\begin{pmatrix}x\\ -x\end{pmatrix} \in \mathbb{R}^2 \middle| x \in \mathbb{R}\right\}$$

$$= \left\{x\begin{pmatrix}1\\ -1\end{pmatrix} \in \mathbb{R}^2 \middle| x \in \mathbb{R}\right\}.$$

We take $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Similarly,

$$\begin{aligned} E_{\lambda_2} &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, (A - \lambda_2 I) \begin{pmatrix} x \\ y \end{pmatrix} = 0_{\mathbb{R}^2} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, (A + I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

We take $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Recall that if $ad - bc \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So,

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^A = e^{PDP^{-1}} &= P\, e^D\, P^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{-1} + e & e^{-1} - e \\ e^{-1} - e & e^{-1} + e \end{pmatrix}. \end{aligned}$$

Definition 3.3.

Let $N \in M_n(\mathbb{R})$. We say that N is a **nilpotent** matrix of index $m \in \mathbb{N}^*$ if $N^{m-1} \neq 0_n$ and $N^m = 0_n$.

Example 3.1.4.

The matrix

$$N = \begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}$$

is a nilpotent matrix of index $m = 3$. Indeed:

$$N^2 = \begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 18 & -18 \\ 6 & 18 & 18 \end{pmatrix} \neq 0_3, \quad \text{and} \quad N^3 = N^2 \cdot N = 0_3.$$

Remark 3.1.2.

Every upper triangular matrix whose diagonal entries are all zero is nilpotent.

For example, the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent because it is upper triangular with zeros on the diagonal.

Theorem 3.1.4.

Let N be a nilpotent matrix of index $m \in \mathbb{N}^*$. Then:

$$e^N = I_n + \frac{N}{1!} + \cdots + \frac{N^{m-1}}{(m-1)!}.$$

Proof 3.1.4.

We have:

$$e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!} = I_n + \frac{N}{1!} + \frac{N^2}{2!} + \cdots + \frac{N^{m-1}}{(m-1)!} + \sum_{k=m}^{\infty} \frac{N^k}{k!}.$$

But since N is nilpotent of index m, for all $k \geq m$ we have $N^k = 0_n$. Indeed:

$$k \geq m \implies N^k = N^{(k-m)+m} = N^{k-m} \cdot N^m = N^{k-m} \cdot 0_n = 0_n.$$

Therefore,

$$e^N = I_n + \frac{N}{1!} + \cdots + \frac{N^{m-1}}{(m-1)!}.$$

Example 3.1.5.

Consider the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is nilpotent. Let us find its index m. We compute:

$$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_2,$$

so $m = 2$. Thus,

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = I_2 + \frac{1}{1!} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

3.2 Second-Order Differential Systems

3.2.1 General Form

A second-order differential equation or system can be written as:

$$y''(t) = f(t, y(t), y'(t)), \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \tag{3.2}$$

where $y(t) \in \mathbb{R}^n$ and $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

3.2.2 Reduction to a First-Order System

To study existence and uniqueness, a second-order system can be reduced to a first-order system by setting:

$$z(t) = y'(t).$$

Then (3.2) becomes:

$$\begin{cases} y'(t) = z(t), \\ z'(t) = f(t, y(t), z(t)), \\ y(t_0) = y_0, \quad z(t_0) = y_1. \end{cases}$$

This is now a first-order system in the variable $X(t) = (y(t), z(t))^T \in \mathbb{R}^{2n}$.

3.2.3 Existence and Uniqueness

- If f is continuous in (t, y, z), there exists at least one local solution (Peano theorem).
- If f is Lipschitz continuous in (y, z) uniformly in t, the local solution is unique (Cauchy-Lipschitz theorem).

Example

Consider the second-order differential equation:

$$y'' + y = 0, \quad y(0) = 1, \; y'(0) = 0.$$

Reducing it to a first-order system:

$$\begin{cases} y_1' = y_2, \\ y_2' = -y_1, \\ y_1(0) = 1, \ y_2(0) = 0, \end{cases}$$

which has the unique global solution:

$$y(t) = \cos(t), \quad y'(t) = -\sin(t).$$

3.3 Resolvent of a Linear System

Consider the system:[?, ?]

$$y' = A(t)y.$$

Lemma 3.3.1.

If for all $t \in I$, we have:

$$R(t, t_0) = e^{(t-t_0)A}.$$

Proof 3.3.1.

From the definition of the resolvent, we have:

$$R(t, t_0) = e^{\int_{t_0}^{t} A(u)du} = e^{\int_{t_0}^{t} A\,du} = e^{A\int_{t_0}^{t} du} = e^{(t-t_0)A}.$$

Example 3.3.1.

The resolvent of the system $y' = Ay$ with

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus:

$$R(t,t_0) = e^{(t-t_0)A}$$
$$= e^{(t-t_0)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} = e^{\begin{pmatrix} 2(t-t_0) & 0 \\ 0 & (t-t_0) \end{pmatrix}} = \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{(t-t_0)} \end{pmatrix}.$$

Lemma 3.3.2.

[?, ?] The solution of the system $y' = Ay$ is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{tA}c,$$

with $c \in \mathbb{R}^n$ such that

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Proof 3.3.2.

Since $I = \mathbb{R}$, we have $0 \in I$. Thus, from the previous result, the solution of the homogeneous system is:

$$\forall t \in \mathbb{R}: \quad y(t) = R(t,0)c, \quad c \in \mathbb{R}^n.$$

But from the previous lemma:

$$R(t,0) = e^{(t-0)A} = e^{tA}.$$

Example 3.3.2.

The solution of the system

$$y' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix} y$$

is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{tA}c, \quad c \in \mathbb{R}^2.$$

Here,

$$A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}, \quad \text{i.e. } y(t) = e^{tA}c,$$

but

$$e^{t\begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}} = e^{\begin{pmatrix} 4t & 3t \\ 0 & 4t \end{pmatrix}} = e^{4t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix}} = e^{4tI_2 + \begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix}}.$$

We can show that

$$\begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix}$$

is a nilpotent matrix of index $m = 2$. Therefore:

$$\begin{aligned} e^{tA} = e^{4t} \cdot e^{\begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix}} &= e^{4t}\left[I_2 + \frac{1}{1!}\begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix} \right] \\ &= e^{4t}\begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{4t} & 3te^{4t} \\ 0 & e^{4t} \end{pmatrix}. \end{aligned}$$

Thus:

$$\forall t \in \mathbb{R}: \quad y(t) = \begin{pmatrix} e^{4t} & 3te^{4t} \\ 0 & e^{4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1e^{4t} + 3c_2te^{4t} \\ c_2e^{4t} \end{pmatrix}.$$

Lemma 3.3.3.

The solution of the following system:

$$\begin{cases} y' = Ay \\ y(t_0) = y_0 \end{cases}$$

is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{(t-t_0)A}y_0.$$

Example 3.3.3.

The solution of the system:

$$\begin{cases} y' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix} y \\ y(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad \text{is given by: } \forall t \in \mathbb{R} : y(t) = e^{(t-t_0)A} y_0.$$

With $A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$, $t_0 = 1$ and $y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e.:

$$\begin{aligned} y(t) = e^{(t-1)\begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}} &= e^{4(t-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \cdot e^{\begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \left(I_2 + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix} \right) \\ &= e^{4(t-1)} \begin{pmatrix} 1 & 3(t-1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} y(t) &= \begin{pmatrix} e^{4(t-1)} & 3(t-1)e^{4(t-1)} \\ 0 & e^{4(t-1)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{4(t-1)} \\ 0 \end{pmatrix} \end{aligned}$$

3.3.1 Non-homogeneous differential system

Theorem 3.3.1.

If the matrix A admits n linearly independent eigenvectors $v_1, v_2, \cdots, v_n$ associated with the real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the general solution of (H) is given by:

$$y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots + c_n v_n e^{\lambda_n t}, \quad \text{with } v_1, v_2, \cdots, v_n \in \mathbb{R}.$$

Example 3.3.4.

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} y$$

Let us compute y_H:

$\lambda_1 = 1$ and $\lambda_2 = 2$ are the two distinct eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the eigenvectors of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ associated respectively with λ_1, λ_2.

$$y_H(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{2t} \end{bmatrix}.$$

- **Computation of y_p:**

Using the method of variation of constants, there exists a particular solution of the form:

$$y_p(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots, \text{ with } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}' = \left(v_1 e^{\lambda_1 t} \quad v_2 e^{\lambda_2 t}, \cdots, v_n e^{\lambda_n t} \right)^{-1} \cdot B(t).$$

Recall that if $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$, then:

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{bmatrix}$$

Example 3.3.5.

Solve the system: $y' = Ay + B$ where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The general solution of the system $y' = Ay + B$ is:

$y = y_H + y_p$, where y_H is the general solution of the associated homogeneous system $y' = Ay$, and y_p is a particular solution of the non-homogeneous system.

• **Computation of y_H:**

We have: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$ are the three distinct eigenvalues of A.
$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ are the eigenvectors of A associated respectively with $\lambda_1, \lambda_2, \lambda_3$. Thus:

$$\begin{aligned} y_H(t) &= c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} e^{4t} = \begin{bmatrix} c_1 e^t \\ 2c_2 e^{2t} \\ 4c_3 e^{4t} \end{bmatrix}, \end{aligned}$$

for all $t \in \mathbb{R}$, with $c_1, c_2, c_3 \in \mathbb{R}$.

• **Computation of y_p:**

Using the method of variation of constants, there exists a particular solution of the form:

$$\begin{aligned} &y_p(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}, \text{ with} \\ &\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{pmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & v_3 e^{\lambda_3 t} \end{pmatrix}^{-1} \cdot B(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Then:

$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 2e^{2t} & 0 \\ 0 & 0 & 4e^{4t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{bmatrix},$$

we get

$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} e^{-2t} \\ 0 \end{bmatrix}.$$

Thus $c_1' = c_3' = 0$ and $c_2' = \frac{1}{2}e^{-2t}$.
Taking integration constants suitably, one particular solution is:

$$y_p(t) = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

Hence the full solution is:

$$y(t) = y_H(t) + y_p(t) = \begin{bmatrix} c_1e^t \\ 2c_2e^{2t} \\ 4c_3e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} c_1e^t \\ 2c_2e^{2t} - \frac{1}{2} \\ 4c_3e^{4t} \end{bmatrix}.$$

• **Problem (P$_1$):**

$$\begin{cases} y' = Ay \\ y(t_0) = y_0 \end{cases}$$

The solution of (P_1) is:

$$y(t) = e^{(t-t_0)A} \cdot y_0, \quad \forall t \in \mathbb{R}.$$

• **Problem (P$_2$):**

$$\begin{cases} y' = Ay + B(t) \\ y(t_0) = y_0 \end{cases}$$

The solution of (P_2) is:

$$y(t) = e^{(t-t_0)A} \cdot y_0 + \int_{t_0}^{t} e^{(t-u)A}B(u)\,du, \quad \forall t \in I.$$

Example 3.3.6.

The solution of the system:

$$\begin{cases} y' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix} y \\ y(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad \text{is given by: } \forall t \in \mathbb{R} : y(t) = e^{(t-t_0)A}y_0.$$

With $A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$, $t_0 = 1$ and $y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is:

$$
\begin{aligned}
y(t) &= e^{(t-1)\begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}} \\
&= e^{4(t-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\
&= e^{4(t-1)} \cdot e^{\begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\
&= e^{4(t-1)} \left(I_2 + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix} \right) \\
&= e^{4(t-1)} \begin{pmatrix} 1 & 3(t-1) \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \begin{pmatrix} e^{4(t-1)} & 3(t-1)e^{4(t-1)} \\ 0 & e^{4(t-1)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{4(t-1)} \\ 0 \end{pmatrix}.
\end{aligned}
$$

3.3.2 The solution of the homogeneous system (H)

Lemma 3.3.4.

Let $t, t_0 \in I$. Consider the function f_{t,t_0} defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ by:

$$f_{t,t_0}(y_0) = y(t, t_0, y_0).$$

The mapping f_{t,t_0} is linear.

Definition 3.4.

The matrix associated with f_{t,t_0} is called the **resolvent matrix** of (H). It is denoted by:

$$R(t, t_0).$$

Theorem 3.3.2.

We have:

(a) $\forall t, t_0 \in I; \quad R(t, t_0) \in M_n(\mathbb{R})$.

(b) $\forall t, t_0 \in I; \quad R(t, t_0)y_0 = y(t, t_0, y_0)$.

(c) $\forall t_0 \in I; \quad R(t_0, t_0) = I_n$, where I_n denotes the identity matrix.

(d) $\forall t, s, r \in I; \quad R(t, s)R(s, r) = R(t, r)$.

(e) $\forall t, s \in I; \quad R(t, s)$ is invertible and:

$$(R(t, s))^{-1} = R(s, t).$$

(f) $\forall t, t_0 \in I; \quad \dfrac{d}{dt}R(t, t_0) = A(t) \cdot R(t, t_0)$.

(g) $\forall t, t_0 \in I; \quad \dfrac{d}{dt}R(t, t_0) = -R(t_0, t)A(t)$.

The fundamental system of (H)

Let $y_1, y_2, \ldots, y_n \in F(I, \mathbb{R}^n)$.

Definition 3.5.

We say that $\{y_1, y_2, \ldots, y_n\}$ is a **fundamental system** of (H) if:

(a) $y_1, y_2, \ldots, y_n$ are solutions of (H).

(b) $y_1, y_2, \ldots, y_n$ are linearly independent, that is:

$$\forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \quad (\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n) = 0$$
$$\Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Example 3.3.7.

For all $t \in \mathbb{R}$, let:

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

(1) Let us show that y_1 and y_2 are two solutions of $y' = A(t)y$. We have:

$$y_1'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and}$$

$$A(t)y_1(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} = \frac{1}{1+t^2}\begin{bmatrix} t^2+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus:

$$y_1'(t) = A(t)y_1(t), \quad \forall t \in \mathbb{R}.$$

So y_1 is a solution of (H). Similarly, one shows that y_2 is also a solution of (H).

(2) Let us show that y_1 and y_2 are linearly independent. Suppose $\alpha, \beta \in \mathbb{R}$ such that $\alpha y_1 + \beta y_2 = 0$. Then:

$$\begin{aligned} \alpha y_1 + \beta y_2 = 0 &\Longrightarrow \alpha y_1(t) + \beta y_2(t) = 0, \quad \forall t \in \mathbb{R}, \\ &\Longrightarrow \alpha \begin{bmatrix} t \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ t \end{bmatrix} = 0, \quad \forall t \in \mathbb{R}, \\ &\Longrightarrow \begin{bmatrix} \alpha t - \beta \\ \alpha + \beta t \end{bmatrix} = 0, \quad \forall t \in \mathbb{R}. \end{aligned}$$

This represents infinitely many equations in the two unknowns α and β. It suffices to take $t = 0$, which gives $\alpha = \beta = 0$.

Hence, y_1 and y_2 are linearly independent.

Theorem 3.3.3.

Let $\{y_1, y_2, \dots, y_n\}$ be a fundamental system of (H). Then:

$$\begin{aligned} S_H &= \Big[\{y_1, y_2, \dots, y_n\}\Big] \\ &= \Big\{ y \in F(I, \mathbb{R}^n) \mid y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n,\ \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \Big\}. \end{aligned}$$

Proof 3.3.3.

If $\{y_1, y_2, \dots, y_n\}$ is a fundamental system of (H), then it is linearly independent. Since $\#\{y_1, y_2, \dots, y_n\} = n = \dim S_H$, the set $\{y_1, y_2, \dots, y_n\}$ is a basis of S_H.

Example 3.3.8.

For all $t \in \mathbb{R}$, let

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2} \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

Since $\{y_1, y_2\}$ is a fundamental system of $y' = A(t)y$, we have:

$$\begin{aligned} S_H &= [\{y_1, y_2\}] \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid y = \alpha_1 y_1 + \alpha_2 y_2,\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\} \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid \forall t \in \mathbb{R},\ y = \alpha_1 y_1(t) + \alpha_2 y_2(t),\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\} \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid \forall t \in \mathbb{R},\ y = \begin{bmatrix} \alpha_1 t - \alpha_2 \\ \alpha_1 + \alpha_2 t \end{bmatrix},\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\}. \end{aligned}$$

Remark 3.3.1.

The general solution of (H) is:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \quad c_1, c_2, \dots, c_n \in \mathbb{R}.$$

3.3.3 The fundamental matrix of (H)

Definition 3.6.

The matrix whose columns form a fundamental system of (H) is called a **fundamental matrix** of (H). That is, M is a fundamental matrix if:

$$M = (y_1, y_2, \ldots, y_n), \quad \{y_1, y_2, \ldots, y_n\} \text{ is a fundamental system of } (H).$$

Example 3.3.9.

For all $t \in \mathbb{R}$, let

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2} \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

Since $\{y_1, y_2\}$ is a fundamental system of $y' = A(t)y$ for $t \in \mathbb{R}$, we set:

$$M(t) = (y_1(t), y_2(t)) = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}.$$

Thus M is a fundamental matrix of $y' = A(t)y$.

Theorem 3.3.4.

Let M be a fundamental matrix of (H). Then:

(a) For all $t \in \mathbb{R}$,

$$M'(t) = A(t)M(t).$$

(b) The general solution of (H) is:

$$y = Mc, \quad c \in \mathbb{R}^n.$$

Proof 3.3.4.

If M is a fundamental matrix of (H), then:

(a) We have:

$$\begin{aligned} M'(t) &= (y_1(t), y_2(t), \dots, y_n(t))' \\ &= (y_1'(t), y_2'(t), \dots, y_n'(t)) \\ &= (A(t)y_1(t), A(t)y_2(t), \dots, A(t)y_n(t)) \\ &= A(t)(y_1(t), y_2(t), \dots, y_n(t)) \\ &= A(t)M(t). \end{aligned}$$

(b) And:

$$\begin{aligned} y(t) &= c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) \\ &= (y_1(t), y_2(t), \dots, y_n(t)) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= M(t)c, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n. \end{aligned}$$

3.3.4 The Wronskian of a system of solutions of (H)

Let $y_1, y_2, \dots, y_n \in S_H$ (the set of solutions of (H)).

Definition 3.7.

The **Wronskian** of $\{y_1, y_2, \dots, y_n\}$, denoted W, is the determinant of the matrix whose columns are $y_1, y_2, \dots, y_n$:

$$\forall t \in I, \quad W(t) := \det[y_1(t), y_2(t), \dots, y_n(t)].$$

Theorem 3.3.5.

Let $y_1, y_2, \dots, y_n \in S_H$. The following statements are equivalent:

(a) $\forall t \in I, \quad W(t) \neq 0.$

(b) $\exists t_0 \in I$ such that $W(t_0) \neq 0$.

(c) $y_1, y_2, \ldots, y_n$ are linearly independent.

Proof 3.3.5.

(a) (1) $\Rightarrow$ (2): Trivial since if $W(t) \neq 0$ for all t, then in particular at some t_0 we have $W(t_0) \neq 0$.

(b) (2) $\Rightarrow$ (3): If $W(t_0) \neq 0$, then the vectors $(y_1(t_0), y_2(t_0), \ldots, y_n(t_0))$ are linearly independent.

(c) (3) $\Rightarrow$ (1): Since $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (H), they remain linearly independent for all $t \in I$, hence $W(t) \neq 0$ for all $t \in I$.

3.3.5 The solution of the non-homogeneous system

Theorem 3.3.6.

Let $(t_0, y_0) \in I \times \mathbb{R}^n$. The solution of the system (E) is:

$$\forall t \in I, \quad y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, u)B(u)\, du.$$

Proof 3.3.6.

Consider the function defined on I by:

$$Z(u) = R(t_0, u)y(u).$$

For $u \in I$, we have:

$$\begin{aligned}
Z'(u) &= \frac{d}{du}\Big[R(t_0, u)y(u)\Big] \\
&= \frac{d}{du}(R(t_0, u)) \cdot y(u) + R(t_0, u)y'(u) \\
&= -R(t_0, u)A(u) \cdot y(u) + R(t_0, u)(A(u)y(u) + B(u)) \\
&= R(t_0, u)B(u).
\end{aligned}$$

That is, $\forall u \in I; \quad Z'(u) = R(t_0, u)B(u)$, which implies:

$$\forall t \in I; \quad Z(t) = Z(t_0) + \int_{t_0}^{t} R(t_0, u)B(u)\, du$$

Thus,

$$\forall t \in I; \quad R(t_0,t)y(t) = R(t_0,t_0)y_0 + \int_{t_0}^{t} R(t_0,u)B(u)\,du$$

Therefore,

$$\begin{aligned}\forall t \in I; \quad y(t) &= R(t,t_0)I_n y(t_0) + \int_{t_0}^{t} R(t,t_0)R(t_0,u)B(u)\,du \\ &= R(t,t_0)y(t_0) + \int_{t_0}^{t} R(t,u)B(u)\,du.\end{aligned}$$

Hence the result.

Theorem 3.3.7.

Let $(t_0, y_0) \in I \times \mathbb{R}^n$. The solution of the system (H) is given by:

$$\forall t \in I; \quad y(t) = R(t,t_0)y_0.$$

Proof 3.3.7.

It is enough to apply the previous theorem with $B = 0$.

Computation of Y_H and Y_P for two non-homogeneous systems

1. Consider the system

$$Y' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}.$$

Compute Y_H. The diagonal matrix has eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2$ with eigenvectors

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the homogeneous solution is

$$Y_H(t) = c_1V_1e^{\lambda_1 t} + c_2V_2e^{\lambda_2 t} = \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

Compute Y_P. Using variation of constants, seek a particular solution of the form

$$Y_P(t) = c_1(t)V_1e^{\lambda_1 t} + c_2(t)V_2e^{\lambda_2 t},$$

with c_1, c_2 differentiable functions satisfying

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \left(V_1e^{\lambda_1 t}\; V_2e^{\lambda_2 t}\right)^{-1} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Thus $c_1'(t) = 1$ and $c_2'(t) = t$, so one antiderivative choice is

$$c_1(t) = t, \qquad c_2(t) = \tfrac{1}{2}t^2.$$

Then

$$Y_P(t) = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \tfrac{1}{2}t^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} te^t \\ \frac{1}{2}t^2e^{2t} \end{pmatrix}.$$

Therefore the general solution is

$$\boxed{Y(t) = Y_H(t) + Y_P(t) = \begin{pmatrix} (c_1 + t)e^t \\ (c_2 + \frac{1}{2}t^2)e^{2t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.}$$

2. Consider the initial-value problem

$$\begin{cases} Y' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} Y + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}, \\ Y(0) = Y_0. \end{cases}$$

The variation-of-constants formula gives

$$Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-u)A}B(u)\,du, \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B(u) = \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix}.$$

To compute e^{tA}, write $A = I_3 + S$ where

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^3 = 0.$$

Then

$$e^{tA} = e^{t(I_3+S)} = e^t e^{tS} = e^t\left(I_3 + tS + \tfrac{t^2}{2}S^2\right) = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$Y(t) = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)} \begin{pmatrix} 1 & t-u & \frac{1}{2}(t-u)^2 \\ 0 & 1 & t-u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} du.$$

One may evaluate the integral explicitly componentwise if a closed-form particular solution is desired; otherwise the integral representation above is the standard variation-of-constants solution. **Let us compute** Y_H : we have $\lambda_1 = 1$ and $\lambda_2 = 2$ which are the two distinct eigenvalues of $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors. Thus:

$$\begin{aligned} Y_H(t) &= c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}, \text{ with } c_1, c_2 \in \mathbb{R}. \end{aligned}$$

Let us compute Y_P : we use the method of variation of constants, there exists a particular solution of the form:

$$Y_P(t) = c_1(t) V_1 e^{\lambda_1 t} + c_2(t) V_2 e^{\lambda_2 t}$$

where $c_1(t), c_2(t)$ are two differentiable functions such that:

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \left(V_1 e^{\lambda_1 t}\ V_2 e^{\lambda_2 t}\right)^{-1} \cdot B(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}^{-1} \cdot \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}$$

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix} \Longrightarrow \begin{cases} c_1(t) = t \\ c_2(t) = \dfrac{1}{2}t^2 \end{cases} . \text{ Thus}$$

$$\begin{aligned} Y_P(t) &= t \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \frac{1}{2}t^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} te^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{2}t^2 e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} te^t \\ \dfrac{1}{2}t^2 e^{2t} \end{pmatrix} \end{aligned}$$

Hence the general solution:

$$Y(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} te^t \\ \dfrac{1}{2}t^2 e^{2t} \end{pmatrix}.$$

5.

$$\begin{cases} Y' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} Y + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \\ Y(0) = Y_0. \end{cases}$$

We have the general solution:

$$Y(t) = e^{tA} Y_0 + \int_0^t e^{(t-u)A} \cdot B(u)\, du.$$

Let us compute

$$e^{tA} = e^{\begin{pmatrix} t & t & 0 \\ 0 & t & t \\ 0 & 0 & t \end{pmatrix}} = e^{tI_3 + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}} = e^t e^{\begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}}$$

Let $N = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$, which is a nilpotent matrix of order 3. We have:

$$\begin{aligned} e^N &= I_3 + \frac{N}{1!} + \frac{N^2}{2!} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore:

$$e^{tA} = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} Y(t) &= e^{tA}Y_0 + \int_0^t e^{(t-u)A} B(u)\, du \\ &= e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} B(u)\, du \\ &= e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)} \begin{pmatrix} 1 & (t-u) & \frac{1}{2}(t-u)^2 \\ 0 & 1 & (t-u) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} du. \end{aligned}$$